MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
4, 30, 1992

ON FREE VIBRATION DAMPING ANALYSIS OF LAYERED BEAMS AND BANDS COMPOSED OF ISOTROPIC VISCOELASTIC LAYERS

STANISŁAW KARCZMARZYK

ZBIGNIEW OSIŃSKI

Warsaw University of Technology

This paper deals with some problems appearing in sinusoidal free vibration damping analysis of linear, viscoelastic, both homogeneous and layered, beams and bands. A homogenity of description of displacements of both discrete and continuous systems is shown. A manner of viscoelastic properties of materials description in formulation of any boundary value problem of sinusoidal vibration is discussed. A procedure of accurate calculations of eigenfrequency, for a given form of characteristic equation, of the structures composed of viscoelastic materials is proposed. Some rules of transformation of any boundary problem formulation for band/beam consisting of isotropic layers into formulation of sucha problem for beam/band is revealed. A difference in vibration damping analysis of layered beam and band, respectivelly is proved and discussed.

1. Introduction

In the paper of Pearce and Baumgarten (cf [11]) one can find a conclusion that damping of vibrations of two–layers beam consisting of elastic strip and viscoelastic coating will be the same as damping of vibration of two–layers plate composed of the same materials when both the coating to strip thickness ratio and eigenfrequencies of the structures are identical. It means that we can analyse the vibration damping of the plate by means of formulas for the beam. The foregoing conclusion has been proved experimentally for the layered structures composed of stiff (metal) elastic strip and supple viscoelastic coating.

In this paper we prove that the rule given by Pearce and Baumgarten will be deficient when a layered structure is composed of stiffness–comparable materials. We show that formulations of the damping problems for beam and band are different and how the vibration damping depends on the difference. Besides we discuss some important problems as for instance the procedure of exact calculating of eigenfrequency of viscoelastic structure. In order to achieve the clarity of the considerations we introduce a reader, in sections 2 and 3, into problems of vibration damping of discrete and continuous systems.

2. Homogeneity of description of displacements for discrete and continuous viscoelastic systems

Some considerations dealing with a linear, sinusoidal vibration damping of discrete systems can be useful in formulation of vibration problems of continuous, viscoelastic structures such as layered beams and bands. Upon basis of a closed form solution for the simplest, vibrating freely, discrete system, consisting of a rigid body of mass m, a spring – characterized by the stiffness coefficient k and a dashpot characterized by the coefficient c, both a way of including of viscoelastic properties in eigenvalue problem formulation and a way of calculating of the logarithmic decrement of a continuous structure is discussed.

The equation of motion of the discrete system vibrating freely is as follows

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \qquad c > 0 \tag{2.1}$$

where x denotes a displacement as well as a space variable while t denotes time. The closed form solution to the equation, assuming that k/m is greater than $(c/2m)^2$, is as follows

$$x(t) = A\cos(\lambda t + \phi)e^{-ht} \equiv A\cos\phi[\cos(\lambda t) + C\sin(\lambda t)]e^{-ht} \equiv AT(t) \tag{2.2}$$

where

$$h = \frac{c}{2m} \qquad \lambda^2 = \omega_0^2 - h^2$$
$$\omega_0^2 = \frac{k}{m} \qquad C = -\tan\phi \tag{2.3}$$

while A, ϕ are constant values dependent on initial conditions [10]. Let us notice that ω_0 is the eigenfrequency of the discrete system for $c = h = 0$ however λ is the eigenfrequency of the system for $h \not\equiv 0$. In the latter case one can obtain, starting with the definition of periodic logarithmic decrement, the following expression for the parameter

$$\delta_T \stackrel{\text{df}}{=} \ln\frac{x(t)}{x(t+T)} = \ln\frac{T(t)}{T(t+T)} = 2\pi\frac{h}{\lambda} \tag{2.4}$$

where T denotes the period of the oscillations. It can be noticed that the product of the following complex quantities

$$\omega_1 = \lambda + \mathrm{i}h \qquad \omega_2 = \lambda - \mathrm{i}h \qquad \mathrm{i}^2 = -1 \tag{2.5}$$

is equivalent to ω_0^2 i.e., $\omega_1\omega_2 = \omega_0^2$. By using the left–hand side equation in the set (2.5) and assuming that $C = \mathrm{i}$ where $i^2 = -1$ one can transform right–hand side of Eq (2.2) to the form

$$\bar{x}(t) = A\bar{T}(t) \equiv A\cos\phi\exp(\mathrm{i}\omega_1 t) = A^*\exp(\mathrm{i}\omega t) \tag{2.6}$$

where A^* is any constant value. We note that the real displacement of the mass m in direction x is defined by the following equation [1]

$$x(t) = \mathrm{Re}(\bar{x}(t)) \tag{2.7}$$

however the formula (2.4) can be written in the form

$$\delta_T = \frac{2\pi h}{\lambda} \equiv 2\pi\frac{\mathrm{Re}(\omega)}{\mathrm{Im}(\omega)} \tag{2.8}$$

where $\omega \equiv \omega_1$ while $\mathrm{Re}(\omega)$, $\mathrm{Im}(\omega)$ are real and imaginary parts of the complex eigenfrequency ω, respectively.

The formulas (2.6), (2.8) have been obtained within the theory of discrete system (by direct using of the classical solution for the simple model) however they are widely used in papers on damping of vibration of continuous systems. Let the function $\bar{x}(t)$ be of the form

$$\bar{x}(t) = X(\boldsymbol{x})\bar{T}(t) = X(\boldsymbol{x})\exp(\mathrm{i}\omega t) \tag{2.9}$$

where $X(\boldsymbol{x})$ denotes a continuous and genarally complex field of the space variables $\boldsymbol{x} \equiv [x_1, x_2, x_3]$. In the papers on vibration analysis of continuous, viscoelastic systems the expressions of the form (2.6), (2.9) are called kinematical assumptions and the parameter $\omega \equiv \omega_1$ is called complex eigenfrequency (cf [6]). The factor $X(\boldsymbol{x})$ has to be evaluated by taking into account equations of motion and boundary conditions. In order to calculate the logarithmic decrement the formula (2.8) is used.

Due to the formal homogenity of description of displacements of both discrete and continuos sinusoidally vibrating systems one can apply some conclusions relating to the simple discrete model in the vibration damping analysis of continuous, viscoelastic, structures.

3. A way of including the viscoelastic properties of material into formulation of a boundary value problem of sinusoidal vibration

Let us discuss a problem of including the viscoelastic properties into formulation of the three-dimensional eigenvaluevalue problem of a continuous structure. By using displacement functions in the form (2.6), (2.9) one obtains infinitesimal strains, in the three-dimensional case, in the form

$$\bar{\varepsilon}_{kl} = (\mathcal{E}(\mathbf{x},\omega))_{kl}\exp(\mathrm{i}\omega t) \tag{3.1}$$

After substituting Eq (3.1) into constitutive equations of linearly viscoelastic material [13]

$$\boldsymbol{\sigma} \equiv \mathsf{C}(t=0)\boldsymbol{\varepsilon}(t) - \int\limits_0^t \frac{\partial \mathsf{C}(t-\tau)}{\partial \tau}\boldsymbol{\varepsilon}(\tau)d\tau \tag{3.2}$$

one obtains the following expressions for stresses

$$\bar{\sigma}_{kl} = (\Sigma(\mathbf{x},\omega))_{kl}\exp(\mathrm{i}\omega t) \equiv \boldsymbol{\sigma} = \mathsf{C}^*(\omega)\boldsymbol{\varepsilon}\exp(\mathrm{i}\omega t) \tag{3.3}$$

where $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$ denote the complex stress and the strain vector respectively, $\mathsf{C}(t)$ is the stiffness matrix of viscoelastic material however elements of the matrix $\mathsf{C}^*(\omega)$ are in fact the Fourier transforms of corresponding elements of the matrix $\mathsf{C}(t)$.

The tensors $(\mathcal{E}(\mathbf{x},\omega))_{kl}$, $(\Sigma(\mathbf{x},\omega))_{kl}$, for continuous system, are generally dependent on the space variables and the frequency ω since material parameters of a viscoelastic material are dependent on a frequency [8]. The tensors are complex thus to obtain real stresses and strains one need to use the equations

$$\sigma_{kl} = \mathrm{Re}(\bar{\sigma}_{kl}) \qquad \varepsilon_{kl} = \mathrm{Re}(\bar{\varepsilon}_{kl}) \tag{3.4}$$

Within the complex stiffness matrix $\mathsf{C}^*(\omega)$ of the isotropic viscoelastic material we have two independent complex material parameters. One of them is the Kirchhoff modulus which can be written in the form

$$\mu^*(\omega) = \mu_1(\omega) + \mathrm{i}\mu_2(\omega) \equiv \mu_1(\omega)[1+\eta_m(\omega)] \tag{3.5}$$

The second independent parameter can be either the Poisson ratio ν^* or the Young modulus E^*

$$\nu^*(\omega) = \nu_1(\omega) + \mathrm{i}\nu_2(\omega) \equiv \nu_1(\omega)[1+\eta_n(\omega)] \tag{3.6}$$

$$E^*(\omega) = E_1(\omega) + \mathrm{i}E_2(\omega) \equiv E_1(\omega)[1+\eta_E(\omega)] \tag{3.7}$$

The functions $\eta_m(\omega)$, $\eta_n(\omega)$, $\eta_E(\omega)$ in Eqs (3.5), (3.6) and (3.7), are called material loss factors. We notice that the expressions (3.1)÷(3.7) will also be valid when behaviour of the viscoelastic material is described by the structural (springs–dashpots) models [1] or a continuous system is excited by an external force to vibrate sinusoidally.

The function $\mu^*(\omega)$ as well as the remaining complex characteristics of a viscoelastic material can be obtained experimentally in domain of frequency ω (cf [2]). The experimental results for each complex material parameter one obtains in two charts as shown in Figure 1. In order to calculate any damping parameter for the continuous structure composed of viscoelastic materials, one needs to read from the experimental charts several material parameters which will afterwards be the input data necessary for making calculations of the damping parameter of the structure. In the case of isotropic material one should read either the values of μ_1, μ_2, ν_1, ν_2 or the values of μ_1, μ_2, E_1, E_2. Since the material characteristics can be defined in several ways we sometimes need to apply the quantities of μ_1, μ_2, K_1, K_2, where K_1 and K_2 are the real and imaginary part of so–called Helmholtz modulus.

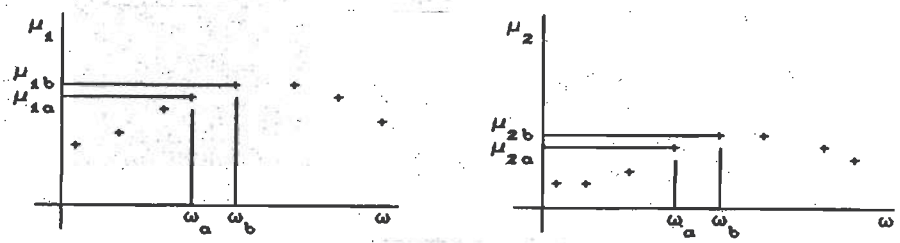

Fig. 1. Typical charts for frequency dependent Kirchhoff modulus

We notice that for the accurate reading of the complex values of any material parameter from the experimental charts, one should know the exact value of vibration frequency. However in the case of free vibration of any system the condition is not fulfilled – an eigenfrequency is unknown. There are two ways to overcome the difficulty i.e., (a) an exact approach and (b) a trial–and–error method. In both cases however the following procedure is proposed

1. by applying either the exact approach or the trial–and–error method the exact value of eigenfrequency ω_E (corresponding with ω_0 in expressions (2.3)) of undamped (elastic) structure is evaluated,

2. for the eigenfrequency ω_E the complex material parameters (for instance $\mu_1 = \mu_1(\omega_E)$, $\mu_2 = \mu_2(\omega_E)$) are found from the experimental charts – such as those given in Fig.1,

3. the complex material parameters are used to calculate the eigenfrequency ω_V (corresponding with λ in expressions (2.2)) and the logarithmic decrement of the continuous system.

The procedure given here is exact enough. Upon basis of Eqs (2.3), (2.4) one can obtain the following relationship

$$\omega_E^2 = \omega_V^2\Big(1 + \frac{\delta_T^2}{4\pi^2}\Big) \tag{3.8}$$

Let us notice that for $\delta_T < 1.0$ we will obtain $\omega_E \cong \omega_V$. Thus we conclude that for exact calculating of the logarithmic decrement one should at first calculate as exactly as possible an eigenfrequency of the continuous, purely elastic system. To prove the importance of the conclusion and the procedure proposed here we present in Table 1 the loss factor η_E versus the frequency for special viscoelastic material LD4 [7,8]. It can be seen that the Young modulus of the material strongly depends on frequency. In such a case it is very important to evaluate exactly the eigenfrequency ω_E of the continuous system.

Table 1. Dependence on frequency of the loss factor $\eta_E = E_2/E_1$ for viscoelastic material LD4 [7,8] where $E^* = E_1 + \mathrm{i}E_2$ denotes complex Young's modulus

ω_E [rad/s]	125.6	251.2	376.8	502.4	628.0	753.6
$\eta_E = E_2/E_1$	0.3243	0.5267	0.6995	0.8550	0.9900	1.360
E_1 [MPa]	1556.0	1723.7	1830.0	1909.0	1973.3	2027.1

Let us explain finally the exact approach and the trial-and-error method of evaluating of the eigenfrequency ω_E. In the first case we have to approximate the material characteristics (like for instance the experimental charts given in Figure 1) by analytic functions in domain of frequency. Thus for any isotropic and viscoelastic material we have four functions – for instance $\mu_1(\omega)$, $\mu_2(\omega)$, $\nu_1(\omega)$, $\nu_2(\omega)$. Since generally both material parameters occur in characteristic equation of eigenvalue problem then for the exact evaluating of the eigenfrequency of purely elastic system one need to solve the following set of nonlinear algebraic equations

$$\det\mathbf{A} = 0 \qquad \mu_1 = f_1(\omega) \qquad \nu_1 = f_2(\omega) \tag{3.9}$$

where $\mathbf{A}$ is a matrix of the boundary eigenvalue problem of the continuous, purely elastic system, however the functions f_1, f_2 are known – as the approximation of the experimental material characteristics. In the case when behaviour of the viscoelastic material is described by the structural (springs–dashpots) Voight model the functions f_1, f_2 are independent of frequency. Elements of the matrix $\mathbf{A}$ depend on both the eigenfrequency ω_E and the parameters μ_1, ν_1 i.e., $A_{kl} = A_{kl}(\omega_E, \mu_1, \nu_1)$. We stress that the exact approach in the calculating of the eigenfrequency of purely elastic, continuous system (discussed here) is entirely and directly connected with the problem of including the viscoelastic properties of materials i.e., dependence their properties on frequency. We do not refer in the considerations to the quality of formulation of the eigenvalue problem itself. Thus we have not considered either form or manner of the numerical solving of the equation $\det\mathbf{A} = 0$.

The trial-and-error method can be easier in some cases. At first we have to take any value of the eigenfrequency ω_a in order to find for instance μ_{1a} from the left-hand side chart in Fig.1. After substituting the value μ_{1a} into the characteristic equation of eigenvalue problem considered we calculate (assuming $\mu_{2a} = 0$) $\omega_{cal} = \omega_b$ where ω_{cal} denotes the calculated eigenfrequency. Since ω_a is not equal to ω_b we have to repeat the procedure which will be interrupted when $\omega_a \cong \omega_b$.

4. A difference between formulations of eigenvalue problem for beam and band

We notice that we have the plane stress within a beam and the plane strain within a band. Because of this eigenfrequencies of any beam of rectangular cross-section are lower than the eigenfrequencies of the band having the same material and geometrical parameters. It can easily be observed by comparing the equations of motion and their solutions obtained, for instance for homogeneous beam and band, within so-called classical theory which is based on Kirchhoff's assumption of flat cross-sections. Thus for the rectangular cross-sections of the structures considered the equations of motion are as follows

– for beam

$$\frac{\partial^4 u_z}{\partial x^4} + \Big(\frac{2}{h}\Big)^2\Big(\frac{3\rho}{E^*}\Big)\frac{\partial u_z^2}{\partial t^2} = 0 \tag{4.1}$$

– for band

$$\frac{\partial^4 u_z}{\partial x^4} + [1-(\nu^*)^2]\Big(\frac{2}{h}\Big)^2\Big(\frac{3\rho}{E^*}\Big)\frac{\partial u_z^2}{\partial t^2} = 0 \tag{4.2}$$

where u_z, h, ρ, E^*, ν^* denote the deflection, the total thickness, the mass density, the complex Young modulus of the beam/band and the complex Poisson ratio, respectively. We notice that Eq (4.2) can be obtained directly from the equation of motion of the classical plate theory. Eigenfrequencies of the beam and the band in the case of simple supports of the structures can be calculated according to the formulas

– for beam

$$\omega_m^2 = \alpha_m^4\Big(\frac{h}{2}\Big)^2\Big[\frac{E^*}{3\rho} \equiv \frac{2\mu^*}{\rho}(1+\nu^*)\Big] \tag{4.3}$$

– for band

$$\omega_m^2 = \alpha_m^4\Big(\frac{h}{2}\Big)^2\Big[\frac{E^*}{3\rho[1-(\nu^*)^2]} \equiv \frac{2\mu^*}{3\rho(1-\nu^*)}\Big] \tag{4.4}$$

where μ denotes the Kirchhoff modulus, $\alpha_m = m\pi/L$, $m = 1, 2, 3, ...$ and the L is the length of the beam/band. It can be seen by comparing the Eqs (4.3), (4.4) that, for given values of the α_m, h, ρ, E^* and $\nu^* \neq 0$, eigenfrequencies calculated according to formula (4.4) are higher than eigenfrequencies calculated according to formula (4.3).

We notice that replacing the Poisson ratio ν^* in Eqs (4.1), (4.3) by the following quotient $\nu^*/(1 - \nu^*)$ we will derive Eqs (4.2) and (4.4), respectively. On the other hand by replacing the Poisson ratio in Eqs (4.2), (4.4) by the quotient $\nu^*/(1 + \nu^*)$ one will obtain Eqs (4.1) and (4.3), respectively. Let us write the rules as follows

$$\text{plane stress} \quad \nu^* \to \frac{\nu^*}{1 - \nu^*} \quad \text{plane strain} \tag{4.5}$$

$$\text{plane strain} \quad \nu^* \to \frac{\nu^*}{1 + \nu^*} \quad \text{plane stress} \tag{4.6}$$

It can be veryfied that the rules of transformation of the plane stress/strain eigenvalue problem into the plane strain/stress eigenvalue problem are valid for any problem formulated in the plane strain/stress for any system composed of viscoelastic, isotropic continua.

5. Differences in vibration damping analyses of beams and bands

The differences appearing in vibration damping analysis of beams and bands result from different equations of motion in both cases. The problem has been discussed in the previous section of the present paper. In this section we discuss some additional aspects relating to homogeneous and layered beams and bands.

In order to calculate the logarithmic decrement δ_T for a homogeneous and viscoelastic beam or band one should apply the formulas (4.3) or (4.4) and (2.8). At first we have to calculate (for instance according to the trial–and–error method) exact values of eigenfrequencies of the structures ignoring damping. As it has been stated in section 3 the eigenfrequencies are necessary for accurate reading from the experimental charts the complex material parameters appearing in Eqs (4.3), (4.4). By means of the equations (i.e. the transformation rules (4.5) and (4.6)) we discuss here differences appearing in vibration damping analysis of homogeneous beams and bands. In this case we can meet two typical situations described below.

1. Let us assume that both a homogeneous beam and a band are made of the same material and their geometrical parameters (excluding their widths) are the same. The eigenfrequencies of the structures and values of the complex material parameters, found from the experimental charts such as those shown

in Figure 1, will be different. Consequently we obtain different values of the logarithmic decrement in both cases.

2. Let both values of the eigenfrequencies and material of the beam and the band be the same. We notice that an eigenfrequency of the beam depends on the Young modulus E^* however an eigenfrequency of the band depends on both the Young modulus and the Poisson ratio ν^*. When both the parameters depend on frequency (i.e., $E^* = E^*(\omega)$ and $\nu^* = \nu^*(\omega)$) we will obtain different values of the logarithmic decrement for the structures. However if the Poisson ratio is not dependent on frequency (i.e., $\nu^* \neq \nu^*(\omega)$) the decrement will be identical in both cases.

The second case of the situation (2) occurs during vibration of classical sandwich beam/band. Such a structure consists of thin facings and a thick and light core. Moreover the facings are of higher stretching stiffness than the core. The eigenfrequencies of the structure, when it is simply supported, can be calculated by using the following equation [14]

$$\omega_m^2 = \alpha_m^2 \frac{K_S}{M} \frac{1 + \frac{K_Z}{K_M} + \alpha_m^2 \frac{K_Z}{K_S}}{1 + \frac{K_S}{\alpha_m^2 K_M}} \tag{5.1}$$

where

$$K_S = \mu_2^* \frac{(a_1 + a_3)^2}{h_2} \qquad K_Z = \frac{E_1^* h_1^3 + E_3^* h_3^3}{12} \tag{5.2}$$

$$K_M = E_1^* h_1 a_1^3 + E_3^* h_3 a_3^3 \qquad M = \sum_j \rho_j h_j \qquad j = 1,2,3 \tag{5.3}$$

$$a_1 = \frac{h_1 + h_2}{2} \qquad a_3 = \frac{h_3 + h_2}{2} \tag{5.4}$$

while $\alpha_m = m\pi/L$ for $m = 1, 2, 3, ..., L$ is the length of the beam/band, h_j is the thickness of the jth layer. a_1, a_3 stand for the distances between the centre of gravity of the core and analogous points of the faces. The subscript $j = 2$ in Eqs (5.1)÷(5.3) refers to parameters of the core. We note that under two assumptions, first $h_1 = h_3$ and second $E_1^* = E_3^*$ one can derive Eq (5.1) directly from the equation of motion given by Mead and Markus [9].

Formula (5.1) predicts that the eigenfrequency of the structure considered depends on the Kirchhoff modulus of the core and the Young moduli of the facings. Since the transformation rules (4.5) and (4.6) do not refer to the Kirchhoff modulus then one can conclude: when the Young moduli of the facings do not depend significantly on frequency and the values of all parameters, excluding α_m, are the same for both the sandwich beam and the band then, for equal eigenfrequencies of the structures, the logarithmic decrement values for the beam and band will be the same.

Basing upon the theory given by Raville et all. in paper [12] one may assume that the conclusion also refers to the three–layers beam and band when adjoining layers of the structures are of stiffness–comparable materials and the cores are much more thick than the facings. In such a case eigenfrequencies of the beam/band, which is clamped at the ends, can be calculated from the following equation [12]

$$\Omega_m + \sum_m = \frac{1}{\alpha_m^4 \frac{E^*}{(1-(\nu^*)^2)\rho}\Big(I_F + \frac{I_T}{1+m^2 S}\Big) - \omega_m^2} = 0 \tag{5.5}$$

where

$$\begin{aligned} \Omega_m &= 0 \qquad && \text{for} \quad m = 1,3,5,... \\ \Omega_m &= -\frac{1}{2\omega_m^2} \qquad && \text{for} \quad m = 2,4,... \end{aligned} \tag{5.6}$$

$$I_F = I_F(h_1, h_3) \qquad I_T = I_T(h_1, h_2, h_3) \tag{5.7}$$

$$S = S(h_1, h_2, h_3, L, E^*, \nu^*, \mu^*) \tag{5.8}$$

while $\alpha_m = m\pi/L$ for $m = 1,2,3,...$, E^*, ν^* are the material parameters of the facings, μ^* is the Kirchhoff modulus of the core, h_2 is the thickness of the middle layer however h_1, h_3 are thicknesses of the facings. The symbol ρ denotes here the mass density of the composite beam/band per unit length and width.

On the basis of expressions (4.5), (4.6) and (5.5)÷(5.8) we conclude: when dependence on frequency of the Young moduli of the facings is negligible and values of all parameters, except of α_m, are the same both for the clamped three–layer beam and band then the logarithmic decrement values for the layered structures will be equal.

Both formula (5.1) and (5.5) have been derived by applying several simplifications refering to both geometry and materials of the layered structures. Because of this it is difficult to notice a difference in vibration damping analysis of the beams and bands. However when the foregoing simplifications can not be applied the difference can easily be proved. For this purpose we apply the characteristic equation of eigenvalue problem of isotropic simply supported plate derived within the linear theory of elasticity by Levinson [6]. The equation is as follows

$$4\beta_1\beta_2(M\pi)^2 \tanh\frac{\beta_1 h}{2} - P^4 \tanh\frac{\beta_2 h}{2} = 0 \tag{5.9}$$

where

$$\beta_1^2 = (M\pi)^2 - \frac{\rho\omega^2}{\mu^*} \qquad \beta_2^2 = (M\pi)^2 - \frac{\rho\omega^2}{\lambda^* + 2\mu^*} \tag{5.10}$$

$$M^2 = \Big(\frac{m}{a}\Big)^2 + \Big(\frac{n}{b}\Big)^2 \qquad P^2 = (M\pi)^2 + \beta_1^2 \tag{5.11}$$

while μ^*, λ^* are the complex Lame constants for viscoelastic material, ρ is the density, a, b, h denote the length, the width and the thickness of the plate, respectively however m, n are integer numbers denoting the modes of vibration.

We notice that after substitution for $M^2 = (m/L)^2$ into Eq (5.9), where L denotes the distance between supports, we will obtain the characteristic equation for isotropic and simply supported band. As stated in section 4 the equations of motion of plane strain can be transformed, according to the rule defined by expression (4.6), into the equations of motion of the plane stress. Because of this the characteristic equation for plane stress problem is not the same as in the case of the plane strain. It is because of the dependence of Lame constant λ^* on μ^* and ν^* (i.e., $\lambda^* = \mu^* 2\nu^*/(1-2\nu^*)$ whereas the ν^* has to be transformed according to Eq (4.6). We can conclude finally that formulation, within the linear theory of elasticity, of the eigenvalue problem for homogeneous, isotropic beam is different than formulation of the problem for such a band. Upon basis of the correspondence principle the rule can be refered to viscoelastic beam and band.

Following the procedure given in [3] one can extend Levinson's approach to formulate the eigenvalue problem for layered band without introducing any simplifications. The formulation for layered beam, derived by taking into account Eq (4.6), differs from that for band however in both cases the eigenfrequency is formally dependent on both the Kirchhoff modulus and the Poisson ratio of each layer. Thus for any layered structure we have

$$\omega = \omega(\mu_1^*, \nu_1^*, \mu_2^*, \nu_2^*, \mu_3^*, \nu_3^*, \mathbf{g}) \tag{5.12}$$

where subscripts 1,2,3 refer to layer 1,2,3, respectively and $\mathbf{g}$ denotes vector of geometrical parameters. Unlike in Eqs (5.1) and (5.5) we have in Eq (5.12) the Poisson ratios of both purely elastic and viscoelastic layers. Since the parameters ν_j^* have to be transformed according to expressions (4.5), (4.6) we obtain different values of the logarithmic decrement for the layered band and beam, respectively. The latter conclusion results directly from Eqs (2.8), (4.5), (4.6) and (5.12). Thus the difference in vibration damping analysis of homogeneous and layered beams and bands has been proved. On the other hand it is impossible, in this case, to derive an explicit relationship of the form (5.12). Therefore one can not estimate an influence of a particular material parameter on a value of the logarithmic decrement of the structure without making calculations.

6. Numerical calculations

In Table 2 of this paper we present several values of the logarithmic decrement for rectangular simply supported three-layer bands and percentage differences between vibration dampings of the bands and the three-layer-beams – corresponding both geometrically and physically to the bands. The differences of intensity of vibration damping are defined by means of the logarithmic decrement i.e.,

$$\varepsilon_1 = \frac{|(\delta_T)_{band} - (\delta_T)_{beam}|}{(\delta_T)_{beam}} 100 \tag{6.1}$$

The quantities $(\delta_T)^{(1)}_{band}$ have been calculated for $(\eta_E)_2 = 0.1$ and $(\eta_n)_2 = 0$ while the values of $(\delta_T)^{(2)}_{band}$ are obtained for $(\eta_E)_2 = 0.1$ and $(\eta_n)_2 = 333$, where subscript 2 denotes the middle layer. In order to investigate an influence of material characteristic $\nu^*(\omega)$ of the middle layer on the logarithmic decrement values for the layered beam the following factor has been calculated

$$\varepsilon_2 = \frac{|(\delta_T)^{(1)}_{beam} - (\delta_T)^{(2)}_{beam}|}{(\delta_T)^{(2)}_{beam}} 100 \tag{6.2}$$

where the values of $(\delta_T)^{(1)}_{beam}$ have been calculated for $(\eta_E)_2 = 0.1$ and $(\eta_n)_2 = 0$ while the values of $(\delta_T)^{(2)}_{beam}$ are obtained for $(\eta_E)_2 = 0.1$ and $(\eta_n)_2 = 333$, where subscript 2 denotes the middle layer.

Table 2. The logarithmic decrement values for 1–st mode of vibration of three–layer bands and the percentage differences defined by expressions (6.1), (6.2). Parameters of the middle layer (core): $h_2 = 120$ [mm], $(E_1)_2 = 1.6 \cdot 10^{10}$ [Pa], $(\nu_1)_2 = 0.3$, $\rho_2 = 1750$ [kg/m^3]. Parameters of the outer layers (facings): $h_1 = h_3 = 2.5$ [mm], $(E_1)_1 = (E_1)_3 = 2.07 \cdot 10^{11}$ [Pa], $\rho_1 = \rho_3 = 7860$ [kg/m^3]. L is the length of the beam/band, $\eta_n \equiv (\eta_n)_2$.

L [mm]	1000	1500	2000	2500
$(\delta_T)^{(1)}_{band}$	0.1336	0.1260	0.1231	0.1217
$(\delta_T)^{(2)}_{band}$	0.1869	0.1914	0.1931	0.1939
$\varepsilon_1^{(1)}$ for $\eta_n = 0$	1.93	1.99	2.11	2.30
$\varepsilon_1^{(2)}$ for $\eta_n = 0.33$	63.72	65.48	66.15	66.48
ε_2	14.36	6.65	3.87	2.56

Parameters of the structures considered here are written above in Table 2. We note that the middle layer (core) of the structures is stiffness–comparable with the outer layers (facings). It has been veryfied that in such a case the theories [9,11,14] predict inaccurate values of both the eigenfrequencies and the logarithmic decrement (cf [3]). Therefore the results given here have been calculated according to the method presented elsewhere [3,4,5].

It can be observed in Table 2 that for $(\eta_n)_2 = 0$ the damping of vibration of both the beams and the bands is almost the same – $\varepsilon_1^{(1)} \cong 2\%$. The result coincides very well with the theortical predictions. We notice that the Young modulus E^*

depnds on the Poisson ratio ν^* and the Kirchhoff modulus μ^* according to the formula $E^* = 2\mu^*(1+\nu^*)$. Assuming $(\eta_n)_2 = 0$ one obtains for the middle (viscoelastic) layer $\nu^* \equiv \nu_1$ (i.e. a real number) and $\eta_E \equiv \eta_m$. It means that one includes only the damping resulting from the shearing deformations as in the (5.1) and (5.5). This is the reason of almost equal intensities of vibration damping of the bands and the beams. However for $(\eta_n)_2 = 0.333$ the damping of vibration of the bands is much more intensive – $\varepsilon_1^{(2)} \cong 66\%$. It seems to be unquestionable that vibration damping of layered bands depends significantly on both characteristics of viscoelastic (isotropic) strips i.e., $\nu^*(\omega)$ and $\mu^*(\omega)$. However in the case of layered beams one needs only to take into account the function $\mu^*(\omega)$ to evaluate material damping of sinusoidal vibrations. The logarithmic decrement of the beams depends weakly on the material loss factor η_n. We note finally that the differences of intensity of vibration damping ($\varepsilon_1^{(1)}$ and $\varepsilon_1^{(2)}$) depend slightly on the length of the structures considered. The same conclusion refers to the factor ε_2 provided that the quotient of L by h (length over total thickness) is greater than 10. For thickset layered beam $(L/h = 8)$ ε_2 is equal to 14.36% thus in the case the influence of $\nu^*(\omega)$ on vibration damping seems to be more significant. However in order to establish the latter conclusion precisely one should compare the results given here with theresults obtained by using other (for instance an experimental) method.

7. Final remarks and conclusions

On the ground of both theoretical considerations and numerical results given in the paper we conclude: (1) in order to calculate exactly the logarithmic decrement of any viscoelastic structure one has first to calculate exactly the eigenfrequency of corresponding purely elastic structure, (2) formulations of the eigenvalue problems, as well as another elastic and viscoelastic problems, for both layered and homogeneous bands/beams composed of isotropic layers can be transformed into corresponding problems for beams/bands according to formulas (4.5) and (4.6), (3) the logarihmic decrement for layered band composed of isotropic strips depends significantly on complete material characteristics of viscoelastic strips however the logarithmic decrement for layered beams is weekly dependent on imaginary part of complex Poisson's ratio of viscoelastic layers.

References

1. BLAND D.R., 1960, *The theory of linear viscoelasticity*, Pergamon Press, Oxford, London, Paris, New York
2. FERRY J.D., 1961, *Viscoelastic properties of polymers*, John Willey and Sons Inc., New York, London
3. KARCZMARZYK S., 1989, *An analysis of vibration damping by internal friction in sandwich beams*, Ph.D., Warsaw University of Technology, Warsaw
4. KARCZMARZYK S., OSIŃSKI Z., 1990, *A new approach to flexural free vibration damping analysis of simply supported sandwich beams*, Proceedings of 28–th Polish Solid Mech.Conf., 112-115
5. KARCZMARZYK S., 1991, *A new approach to flexural free vibration damping analysis of a two–layer viscoelastic plate*, Machine Dynamics Problems, 6, 5-17
6. LEVINSON M., 1985, *Free vibrations of a simply supported rectangular plate: an exact elasticity solution*, Journal of Sound and Vibr., 98, 289-297
7. LUNDEN R., 1977, *Ship vibration attenuation using unconventional methods – damping devices, resilient mountings and active control*, Chalmers Univ. of Techn., S. F51, Goeteborg
8. MATYJASZEK M., 1985, *Optymalne kształtowanie prętowych układów drgających z materiałów lepkosprężystych przy wymuszeniach harmonicznych*, Praca dokt. Pol.Krakowska, Kraków
9. MEAD D.J., MARKUS S., 1969, *The forced vibration of a three–layer damped sandwich beam with arbitrary boundary conditions*, Journal of Sound and Vibr., 10, 163-175
10. OSIŃSKI Z., 1978, *Teoria drgań*, PWN, Warszawa
11. PEARCE B.K., BAUMGARTEN J.R., 1971, *The damping effects of viscoelastic materials, part 2 – transverse vibrations of plates with viscoelastic coatings*, Journal of Eng. for Ind., May, 651-655
12. RAVILLE M.E., UENG E-S, LEI M-M, 1961, *Natural frequencies of vibration of fixed-fixed sandwich beams*, Journal of Appl.Mech., September
13. SKRZYPEK J., 1986, *Plastyczność i pełzanie*, PWN Warszawa
14. ZIEMIAŃSKI L., 1984, *Analiza widma częstości własnych belek warstwowych*, Materiały VII Symp.Techn.Wibr.i Wibroakustyki, AGH Kraków

Analiza drgań swobodnych belek i pasm warstwowych utworzonych z izotropowych warstw lepkosprężystych

Streszczenie

Praca dotyczy pewnych problemów pojawiających się w analizie tłumienia swobodnych i liniowych drgań sinusoidalnych belek i pasm lepkosprężystych – zarówno jednorodnych jak i warstwowych. Przedstawiono jednorodność opisu przemieszczeń w układach dyskretnych i ciągłych. Przedyskutowano sposób uwzględnienia lepkosprężystych właściwości materiałów w sformułowaniu dowolnego problemu brzegowego drgań sinusoidalnych. Zaproponowano procedurę dokładnych obliczń częstości drgań własnych, dla danej postaci równania charakterystycznego, struktur złożonych z materiałów lepkosprężystych. Wyprowadzono zasady transformacji sformułowania dowolnego problemu brzegowego dla pasma/belki składającego(j) się z warstw izotropowych do sformułowania takiego problemu dla belki/pasma. Przedyskutowano i udowodniono różnice w analizie tłumienia drgań belki i pasma.

Manuscript received February 3, 1991; accepted for print May 11, 1992