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THE PARADOX OF TORSIONAL BUCKLING

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Within this paper, the background and the substance of a surprising formation of the torsional buckling loads for thin-walled columns with variable, bisymmetric I-sections is presented. Appropriate reference to flexural buckling and to the originally investigated neighbouring problem of free torsional vibrations is made. Author's opinion on possible existence of similar features in coupled flexural-torsional eigenvalue problems is expressed.

1. Preface

Although in the past some reference to the title issue has already been made (cf [11,12,13,14]), the corresponding papers, except the last quoted, appeared as non-English written publications and were edited partly by a private foundation, so that they must be regarded to have only an extremely limited circulation. Therefore, by presenting this problem now, still open, to the direct insight of a highly specialized readership, it is hoped to achieve further development of the governing theory with an expected impact on the practice of engineering design.

2. Background

As far as the classic theory of Vlasov [34] is concerned, the generalization for thin-walled beam-shells with variable cross-sections was achieved a quarter of a century ago (cf [4,5,6]). This theory was later extended, including free vibration analysis, for cases of variable, built-up sections [7], with particular emphasis on bisymmetric I-profiles [8]. In both the latter papers, for the numerical example of an I-beam with variable bisymmetric cross-section, shown in Fig.1, the lowest natural torsional frequencies $k_{1\theta}$ have been determined by the finite difference method. They were compared with those corresponding to cases of both the enveloping

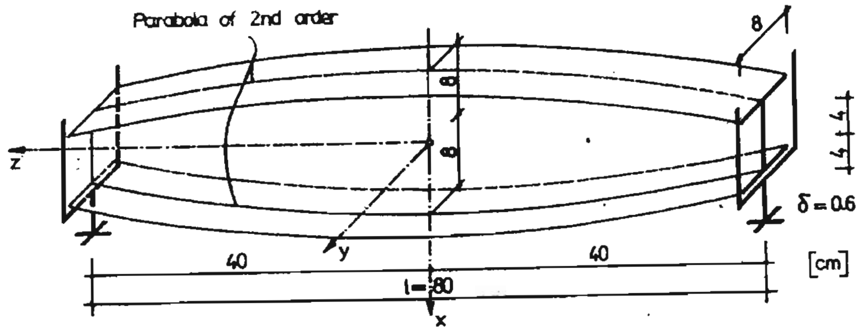


Fig. 1. Beam investigated for natural torsional frequencies

constant sections: minimal and maximal – calculated similarly, and on grounds of an exact solution. The relevant results are given in Table 1.

Table 1. Natural torsional frequencies for beam of Fig.1

Cross--section	Type of solution			
	Exact	FDM 20 segm.	FDM 10 segm.	FDM 4 segm.
Constant minimal	544	543	532	509
Variable	–	321	316	289
Constant maximal	494	493	489	466

It follows from Table 1 that the $k_{1\theta}$–value, obtained for the variable cross–section, falls beyond the interval defined by both the extreme constant sections. This fact, in view of earlier conventional results of static torsion analysis was regarded as uncommon and, consequently, treated with much reservation. Therefore, in the next step, the related question of of torsional buckling has become a point of concern: the controversial result of the dynamic inquiry could be, eventually, explained by qualitative differences in the intensity of mass distribution for the case of the variable section against both of the constant ones, but a similar phenomenon within the problem of torsional buckling was considered as entirely impossible. This opinion was strengthened by the existing investigations concerning flexural buckling of thin–walled bars with constant versus variable sections (cf [2]).

3. Theory

3.1. Introduction

The general view of the investigated, axially compressed, thin–walled I–column with variable, bisymmetric cross–section is presented in Fig.2. It is obvious that in this case the problems of flexure and torsion are uncoupled and the determination of all three critical forces, both the flexural and the torsional, can be approached separately; in the continuation, the focus is put on torsional buckling alone.

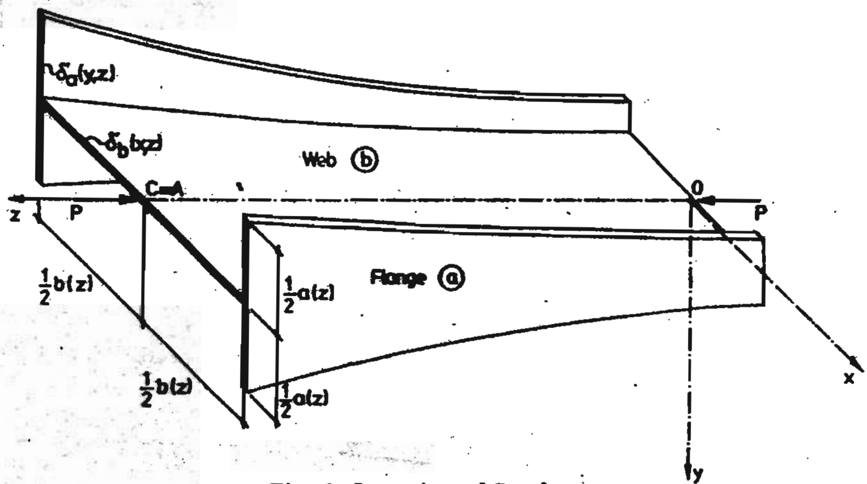


Fig. 2. Investigated I–column

In practice the determination of the torsional buckling load can be performed by applying the usual finite element method based upon step wise constant sections. For reasons of a more general treatment, it is worthwhile to define the governing differential equation of torsional buckling and to analyze its practical performance. the quoted differential equation has been derived in the previous paper [9] – on the basis of static equilibrium and, independently, utilizing the energy method of Bleich [1]; following this, the latter, together with the corresponding finite difference solution, has been demonstrated by Cywiński and Kollbrunner [13]. In the present paper, the established theory will be reflected on with due reference to the last paper cited.

3.2. Torsional deformation

To find the fundamental differential equation of the problem, first the torsion associated strain-displacement relations for an arbitrary point of the thin–walled bar middle surface must be established; a reference illustration is given in Fig.3. In particular, it is necessary to determine the flange strains ε_a^* assumed to be tangent to the corresponding flange curves; the z–axis parallel web strains ε_b are here equal to zero.

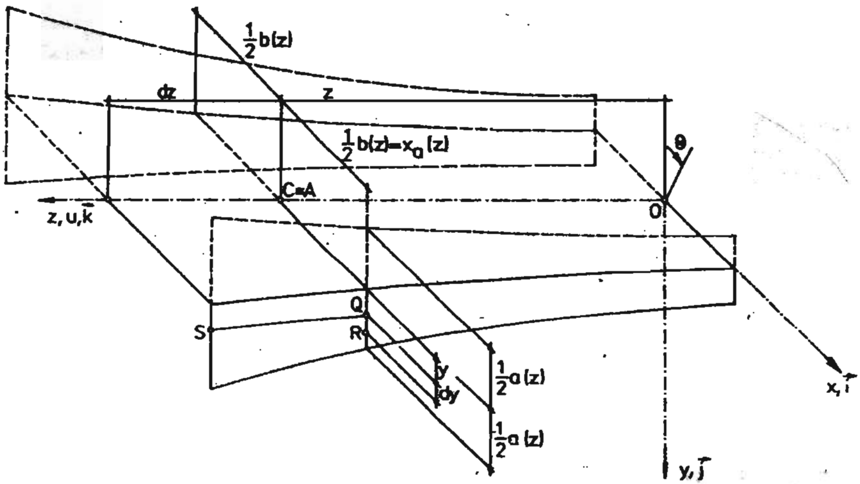


Fig. 3. Reference illustration for strain–displacement analysis

According to Fig.3, the coordinates of an arbitrary flange point Q and of both the adjacent ones, R and S, are as follows

$$\begin{aligned} &Q(x_a, y, z)\\ &R(x_a, y + dy, z)\\ &S\Big(x_a + \frac{dx_a}{dz}, y, z + dz\Big) \end{aligned} \tag{3.1}$$

The foregoing points, due to the bar torsion about the axis of the centers of twist A, move into new positions with the coordinates as follows

$$\begin{aligned} &Q'(x_a - \Theta y, y + \Theta x_a, z + u)\\ &R'\Big(x_a - \Theta y - \Theta dy, y + dy + \Theta x_a, z + u + \frac{\partial u}{\partial y}dy\Big)\\ &S'\Big(x_a + \frac{dx_a}{dz}dz - \Theta y - \frac{d\Theta}{dz}dzy, y + \Theta x_a + \frac{d\Theta}{dz}dzx_a + \Theta\frac{dx_a}{dz}dz,\\ &\qquad z + dz + u + \frac{\partial u}{\partial z}dz\Big) \end{aligned} \tag{3.2}$$

hereby $u(y, z)$ represents the temporarily unknown displacement of point Q along the bar axis z.

The flange tangent directed strain ε_a^* of point Q can be expressed in the following way

$$\varepsilon_a^* = \frac{(\overline{Q'S'})^2 - (\overline{QS})^2}{2(\overline{QS})^2} \tag{3.3}$$

with $(\overline{QS})^2$ and $(\overline{Q'S'})^2$ given by the relations

$$\begin{aligned}(\overline{QS})^2 &= \Big[1 + \Big(\frac{dx_a}{dz}\Big)^2\Big]dz^2 \\ (\overline{Q'S'})^2 &= \Big[\Big(\frac{dx_a}{dz} - \frac{d\Theta}{dz}y\Big)^2 + \Big(\frac{d\Theta}{dz}x_a + \Theta\frac{dx_a}{dz}\Big)^2 + \Big(1 + \frac{\partial u}{\partial z}\Big)^2\Big]dz^2 \cong \\ &\cong \Big[\Big(\frac{dx_a}{dz}\Big)^2 - 2\frac{d\Theta}{dz}\frac{dx_a}{dz}y + 1 + 2\frac{\partial u}{\partial z}\Big]dz^2\end{aligned} \tag{3.4}$$

By virtue of Eq (3.4), Eq (3.3) becomes

$$\varepsilon_a^* = \frac{1}{1 + \big(\frac{dx_a}{dz}\big)^2}\Big(\frac{\partial u}{\partial z} - \frac{d\Theta}{dz}\frac{dx_a}{dz}y\Big) \tag{3.5}$$

Expressing now the quantities $\overline{Q'R'}$ and $\overline{Q'S'}$ in vector shape

$$\begin{aligned}\overrightarrow{Q'R'} &= -\Theta dy\vec{i} + dy\vec{j} + \frac{\partial u}{\partial y}dy\vec{k} \\ \overrightarrow{Q'S'} &= \Big(\frac{dx_a}{dz} - \frac{d\Theta}{dz}y\Big)dz\vec{i} + \Big(\frac{d\Theta}{dz}x_a + \Theta\frac{dx_a}{dz}\Big)dz\vec{j} + \Big(1 + \frac{\partial u}{\partial z}\Big)dz\vec{k}\end{aligned} \tag{3.6}$$

and utilizing the usual hypothesis of middle surface zero shear deformation in the form of

$$\cos(\overrightarrow{Q'R'}, \overrightarrow{Q'S'}) = \frac{\overrightarrow{Q'R'}\,\overrightarrow{Q'S'}}{|\overrightarrow{Q'R'}||\overrightarrow{Q'S'}|} = 0 \tag{3.7}$$

one obtains the equation

$$-\Theta\frac{dx_a}{dz} + \Theta\frac{d\Theta}{dz}y + \frac{d\Theta}{dz}x_a + \Theta\frac{dx_a}{dz} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial y} = 0 \tag{3.8}$$

wherefrom, after reduction and negligence of small quantities of higher order, it follows that

$$\frac{\partial u}{\partial y} = -\frac{d\Theta}{dz}x_a \tag{3.9}$$

The integration of Eq (3.9) with respect to y yields the longitudinal displacement u, expressed as follows

$$u = u_0 - \frac{d\Theta}{dz}x_a y \tag{3.10}$$

with the function u_0 representing the z–axis parallel displacement of the web and flange profile lines intersection point $(y = 0)$. Here $u_0 = 0$ and, thus

$$u = -\frac{d\Theta}{dz} x_a y \tag{3.11}$$

The derivative of u with respect to z is given, evidently, by

$$\frac{\partial u}{\partial z} = -\frac{d^2\Theta}{dz^2} x_a y - \frac{d\Theta}{dz}\frac{dx_a}{dz} y \tag{3.12}$$

and, taking this into account, Eq (3.5) becomes the following

$$\varepsilon_a^* = \frac{1}{1 + \left(\frac{dx_a}{dz}\right)^2}\Big(-\frac{d^2\Theta}{dz^2} x_a y - 2\frac{d\Theta}{dz}\frac{dx_a}{dz} y\Big) \equiv -\frac{1}{1 + {x'_a}^2}(\Theta'' x_a y + 2\Theta' x'_a y) \tag{3.13}$$

Regarding ${x'_a}^2$, compared to 1, as small (assumption of moderate variation of cross–section), it can be neglected and, thus, Eq (3.13) reduces to the simple form

$$\varepsilon_a^* = -(\Theta'' x_a y + 2\Theta' x'_a y) \tag{3.14}$$

Finally, introducing the expression for the sectorial area

$$w = x_a y \tag{3.15}$$

Eq (3.14) can be expressed, most concisely, as

$$\varepsilon_a^* = -(\Theta'' w + 2\Theta' w) \tag{3.16}$$

It must be stressed that, when considering variable sections without a more thorough investigation, solely on the basis of the theory appropriate for constant sections, the flange strain ε_a becomes

$$\varepsilon_a = -\eta_a'' y = -(\Theta x_a)'' y = -(\Theta w)'' \tag{3.17}$$

with η_a being the flange deflection in the y–direction. In view of the derived Eq (3.16) it is evident that Eq (3.17) inherits an error that vanishes for x_a being linear (or constant) only.

3.3. Fundamental differential equation

In deriving this equation the application of the energy method will be shown. It is assumed that at any cross–section of the I–column under consideration, besides the usual normal stresses n due to P, also the normal stresses σ and shear stresses τ, as well as the Saint–Venant torsional moments T_v, due to mixed torsion (warping + Saint–Venant), occur, with the obvious restriction on σ and τ to column flanges only. However, the internal strain energy U_i is being produced only by σ and T_v, since shear deformation is not taken into account.

By virtue of Hooke's law and in view of Eq (3.16) the normal stress σ can be expressed as follows

$$\sigma = -E(\Theta''w + 2\Theta'w') \tag{3.18}$$

where E is the Young modulus. The expression for the Saint–Venant torsional moment T_v is similar to that for constant sections

$$T_v = GJ_v\Theta' \tag{3.19}$$

with G and J_v standing for shear modulus and the Saint–Venant moment of inertia, respectively. The energy U_i can be formulated in the following way

$$U_i = \frac{1}{2}\int\limits_V \frac{\sigma^2}{E}dV + \frac{1}{2}\int\limits_l T_v\Theta'dz \tag{3.20}$$

where $dV = dFdz$, being the product of cross–sectional area and length elements dF and dz, respectively, represents the column volume element; V is the column total volume and l denotes its entire length. By putting Eqs (3.18) and (3.19) into Eq (3.20) one obtains

$$U_i = \frac{1}{2}\int\limits_l \Big[E\Big(\Theta''\int\limits_F w^2dF + 4\Theta'^2\int\limits_F w'^2dF + 4\Theta''\Theta'\int\limits_F ww'dF\Big) + GJ_v\Theta'^2\Big]dz \tag{3.21}$$

This equation, with the notations

$$\int\limits_F w^2dF = F_{ww}$$
$$\int\limits_F w'^2dF = F_{w'w'} \tag{3.22}$$
$$\int\limits_F ww'dF = F_{ww'}$$

can be presented finally in the following form

$$U_i = \frac{1}{2}\int\limits_l \Big(EF_{ww}\Theta''^2 + 4EF_{w'w'}\Theta'^2 + 4EF_{ww'}\Theta''\Theta' + GJ_v\Theta'^2\Big)dz \tag{3.23}$$

Taking account of the cross–sectional variation, the potential energy of the external loads U_e can be expressed as follows

$$U_e = \frac{1}{2}\int\limits_l\int\limits_F n(\xi'^2 + \eta'^2)dFdz \tag{3.24}$$

where

$$n = -\frac{P}{F} \tag{3.25}$$

ξ and η are the x- and y–axis oriented displacements of an arbitrary cross–sectional point caused by the column rotation about the z–axis. For the I–column considered Eq (3.24) becomes the following

$$U_e = \frac{1}{2}\int\limits_l n\Big(2\int\limits_{F_a}({\xi'_a}^2 + {\eta'_a}^2)dF_a + \int\limits_{F_b}({\xi'_b}^2 + {\eta'_b}^2)dF_b\Big)dz \tag{3.26}$$

Taking into account the obvious relations

$$\begin{array}{llll} \xi_a = -\Theta y & \eta_a = \Theta x_a & \xi_b = 0 & \eta_b = \Theta x \\ \xi'_a = -\Theta' y & \eta'_a = \Theta' x_a + \Theta x'_a & \xi'_b = 0 & \eta'_b = \Theta' x \end{array} \tag{3.27}$$

and integrating over the areas of flange and web cross–sections F_a and F_b, respectively, Eq (3.26) transforms to the adequate form

$$U_e = \frac{1}{2}\int\limits_l n\Big((F_{xx} + F_{yy})\Theta'^2 + 2F_a(2\Theta\Theta' x_a x'_a + \Theta^2 {x'_a}^2)\Big)dz \tag{3.28}$$

here F_{xx} and F_{yy} are the conventional moments of inertia

$$F_{xx} = \int\limits_F x^2 dF \qquad\qquad F_{yy} = \int\limits_F y^2 dF \tag{3.29}$$

By putting Eq (3.25) into Eq (3.28) and with

$$x_a = \frac{b}{2} \qquad\qquad \frac{F_{xx} + F_{yy}}{F} = r^2 \qquad\qquad \frac{F_a}{2F} = \alpha \tag{3.30}$$

one obtains the potential U_e in the following form

$$U_e = -\frac{1}{2}\int\limits_l \Big(r^2 P\Theta'^2 + 2\alpha P\Theta\Theta' bb' + \alpha P\Theta^2 b'^2\Big)dz \tag{3.31}$$

The total energy U is in general

$$U = U_i + U_e \tag{3.32}$$

becoming in the particular case investigated

$$\begin{aligned} U \;=\; & \frac{1}{2}\int\limits_l \Big(EF_{ww}\Theta''^2 + 4EF_{w'w'}\Theta'^2 + 4EF_{ww'}\Theta''\Theta' + GJ_v\Theta'^2 - \\ & - r^2 P\Theta'^2 - 2\alpha P\Theta\Theta' bb' - \alpha P\Theta^2 b'^2\Big)dz \end{aligned} \tag{3.33}$$

The usual condition of the total energy minimum is expressed by the well known differential equation of Euler [1]

$$\frac{\partial \Phi}{\partial \Theta} - \frac{d}{dz}\left(\frac{\partial \Phi}{\partial \Theta'}\right) + \frac{d^2}{dz^2}\left(\frac{\partial \Phi}{\partial \Theta''}\right) = 0 \tag{3.34}$$

hereby Φ represents the integrand of the integral expression (3.33). This results in the following differential equation

$$\begin{gathered}\left(EF_{ww}\Theta'' + 2EF_{ww'}\Theta'\right)'' - 2\left(EF_{ww'}\Theta'' + 2EF_{w'w'}\Theta'\right)' + \\ + \left((r^2P - GJ_v)\Theta'\right)' + \left((\alpha P)'b' + \alpha P b''\right)\Theta b = 0\end{gathered} \tag{3.35}$$

which, with P regarded as constant, after appropriate transformation becomes

$$\begin{gathered}\left[\frac{EF_{ww}}{b^2}\left((\Theta b)'' - \Theta b''\right)\right]'' b - \frac{EF_{ww}}{b^2}\left((\Theta b)'' - \Theta b''\right)b'' + \\ + \left((r^2P - GJ_v)\Theta'\right)' + P(\alpha b')'\Theta b = 0\end{gathered} \tag{3.36}$$

This equation represents the fundamental differential equation of torsional buckling for the investigated case of a thin-walled I-column with variable, bisymmetric cross-section.

3.4. Particular forms of the fundamental differential equation

The derived Eq (3.36) concerns the I-column with a cross-section that varies in an arbitrary way, i.e. changes can be attributed to its web-height, flange-widths and thicknesses of web and flanges.

Hereby the change of web-height has a special importance since it substantially affects the shape of the fundamental differential equation in force.

For the case of a constant web-height , with all the other changes of the cross-section remaining as mentioned above, one obtains the simple equation

$$\left(EF_{ww}\Theta''\right)'' + \left((r^2P - GJ_v)\Theta'\right)' = 0 \tag{3.37}$$

since $b' = b'' = 0$. In turn, this equation reduces to the well known form of Wagner [35]

$$EF_{ww}\Theta^{IV} + (r^2P - GJ_v)\Theta'' = 0 \tag{3.38}$$

when the entire cross-section remains constant.

Another special case is the tapered I-member investigated by Culver [3]; its differential equation $(6)_3$ can be written in the following form

$$E\left(F_{ww}\Theta''\right)'' + 4E\left(\frac{F_{ww}}{z^2}\Theta'\right)' + \left((r^2P - GJ_v)\Theta'\right)' = 0 \tag{3.39}$$

The same case considered on the basis of Eq (3.36) produces

$$\left(\frac{EF_{ww}}{b^2}(\Theta b)''\right)'' b + \left((r^2P - GJ_v)\Theta'\right)' = 0 \tag{3.40}$$

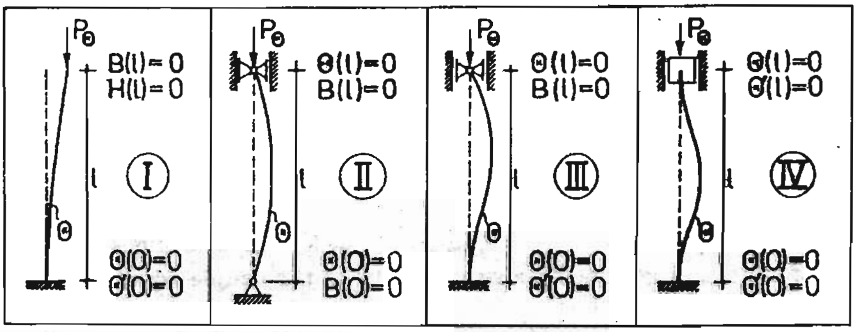
since $b'' = 0$ and $\alpha' = 0$. It has already been shown (cf [9,13]) that both the last equations are identical.

3.5. Method of solution – boundary conditions

Obviously, exact solutions of the generalized equations are not available and, therefore, approximate solution methods must be applied. One of them is the finite difference method. For questions of thin-walled bars it has already been used several times in the past (cf [4,5,7,8,13]), proving successfully its fitness. As far as the torsional buckling problem is concerned the application of the finite difference method was shown in detail in [13]. Because of its general character, it is not discussed in this paper.

The boundary conditions for the four Euler-similar types of support are given in Table 2.

Table 2. Types of support and boundary conditions



It follows from Table 2 that, besides the angle of twist Θ and the warping rate Θ', the bimoment B, and the total torsional moment $T = T_v + T_w$, are also involved; hereby T_w ia a new quantity, namely the flexural-torsional moment. According to [7,8] B and T_w result from the following equations

$$B = -\frac{EF_{ww}}{b}\left((\Theta b)'' - \Theta b''\right) \tag{3.41}$$

$$T_w = -\left[\left[\frac{EF_{ww}}{b^2}\left((\Theta b)'' - \Theta b''\right)\right]' b - \frac{EF_{ww}}{b^2}\left((\Theta b)'' - \Theta b''\right) b'\right] = \left(\frac{B}{b}\right)' b - \frac{B}{b} b' \tag{3.42}$$

4. Numerical example

Applying the developed theoretical basis, the torsional buckling load P_Θ of the considered I–column with variable cross–section can be determined. The practical conduct will be shown on the numerical example of I–column, illustrated in Fig.4. Hereby interest was drawn only to the formal analysis side, abandoning the questions of the column actual performance, which depends upon other features like the stress range, material and geometrical imperfections, limit strength, etc., not considered in this paper.

The type of the assumed cross–sectional variation allowed the computations to be based on the more simple form (3.37) of the fundamental differential equation. It has been solved as an eigenvalue problem by the finite difference method, for the boundary conditions corresponding to fork–type supports. The column has been divided along its length into 20 equal parts. The obtained result is demonstrated in Table 3 – where, for comparison, also the Euler buckling loads P_x and P_y due to flexure about the x- and y–axis, respectively, are given.

Table 3. Comparison of buckling loads (*Approximate according to [18] – Diagram I 5)

Cross--section	Buckling loads [MN]		
	P_x	P_y	P_Θ
Constant minimal	1.73	6.05	3.34
Variable	10.18*	10.39*	9.51
Constant maximal	13.82	11.23	8.54

It follows from Table 3 that for the variable cross–section investigated, in opposition to flexural buckling, the torsional buckling appears to produce the critical load P_Θ located outside the interval, specified by the two values appropriate for both the extreme constant sections. This result confirmed the earlier reported, qualitatively similar effect, obtained within the problem of free vibrations. It was found to be very doubtful: one could get the impression that the weakening of the column body results in its strengthening. This thesis contains all the features of the paradox, i.e. the "paradox first form".

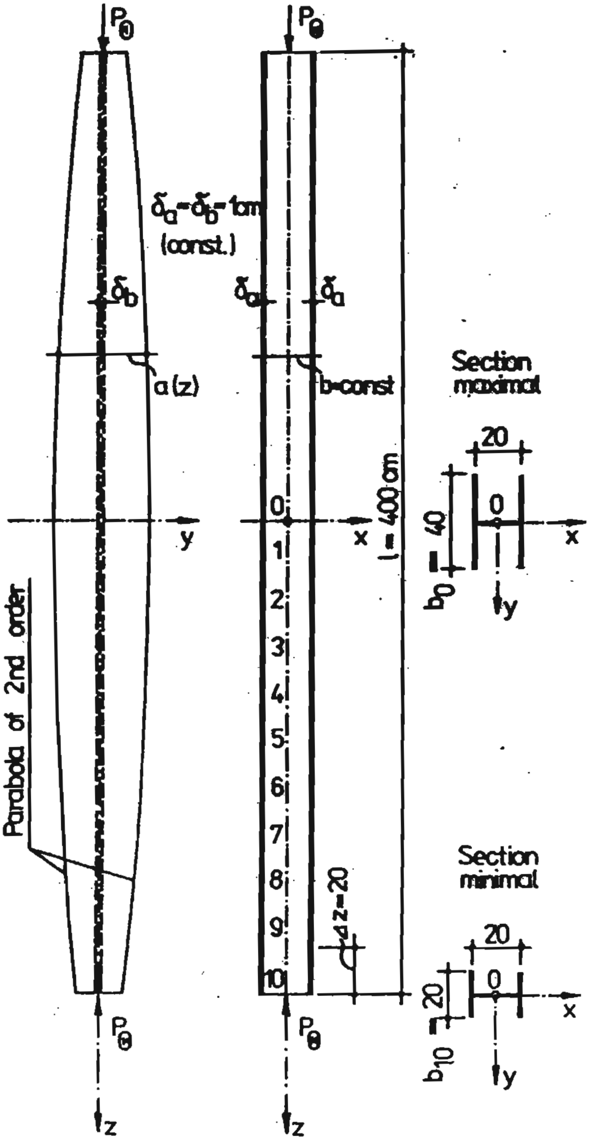


Fig. 4. Variable I–column analysed numerically

5. Post–paradox research

After the paradox was discovered, prior to its first public presentation [13], its existence was carefully verified, also by applying another approach (Galerkin). Each time the buckling load $P_\Theta = 9.51$ MN was found formally legitimate. Moreover, an opposite cross–sectional variation: increase of flange widths towards the member ends, brought up a decrease of the P_Θ–value, below those corresponding to both the constant sections [10]; this properly reflects the "paradox second form". Thus, the quoted paradox appeared now in its complete form.

Because the paper by Cywiński and Kollbrunner [13] was distributed through private channels only, comments were scarce. Corresponding notice was first made by Lind [19]; the paradox existence was fully confirmed by applying the method of Rayleigh and Vianello. Similar observations were reported by Lindner [20], where the Ritz method together with the utilization of Hermite polynomials was used. Finite element procedures in different versions were applied later by Szymczak [27], yielding the same qualitative results. Hereby, independently, besides the paradox first form, also the appearance of its second form has been confirmed. The substantial results of the last two comments, versus those of [13], are pointed out in Table 4; related details are given elsewhere [11,14].

Table 4. Comparison of torsional buckling loads

Cross-section	Solution				P_Θ [MN]
Constant minimal	Exact				3.34
Variable	Approximation	Cywiński/ Kollbrunner	1971 [13]	FDM $\Delta z = 20$	9.51
		Lindner	1976 [20]	Ritz + Hermite	9.29
		Szymczak	1978 [27]	FEM $\Delta z = 20$	9.40
Constant maximal	Exact				8.54

In order to be precise it should be underlined that the P_Θ–values of Table 4 correspond to I–shapes somehow different, as far as the web–height is concerned; in [13] it was specified as the distance between the flange center lines, whereas in [27] and, probably, in [20] – as the distance between the flange outer fibers.

A special investigation of the paradox second form was discussed by Szymczak [27], with simultaneous research concerning the interconnected problem of compression and vibrations as presented in [26]. Applying the criterion of dynamic stability, the value of the compressive force was determined which corresponded to zero–value of the natural frequency in torsion – thus describing the approached buckling load P_Θ; its paradoxical formation has been fully confirmed. The presented correlation of buckling and vibration problems supported the earlier Authors statements [7,8] on the questioned location of the natural frequencies in torsion ($k_{1\Theta}$) of thin walled bars with variable cross–sections. This feature has been later reliably confirmed by Wekezer [36].

An additional confirmation of the paradoxical P_Θ–formation was obtained by Szymczak [28,29,30] within the post–critical analysis of torsional buckling, where the relevant bifurcation points were found symmetric and stable, thus excluding an eventual drop in buckling load due to the initial imperfections.

An advanced investigation of eigenvalue problems for bars with variable cross–sections, related to optimization problems was given by Grinev and Filippov [17]. This monographic study concerned eigenvalues of longitudinal, torsional and flexural vibrations, as well as of flexural buckling, whereby only bars with solid cross–sections were considered. It was shown that in all three cases of free vibrations, the frequencies corresponding to bars with variable cross–sections could be found also outside the ranges determined by the extreme constant sections – on the contrary to flexural buckling where, in such cases the respective buckling loads have been found to be situated, necessarily, inside those ranges. The aforementioned type of approach has been extended later by Szymczak [31] into the family of thin walled sections. The former statements on possible locations of P_Θ and $k_{1\Theta}$ for variable sections outside the ranges specified by both the extreme constant sections, have been fully confirmed in that way, as well.

A particular view of the paradox was demonstrated by Dąbrowski [15]. Expressing the buckling load of mixed torsion P_Θ in the form of

$$P_\Theta = P_v + P_w \tag{5.1}$$

where P_v and P_w represent the effects of Saint–Venant (primary) and warping (secondary) torsion, respectively, the analysis of the P_v–component became a special point of interest. It was proved that the mentioned component behaved paradoxically even when comparing I–columns with constant sections alone. The opinion was expressed that just this P_v–component should be blamed for the existence of the paradox within the class of columns with variable cross–sections considered. It has been shown previously (cf [11,12,14]) that the thesis of Dąbrowski [15] should be rejected. For the case of the variable cross–section under consideration, the proof has been given that the P_w–component had a more decisive impact (than P_v) on the formation of P_Θ, and that P_w was also the subject of a paradoxical performance, although the behaviour of P_w within the class of constant sections was "normal". A graphical illustration of those findings is given in Fig.5.

In the 80–ies some other studies were published on the problems concerned. The mutual relationship of vibration and buckling problems for I–columns with

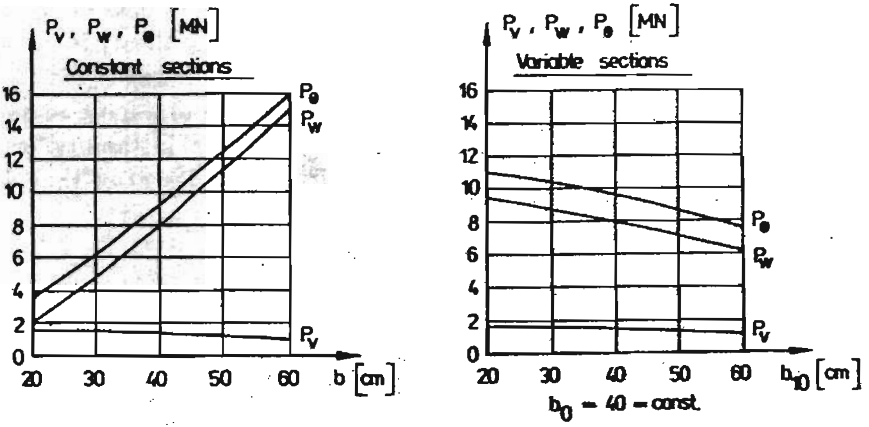


Fig. 5. Behaviour of P_v, P_w and P_Θ for the column considered

variable flange-widths was investigated by Szymczak [32] with the aid of optimization analysis. Szymczak considered also [33] optimal design of those columns. Hereby, the torsional and flexural buckling questions were of interest. To clarify the paradox, a special procedure was applied by Dubas [16]. The model of differently tapered space truss was used – of square mid-span cross-projection, in particular. Applying the method of Engesser-Vianello, the paradoxical formation of P_w was confirmed, formally, but the explanation of its substance was not sufficient.

Later, simultaneous action of torsion and compression of thin-walled I-members with variable cross-sections was considered as a problem of optimization; a paradoxical increase of the angle of twist, together with the increase of the bar volume, was found.

The finite element analysis of lateral instability of thin-walled bars with variable sections has been shown by Wezeker [37] but the paradox has not been established. Finally, the unconventional behaviour of those bars, within the neighbouring problem of free vibrations, was again reported by Wezeker [38,39].

6. Final comments

The recorded facts allow to state that, in the framework of the present theory of thin-walled bars, the formal existence of the reported paradox is evident. However, its essential explanation is, until now, insufficient. The paradox concerns the most simple case of uncoupled torsional buckling but it is possible that similar problems can arise with coupled flexural-torsional stability. The generated "discrepancy" can be the basis for the opinion that the theory itself is imperfect and needs verification. Some recent critics of this theory (cf [21,23,24,25]), based on different data, suggest that a thorough reexamination of the governing theory is extremely desirable. Nevertheless, an experimental verification of the paradox seems to be necessary. In the event of its confirmation or the emergence of a more reasonable theory, efforts should be made to modify the flexural–torsional instability formulas currently observed [2], with an evident effect on practical design of thin–walled structures.

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Paradoks wyboczenia skrętnego

Streszczenie

W ramach szczegółowego przeglądu przedstawiono tło i istotę zaskakującego kształtowania się skrętnych sił krytycznych dla słupów cienkościennych o zmiennych, bisymetrycznych przekrojach dwuteowych. Dokonano stosownego odniesienia do wyboczenia giętnego i do pierwotnie badanego, pokrewnego problemu skrętnych drgań swobodnych. Wyrażono opinię o możliwości istnienia podobnych zjawisk w sprzężonych, giętno-skrętnych zagadnieniach własnych.

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