MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
4, 30, 1992

BOUNDARY ELEMENT METHOD FOR AXISYMMETRIC THERMOELASTIC PROBLEMS

JACEK BŁOCKI

Institute of Nuclear Physics, Cracow

Using the notation of two-point tensors a boundary integral equation for steady-state problems of thermoelasticity is derived. We point out that only the fundamental solution U_{Ki} and the Galerkin tensor $G_K{}^i$ have to be known in order to formulate the problem in an arbitrary curvilinear system of coordinates. After rewriting the integral equation for axisymmetric problems we solve this equation by means of the Boundary Element Method. Parabolic elements are utilized where nonsingular and singular integrals are computed numerically. The stress state in the body region and on its boundary are calculated with the help of suitable formulae which have been derived. Some numerical calculations in comparison with the results obtained analytically are given.

1. Introduction

In recent years, the Boundary Element Method (BEM) appears to be an economical and accurate numerical technique for solving various engineering problems (Banerjee, Butterfield 1981; Brebbia, Telles, Wrobel 1984; Beskos 1987). In this method, the solution to some governing equations, usually partial differential equations, is recast into the solution to integral equations. Such equations apply to the boundary of the domain and incorporates the boundary conditions directly. Consequently, only the boundary of the solution domain needs to be discretized. It is the main difference between the BEM and other numerical procedures, such as the finite element or finite difference method. In addition, the BEM is particularly suitable for problems that involve high local stress gradients.

In order to arrive at the integral formulation for steady-state axisymmetric problems of thermoelasticity, two alternative approaches have been investigated (Bakr, Mihsein, Fenner 1985; Bakr 1986; Rizzo, Shippy 1986). In the first one the three-dimensional fundamental solution of elasticity with the thermoelastic term is integrated around the axis of rotational symmetry. The second approach makes use of the axisymmetric fundamental solutions (Cruse, Snow, Wilson 1977; Bakr, Fenner 1983).

In this paper, on the contrary, by the help of the notation of two-point tensors, we derive the boundary integral equation valid within an arbitrary curvilinear system of coordinates. Then, we rewrite this equation in the cylindrical coordinates system where the fundamental solution U_{Ki} and the Galerkin tensor $G_K{}^i$ are defined as integrals around the axis of rotational symmetry (Section 2). The BEM is employed to solve numerically the problem (Section 3). It is assume that the temperature field in the axisymmetric body is a known function because the solution of such a problem was given in the previous work by Błocki (1991). The formulae for stresses at internal points and on the boundary are described in the Section 4. For the boundary stresses the relations proposed by Lachat (1975) or by Brebbia, Telles and Wrobel (1984) are limited to flat boundary surfaces, therefore, we cannot apply them here. On the other hand, we could apply the relations given by Sládek and Sládek (1986) or Sládek and Sládek (1989). Nevertheless, the relations derived in the Section 4 seem to be less complicated than those by Sládek and Sládek. Comparisons of numerical results for displacements and stresses obtained by the present procedure have been made with results of existing analytical solutions. The method delivers sufficient results as can be seen from the example described in the Section 5.

2. Integral equation formulation

Let us consider a domain Ω which is enclosed by its boundary surface S. We assume that points denoted by q or p lie inside the domain Ω, but points denoted by Q or P lie on the surface S. Our aim is to find steady-state solutions of thermoelastic problems, that is, solutions to the following equations (Nowacki 1970)

$$\nabla^2 T(p) = 0 \qquad p \in \Omega \tag{2.1}$$

$$G\nabla^2 \boldsymbol{u}(p) + \frac{G}{(1-2\nu)}\text{grad div}\boldsymbol{u}(p) + \boldsymbol{F}^*(p) = 0 \qquad p \in \Omega \tag{2.2}$$

where

$$\nabla^2 \boldsymbol{u} = \text{grad div}\boldsymbol{u} - \text{rot rot}\boldsymbol{u} \tag{2.3}$$

$$\boldsymbol{F}^*(p) = \boldsymbol{F}(p) - \frac{\alpha E}{(1-2\nu)}\text{grad}T(p) \tag{2.4}$$

$$G = \frac{E}{2(1+\nu)} \tag{2.5}$$

In the expressions given above the function $T(p)$ represents the temperature, $\boldsymbol{u}(p)$ is a displacement vector, and $\boldsymbol{F}(p)$ is a vector of the body force. The constants E, ν and α are Young modulus of elasticity, Poisson ratio, and the coefficient of thermal expansion, respectively.

For the temperature $T(Q)$ the following boundary conditions should be satisfied

$$T(Q) = \tilde{t}(Q) \qquad Q \in S_t \tag{2.6}$$

$$\frac{\partial T(Q)}{\partial n} = \tilde{q}(Q) \qquad Q \in S_q \tag{2.7}$$

S_t and S_q are different parts of the boundary S such that their total gives S ($S = S_t \cup S_q$ and $S_t \cap S_q = \emptyset$). The symbol $\partial/\partial n$ represents the normal derivative where $\boldsymbol{n}$ is a outward normal vector for the body Ω. The quantities $\tilde{t}(Q)$ and $\tilde{q}(Q)$ are known functions.

For the displacement field $\boldsymbol{u}(Q)$ the boundary conditions are as follows

$$u_i(Q) = \tilde{u}_i(Q) \qquad Q \in S_u \tag{2.8}$$

$$(i = 1,2,3)$$

$$p_i(Q) = \tilde{p}_i(Q) \qquad Q \in S_\sigma \tag{2.9}$$

where

$$p_i(Q) = \sigma_{ij} n^j \tag{2.10}$$

$$\sigma_{ij} = G\left(u_i\Big|_j + u_j\Big|_i\right) + \frac{2G\nu}{(1-2\nu)} g_{ij} u^k\Big|_k - \frac{\alpha E}{(1-2\nu)} T g_{ij} \tag{2.11}$$

Components of the displacement vector $\tilde{u}_i(Q)$ and components of the surface traction $\tilde{p}_i(Q)$ are given functions of space. n^j ($j = 1,2,3$) denote components of the unit vector $\boldsymbol{n}$. The quantities σ_{ij} and g_{ij} are the stress tensor and the metric tensor, respectively. A vertical bar is a symbol of the covariant derivative. As before, S_u and S_σ are different part of the surface S such that their total gives S.

Moreover, it will be useful to define two starred quantities

$$\sigma^*_{ij} = \sigma_{ij} + \frac{\alpha E}{(1-2\nu)} T g_{ij} \tag{2.12}$$

$$p^{*i} = p^i + \frac{\alpha E}{(1-2\nu)} T n^i \tag{2.13}$$

which are called pseudo-stress and pseudo-traction. In the present paper, we will consider only Eq (2.2) because numerical solutions of axisymmetric problems of the heat transfer using the Boundary Element Method were presented in the earlier paper by Błocki (1991). Therefore, having solved Eq (2.1) with the prescribed thermal boundary conditions (2.6) and (2.7), we can regard the temperature as a known function of space and insert it into Eq (2.2).

To begin with we will derive the integral formulation of the problem under consideration. Therefore, we consider a elastic body subject to the action of two systems of the body forces, tractions, and temperatures. The first one (denoted by a bar over them) produces the displacements $\bar{u}_i$. The second one (without bars) produces the displacements u^i. It is well known that using the relations (2.4), (2.12) and (2.13) we can transform each equation for the steady-state problem of thermoelasticity into an appropriate elasticity equation. Therefore, we can make use of the reciprocity principle of the theory of elasticity (Nowacki 1970), that means

$$\int\limits_S (p^{*i}\bar{u}_i - \bar{p}_i^* u^i)dS + \int\limits_\Omega (F^{*i}\bar{u}_i - \bar{F}_i^* u^i)dV = 0 \tag{2.14}$$

The tractions and body forces with bars are of the form

$$\bar{F}_i^* = \delta(q,p)g_{iK}e^K \tag{2.15}$$

$$\bar{p}_i^* = \bar{\sigma}_{ij}n^j \tag{2.16}$$

The quantity $\delta(p,q)$ is the Dirac delta function, and e^K are components of the unit base vectors. The tensor g_{iK} and some of tensors presented later are called the two-point tensors. In order to distinguish between two curvilinear coordinate systems, one connected with the point P or p and the second one connected with the point Q or q, indices of such tensors will be denoted by the upper case characters for the first system and the lower case characters for the second system. Moreover, from the Eq (2.15) we should notice that the temperature $\bar{T}$ is equal to zero. The second system of forces (without bars) is defined as follows

$$F^{*i} = -\frac{\alpha E}{(1-2\nu)}g^{ij}T \tag{2.17}$$

with the tractions defined by Eq (2.13).

The fundamental solution for three dimensional problems of elasticity in an arbitrary curvilinear coordinate system is a second order tensor $U_{Ki}(p,q)$. The components of this tensor in the cartesian coordinate system are known (Nowacki 1970)

$$\widetilde{U}_{Ki}(p,q) = \frac{C}{r}\Big[(3-4\nu)\delta_{Ki} - r|_K r|_i\Big] \tag{2.18}$$

where r is the distance between the point p and the point q

$$r = |q-p| \tag{2.19}$$

and the constant C is equal to

$$C = \frac{1}{16\pi G(1-\nu)} \tag{2.20}$$

From the definition of the fundamental solution, it is straightforward that the following relation is satisfied

$$\bar{u}_i = U_{Ki}e^K \tag{2.21}$$

To calculate the stress field $\bar{\sigma}_{ij}$ we make use of Eqs (2.11), and (2.21), and assumption that $\bar{T} = 0$, i.e.

$$\bar{\sigma}_{ij} = D_{Kij}e^K \tag{2.22}$$

where

$$D_{Kij} = G\Big(U_{Ki}\Big|_j + U_{Kj}\Big|_i\Big) + U_K{}^l\Big|_l g_{ij} \tag{2.23}$$

Next, defining the tensor T_{Ki} as it follows

$$T_{Ki} = D_{Kij}n^j \tag{2.24}$$

we arrive at the relation which has the form

$$\bar{p}_i = T_{Ki}e^K \tag{2.25}$$

Substituting Eqs (2.13), (2.21), (2.25), (2.17) and (2.15) into the reciprocity Eq (2.14) produces the integral equation

$$\begin{aligned} u_K(p) &= \int_S \Big[U_{Ki}(p,Q)p^i(Q) - T_{Ki}(p,Q)u^i(Q) + \\ &+ \frac{\alpha E}{(1-\nu)}T(Q)U_{Ki}(p,Q)n^i\Big]dS(Q) + b^*_K(p) \end{aligned} \tag{2.26}$$

where

$$b^*_K(p) = -\frac{\alpha E}{(1-\nu)}\int_\Omega U_{Ki}(p,q)g^{ij}T\Big|_j(q)dV(q) \tag{2.27}$$

Next, integrating by parts the last integral, gives

$$b^*_K(p) = -\frac{\alpha E}{(1-\nu)}\int_S U_{Ki}(p,Q)T(Q)n^i dS(Q) + b_K(p) \tag{2.28}$$

in which

$$b_K(p) = \frac{\alpha E}{(1-\nu)}\int_\Omega U_{Ki}\Big|_j(p,q)g^{ij}T(q)dV(q) \tag{2.29}$$

The foregoing volume integral above can be transformed into a surface integral, i.e.

$$b_K(p) = \int\limits_S \Big[V_K(p,Q)T(Q) - W_K(p,Q)\frac{\partial T}{\partial n(Q)}\Big]\,dS(Q) \tag{2.30}$$

where

$$V_K(p,Q) = -\frac{\alpha E}{2(1-\nu)} G_K{}^i\Big|_{ij} n^j \tag{2.31}$$

$$W_K(p,Q) = -\frac{\alpha E}{2(1-\nu)} G_K{}^i\Big|_i \tag{2.32}$$

The last two relations will be true provided the tensor $G_K{}^i$ satisfies the following relation

$$G_K{}^j\Big|_{ji} = 2(1-\nu)(U_{Ki} + H_{Ki}) \tag{2.33}$$

and H_{Ki} is such that

$$(H_{Ki}\Big|_l - G_K{}^i\Big|_{jil})g^{il} = 0 \tag{2.34}$$

In the cartesian coordinate system components of the tensor $G_K{}^i$ are known (Danson 1981)

$$\widetilde{G}_K^i = \frac{1+\nu}{4\pi E} r\delta_K^i \tag{2.35}$$

The final form of the integral equation after the latest transformation is as follows

$$u_K(p) = \int\limits_S \Big[U_{Ki}(p,Q)p^i(Q) - T_{Ki}(p,Q)u^i(Q)\Big]\,dS(Q) + b_K(p) \tag{2.36}$$

where $b_K(p)$ is given by Eq (2.30).

We should notice that the steady-state problem of thermoelasticity formulated by such a integral equation can be posed in an arbitrary curvilinear system of coordinates. We only need to know the components of two tensors U_{Ki} and $G_K{}^i$ in a specified system of coordinates.

If we let $p \to P$ and compute the limit of the both side of Eq (2.36) we arrive at the boundary integral equation (Kupradze 1963)

$$c_K{}^J u_J(P) = \int\limits_S \Big[U_{Ki}(P,Q)p^i(Q) - T_{Ki}(P,Q)u^i(Q)\Big]\,dS(Q) + b_K(P) \tag{2.37}$$

where

$$b_K(P) = \int\limits_S \Big[V_K(P,Q)T(Q) - W_K(P,Q)\frac{\partial T}{\partial n(Q)}\Big]\,dS(Q) \tag{2.38}$$

$$c_K{}^J = \frac{1}{2}\delta_K^J \tag{2.39}$$

The last relation is true provided a unique tangent plane exists at P. But if this is not the case the quantity $c_K{}^J$ has to be computed by applying special methods (Hartmann 1981). However, it is known (Bakr 1986) that using the boundary integral equation together with rigid–body movements and known solutions of elasticity problems the components of $c_K{}^J$ can be indirectly calculated. The details of this approach are discussed in the next section.

One of the methods which leads to the fundamental solution in the cylindrical coordinates system (ρ, Θ, z) is to integrate the tensor $\tilde{U}_{Ki}$ with respect to the coordinate $\Theta = \Theta_Q - \Theta_P$, i.e.

$$\bar{U}_{Ki} = \frac{1}{2\pi} \int_0^{2\pi} \tilde{U}_{Ki} \, d\Theta \tag{2.40}$$

The integral above can be evaluated with the help of the following relation

$$r = \sqrt{(z_Q - z_P)^2 + \rho_Q^2 + \rho_P^2 - 2\rho_Q \rho_P \cos(\Theta_Q - \Theta_P)} \tag{2.41}$$

where ρ_Q, Θ_Q, z_Q and ρ_P, Θ_P, z_P represent coordinates of the point Q and P, respectively.

According to Eq (2.40) the result of integrating takes the form

$$\bar{U}_{Ki} = A_{Ki}[K(m) - E(m)] + B_{Ki}E(m) \tag{2.42}$$

where A_{Ki} and B_{Ki} are given in the Appendix. The functions $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and the second kind (Abramowitz i Stegun 1974), i.e.

$$K(m) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - m \sin^2 \Theta}} \, d\Theta \tag{2.43}$$

$$E(m) = \int_0^{\pi/2} \sqrt{1 - m \sin^2 \Theta} \, d\Theta \tag{2.44}$$

where

$$m = \frac{2b}{a + b} \tag{2.45}$$

$$a = \rho_P^2 + \rho_Q^2 + (z_Q - z_P)^2 \tag{2.46}$$

$$b = 2\rho_P \rho_Q \tag{2.47}$$

Using Eqs (2.42) and (2.24) we can calculate the components of $\bar{T}_{Ki}$ in the cylindrical coordinate system (see Appendix).

One can define the components of $\bar{G}^i_K$ in the following way

$$\bar{G}^i_K = \frac{1}{2\pi}\int_0^{2\pi} \tilde{G}^i_K d\theta \tag{2.48}$$

which results in

$$\bar{G}^i_K = \frac{1}{4\pi^2 G}\sqrt{a+b}\, E(m)\delta^i_K \tag{2.49}$$

The tensor $\bar{G}^i_K$ meets the condition (2.33). Therefore, we make use of Eqs (2.31) and (2.32) in order to compute the components of the vectors $\bar{V}_K$ and $\bar{W}_K$ (see Appendix).

The final form of the boundary integral equation posed in the cylindrical coordinates system is of the form

$$\begin{aligned} c_K{}^J(P)u_J(P) &= 2\pi \int_{S^*} \Big[\bar{U}_{Ki}(P,Q)p^i(Q) - \bar{T}_{Ki}(P,Q)u^i(Q) + \\ &+ \bar{V}_K(P,Q)T(Q) - \bar{W}_K(P,Q)\frac{\partial T}{\partial n(Q)}\Big]\rho_Q dS^*(Q) \end{aligned} \tag{2.50}$$

where S^* is a curve lying on the surface S and any radial plane passing through the axis of rotational symmetry. If the domain contains a part of this axis the curve S^* is open and in the opposite case S^* is a close one.

3. Boundary element method

In this section, a general numerical scheme for the solution of the boundary integral Eq (2.50) by the Boundary Element Method is presented. The starting point in this method is to divide the boundary curve S^* into N segments. Each segment is a isoparametric quadratic element. It has three nodal points one at each end, denoted by A and C, and one at the midpoint, denoted by B. Moreover, the geometry of the body and distributions of the unknowns are expressed in terms of quadratic shape functions of the intrinsic coordinate $\zeta \in < -1, 1 >$. The cylindrical variables ρ and z which describe the geometry can be written as

$$\begin{aligned} \rho(\zeta) &= \phi^1\rho_A + \phi^2\rho_B + \phi^3\rho_C \\ z(\zeta) &= \phi^1 z_A + \phi^2 z_B + \phi^3 z_C \end{aligned} \tag{3.1}$$

where ρ_A, z_A, ρ_B, z_B, ρ_C and z_C are coordinates of the nodes A, B and C, respectively. The shape functions ϕ^1, ϕ^2 i ϕ^3 are defined

$$\begin{aligned} \phi^1 &= -\frac{1}{2}\zeta(1-\zeta) \\ \phi^2 &= (1-\zeta)(1+\zeta) \\ \phi^3 &= \frac{1}{2}\zeta(1+\zeta) \end{aligned} \tag{3.2}$$

From the relations above one can see that the nodal values of the variable ζ are -1, 0, and 1, respectively. The Jacobian of transformation is given by the following relation

$$J = \sqrt{J_\rho^2 + J_z^2} \tag{3.3}$$

where

$$J_\rho = \frac{d\rho}{d\zeta} = \phi^1{}_{,\zeta}\rho_A + \phi^2{}_{,\zeta}\rho_B + \phi^3{}_{,\zeta}\rho_C \tag{3.4}$$

$$J_z = \frac{dz}{d\zeta} = \phi^1{}_{,\zeta}z_A + \phi^2{}_{,\zeta}z_B + \phi^3{}_{,\zeta}z_C \tag{3.5}$$

Thus, we have

$$dS^* = J\,d\zeta \tag{3.6}$$

By the help of the foregoing relation above we can compute components of the normal vector $\mathbf{n}$, namely

$$n_\rho = \frac{dz}{dS^*} = \frac{J_z}{J} \qquad n_z = -\frac{d\rho}{dS^*} = -\frac{J_\rho}{J} \tag{3.7}$$

Similarly, we can compute components of the tangent vector to the curve S^*, i.e.

$$m_\rho = -n_z \qquad m_z = n_\rho \tag{3.8}$$

As mentioned before, the variation of the unknown functions are parabolic. Therefore, for the αth element $(\alpha = 1, ..., N)$, we have

$$\begin{aligned} T^\alpha &= \phi^1 T_1^\alpha + \phi^2 T_2^\alpha + \phi^3 T_3^\alpha \\ \left(\frac{\partial T}{\partial n}\right)^\alpha &= \phi^1\left(\frac{\partial T}{\partial n}\right)_1^\alpha + \phi^2\left(\frac{\partial T}{\partial n}\right)_2^\alpha + \phi^3\left(\frac{\partial T}{\partial n}\right)_3^\alpha \end{aligned} \tag{3.9}$$

$$\begin{aligned} u^{i\alpha} &= \phi^1 u_1^{i\alpha} + \phi^2 u_2^{i\alpha} + \phi^3 u_3^{i\alpha} \\ p^{i\alpha} &= \phi^1 p_1^{i\alpha} + \phi^2 p_2^{i\alpha} + \phi^3 p_3^{i\alpha} \end{aligned} \tag{3.10}$$

where the quantities T^α, $(\partial T/\partial n)^\alpha$, $u^{i\alpha}$ and $p^{i\alpha}$ with the subscripts 1, 2, or 3 represent their values at the points A, B or C, respectively.

Now, we assume that the point P is taken successively to every nodal point on the discretized boundary. This leads to the following set of integral equations

$$
\begin{aligned}
c_K{}^J(P_\beta)u_J(P_\beta) &= 2\pi \sum_{\alpha=1}^{N} \int_{S_\alpha^*} \Big[\bar{U}_{Ki}(P_\beta,Q)p^i - \bar{T}_{Ki}(P_\beta,Q)u^i + \\
&+ \bar{V}_K(P_\beta,Q)T - \bar{W}_K(P_\beta,Q)\frac{\partial T}{\partial n}\Big]\rho_Q dS^*(Q) \qquad (3.11)\\
&\qquad (\beta = 1,2,...,M)
\end{aligned}
$$

where S_α^* represents the αth element, and M is a number of nodes.

Substituting the parametric representations (3.1), (3.9), and (3.10) into the last equations, gives

$$
c_K{}^J(P_\beta)u_J(P_\beta) = \sum_{\alpha=1}^{N}\Big[I_{Ki}^l p_l^{i\alpha} - J_{Ki}^l u_l^{i\alpha} + K_K^l T_l^\alpha - L_K^l\Big(\frac{\partial T}{\partial n}\Big)_l^\alpha\Big]
$$

$$
(l = 1,2,3; \quad \beta = 1,2,...,M) \tag{3.12}
$$

where

$$
I_{Ki}^l = 2\pi\int_{-1}^{1}\phi^l\bar{U}_{Ki}\rho_Q J d\zeta \tag{3.13}
$$

$$
J_{Ki}^l = 2\pi\int_{-1}^{1}\phi^l\bar{T}_{Ki}\rho_Q J d\zeta \tag{3.14}
$$

$$
K_K^l = 2\pi\int_{-1}^{1}\phi^l\bar{V}_K\rho_Q J d\zeta \tag{3.15}
$$

$$
L_K^l = 2\pi\int_{-1}^{1}\phi^l\bar{W}_K\rho_Q J d\zeta \tag{3.16}
$$

The functions $\bar{U}_{Ki}$, $\bar{T}_{Ki}$, $\bar{V}_K$, and $\bar{W}_K$ are expressed in terms of the elliptic integrals (2.43) and (2.44). To evaluate this type of integrals we will apply the approximated formulae proposed by Cody (1965).

In order to compute the integrals (3.13)÷(3.16) we make use of the standard Gauss quadrature rules for all elements. However, in the case when the point $Q \to P_\beta$, the integrals I_{Ki}^l, K_K^l and L_K^l become singular, because $K(m)$ is not bounded for $m \to 1$, i.e.

$$
\lim_{m\to 1}\Big\{K(m) - \frac{1}{2}\ln\frac{16}{1-m}\Big\} = 0 \tag{3.17}
$$

The evaluation of such singular integrals by using the standard Gauss method would produce inaccurate results. Therefore, we employ special numerical strategy discussed by Telles (1987). This simple method is based upon the standard Gaussian quadrature scheme and a cubic coordinate transformation, i.e.

$$\int_{-1}^{1} f(\zeta)d\zeta = \int_{-1}^{1} \bar{f}(\gamma)\frac{d\zeta}{d\gamma}d\gamma \tag{3.18}$$

in which

$$\zeta(\gamma) = \frac{(\gamma-\bar{\gamma})^3 + \bar{\gamma}(\bar{\gamma}^2+3)}{1+3\bar{\gamma}^2} \tag{3.19}$$

$$\frac{d\zeta}{d\gamma} = \frac{3(\gamma-\bar{\gamma})^2}{1-3\gamma^2} \tag{3.20}$$

$$\bar{\gamma} = \sqrt[3]{\zeta^*(\zeta^*-1)+|\zeta^*-1|} + \sqrt[3]{\zeta^*(\zeta^*-1)-|\zeta^*-1|} \tag{3.21}$$

where ζ^* represents the value of points at which the kernels of integrals become singular. The important feature of this method is self-adaptiveness of the scheme what means that we can apply this method both for the singular kernels and the nearly singular ones.

Some of the integrals J^l_{Ki} exist only in the Cauchy principal value sense because of their strong singularities involved in their kernels that is of the type $1/r$, where $r = |P_\beta - Q|$. However, there is no need to calculate these integrals explicitly because the evaluation of them can be carried out together with the $c_K{}^J$ coefficients by using rigid-body translations and known solutions of elasticity problems (Bakr 1986).

The relations (3.12) represent the whole set of linear algebraic equations which can be expressed in matrix form as

$$\mathbf{G}p = \mathbf{H}u + b \tag{3.22}$$

where p is a vector of nodal tractions, u is a vector of nodal displacements, and b is a vector of thermal loads.

The rigid-body translation in the z coordinate gives all the nodal displacements u_ρ equal to zero and in addition

$$p = 0 \qquad\qquad b = 0 \tag{3.23}$$

Therefore, we have

$$H_{k-1,k} = -\sum_{\substack{j=1\\k\neq 2j}}^{M} H_{k-1,2j}$$

$$k = 2i \qquad (i = 1,...,M) \tag{3.24}$$

$$H_{kk} = -\sum_{\substack{j=1\\k\neq 2j}}^{M} H_{k,2j}$$

The rest of the elements of the matrix $\mathbf{H}$ which are to determine, that is, H_{kk} and $H_{k+1,k}$ (where $k = 2i - 1$; $i = 1,...,M$) will be computed by means of the following elasticity solution

$$u_\rho = \frac{1-\nu}{E}\rho \qquad u_z = -\frac{2\nu}{E}z \tag{3.25}$$

with the tractions given by

$$p_\rho = n_\rho \qquad p_z = 0 \tag{3.26}$$

After making use of the boundary conditions the final form of the algebraic equations is

$$\mathbf{A}\boldsymbol{X} = \boldsymbol{F} \tag{3.27}$$

where $\boldsymbol{X}$ represents unknown displacements and tractions.

4. Displacements and stresses

After solving the set of Eqs (3.27) all the boundary functions are known. To calculate the value of the displacement field inside the body, that is for $p \in \Omega$, one can make use of the integral Eq (3.12) in which the point P_β is replaced by p and in addition the coefficients of $c_K{}^J$ are set to δ^J_K. By differentiating this equation with respect to the coordinates of p we arrive at the displacement gradients, the strains and finally the stresses at p, namely,

$$\sigma_{JK}(p) = \sum_{\alpha=1}^{N}\Big[M^l_{JKi}p^{i\alpha}_l - N^l_{JKi}u^{i\alpha}_l + O^l_{JK}T^\alpha_l - P^l_{JK}\Big(\frac{\partial T}{\partial n}\Big)^\alpha_l\Big] \tag{4.1}$$

where

$$M^l_{JKi} = 2\pi\int\limits_{-1}^{1}\phi^l D^*_{JKi}\rho_Q J d\zeta \tag{4.2}$$

$$N^l_{JKi} = 2\pi \int_{-1}^{1} \phi^l S_{JKi} \rho_Q J d\zeta \tag{4.3}$$

$$(l = 1, 2, 3)$$

$$O^l_{JK} = 2\pi \int_{-1}^{1} \phi^l X_{JK} \rho_Q J d\zeta \tag{4.4}$$

$$P^l_{JK} = 2\pi \int_{-1}^{1} \phi^l Y_{JK} \rho_Q J d\zeta \tag{4.5}$$

$$D^*_{JKi} = \frac{2G\nu}{1-2\nu} U^L{}_i\big|_L g_{JK} + G(U_{Ji|K} + U_{Ki|J}) \tag{4.6}$$

$$S_{JKi} = \frac{2G\nu}{1-2\nu} T^L{}_i\big|_L g_{JK} + G(T_{Ji|K} + T_{Ki|J}) \tag{4.7}$$

$$X_{JK} = \frac{2G\nu}{1-2\nu} V^L\big|_L g_{JK} + G(V_{J|K} + V_{K|J}) \tag{4.8}$$

$$Y_{JK} = \frac{2G\nu}{1-2\nu} W^L\big|_L g_{JK} + G(W_{J|K} + W_{K|J}) \tag{4.9}$$

In order to avoid the singularities of the integral formulae for stresses (4.1) when $p \to P$, we will evaluate surface stresses from simple relationship between the tractions and the local displacement gradients in the surface near the point P. This is described below.

A local coordinate system with axes designated by 1, 2, and 3 is aligned along the tangent and normal directions at the point P on the surface being considered. For the axisymmetric problem components of the stress and strain tensors in this coordinates system can be written as

$$\begin{aligned} \sigma_{11} &= \frac{E}{1-\nu^2}(\varepsilon_{11} + \nu\varepsilon_{22}) + \frac{\nu}{1-\nu}\sigma_{33} - \frac{\alpha E}{1-\nu}T \\ \sigma_{22} &= E\varepsilon_{22} + \nu(\sigma_{11} + \sigma_{33}) - \alpha E T \\ \sigma_{13} &= p_1 \qquad\qquad \sigma_{33} = p_3 \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} \varepsilon_{11} &= u_{1,1} + \frac{u_3}{R_1} & \varepsilon_{22} &= \frac{\rho_{,1}}{u_1} + \frac{u_3}{R_2} \\ p_1 &= -p_\rho n_z + p_z n_\rho & p_3 &= p_\rho n_\rho + p_z n_z \\ u_1 &= -u_\rho n_z + u_z n_\rho & u_3 &= u_\rho n_\rho + u_z n_z \end{aligned} \tag{4.11}$$

R_1 and R_2 are the principal radii of curvature of S, ρ is a circumferential radius of the origin of the system 1, 2, 3.

Taking into account certain fundamental relations from the differential geometry, namely, the conditions of Codazzi and Gauss, we have

$$\begin{aligned} \varepsilon_{11} &= -n_z u_{\rho,1} + n_\rho u_{z,1} \\ \varepsilon_{22} &= \frac{u_\rho}{\rho} \end{aligned} \tag{4.12}$$

in which

$$\begin{aligned} u_{\rho,1} &= \frac{1}{J}\phi^i_{,\zeta} u_{\rho i} \\ u_{z,1} &= \frac{1}{J}\phi^i_{,\zeta} u_{zi} \qquad (i = 1,2,3) \\ u_\rho &= \phi^i u_{\rho i} \end{aligned} \tag{4.13}$$

where $u_{\rho i}$ and u_{zi} represent the nodal displacements.

Some transformations are needed to define the components of the stress tensor in the cylindrical system of coordinates $(\rho,\ \Theta,\ z)$, i.e.

$$\begin{aligned} \sigma_{\rho\rho} &= \sigma_{11} n_z^2 + \sigma_{33} n_\rho^2 - 2\sigma_{13} n_\rho n_z \\ \sigma_{\Theta\Theta} &= \sigma_{22} \\ \sigma_{zz} &= \sigma_{11} n_\rho^2 + \sigma_{33} n_z^2 + 2\sigma_{13} n_\rho n_z \\ \sigma_{\rho z} &= (\sigma_{33} - \sigma_{11}) n_\rho n_z + \sigma_{13}(n_\rho^2 - n_z^2) \end{aligned} \tag{4.14}$$

The rest of the stress tensor components are equal to zero.

5. Numerical results

A computer program based on the algorithm just described has been developed and used for obtaining numerical results. Comparisons of these results have been made with existing analytical solutions to validate the present formulation. One of such examples is described below.

Let us consider a thick-walled spherical shell (Fig.1). The temperature of the outer surface of the shell is defined in the following way

$$T(R_2,\gamma) = T_0 \sin^2\gamma \tag{5.1}$$

where R_2 is the radius of this surface, T_0 is a constant, and $\gamma \in \langle 0, \pi/2 \rangle$ is an angle measured from the axis of rotational symmetry (Fig.1). The inner shell surface is kept at the temperature equal to zero. The material properties of the

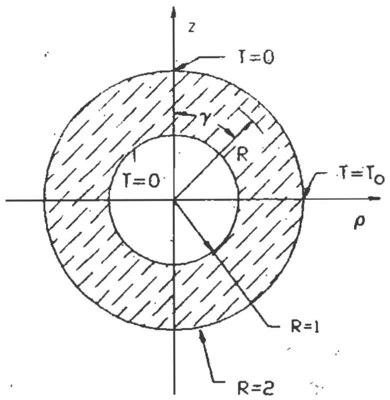

Fig. 1. Spherical shell

shell are Young modulus $E = 2.07 \cdot 10^8$ kN/m^2, Poisson ratio $\nu = 0.3$ and the coefficient of thermal expansion $\alpha = 11 \cdot 10^{-6} \cdot 1/C^o$, respectively.

Hellen, Galluzzo, Kfouri (1977) conducted an analytical study of this example and published results which for the temperature and displacement fields are

$$T = T_0\Big[\frac{4}{3}\Big(1 - \frac{1}{\bar{R}}\Big) + \frac{8}{93}\Big(\bar{R}^2 - \frac{1}{\bar{R}^3}\Big)(1 - 3\cos^2\gamma)\Big] \tag{5.2}$$

$$u = R_1\alpha T_0\Big[0.707\frac{1}{\bar{R}^3} - 1.24 + 1.01\bar{R} + (1.5\cos^2\gamma - 0.5)\Big(0.0161\frac{1}{\bar{R}^4} - 0.183\frac{1}{\bar{R}^2} - 0.122\bar{R} - 0.052\bar{R}^3\Big)\Big] \tag{5.3}$$

$$v = R_1\alpha T_0\Big(0.0161\frac{1}{\bar{R}^4} + 0.234\frac{1}{\bar{R}^2} + 0.183\bar{R} - 0.120\bar{R}^3\Big)\sin\gamma\cos\gamma \tag{5.4}$$

where $\bar{R} = R/R_1$. R is a radius measured from the center of the shell, and R_1 is a radius of the inner surface of the shell.

Taking into account only the upper symmetric half of the sphere, the computations were carried out using the division of the boundary into 20 elements. In the Table 1. the results obtained analytically and numerically for $R \in < R_1, R_2 >$ and $\gamma = \pi/2$ are presented.

Table 1. Temperatures and displacements

R	Temperatures		Displacements	
	Analyt.	Numer.	Analyt.	Numer.
1.0	0.0000	0.0000	0.7122	0.7117
1.1	0.1607	0.1607	0.6898	0.6889
1.2	0.2963	0.2963	0.7048	0.7034
1.3	0.4139	0.4139	0.7470	0.7452
1.4	0.5182	0.5182	0.8097	0.8075
1.5	0.6125	0.6125	0.8883	0.8857
1.6	0.6992	0.6992	0.9799	0.9769
1.7	0.7801	0.7801	1.0820	1.0790
1.8	0.8566	0.8565	1.1940	1.1900
1.9	0.9296	0.9296	1.3130	1.3090
2.0	1.0000	1.0000	1.4400	1.4360

For the equivalent stress defined in the following way

$$\sigma_{eq} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2} \tag{5.5}$$

additional results are shown in the Fig 2. It is assumed that

$$R \in < R_1 = 1.0, R_2 = 2.0 > \qquad \text{and} \qquad \gamma = \pi/2$$

The analytical formulae for σ_{eq} were derived by applying (2.11) together with the solutions (5.3) and (5.4).

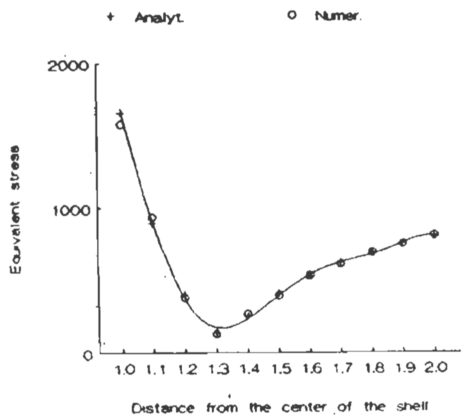

Fig. 2. Equivalent stress

We can notice that reported results obtained from the two methods are quite closed.

References

1. ABRAMOWITZ M., STEGUN I.A., 1974, *Handbook of Mathematical Functions*, Dover, New York
2. BAKR A.A., FENNER, R.T., 1983, *Use of the Hankel Transform in Boundary Integral Methods for Axisymmetric Problems*, Int.J.Num.Meth.Eng., 19, 1765-1769
3. BAKR A.A., MIHSEIN M.J.A., FENNER R.T., 1985, *Treatment of Thermoelastic Effects in the Boundary Integral Equation Method*, J.Thermal Stresses, 8, 291-303
4. BAKR A.A., 1986, *The Boundary Integral Equation Method in Axisymmetric Stress Analysis Problems*, Springer-Verlag, Berlin
5. BANERJEE P.K., BUTTERFIELD R., 1981, *Boundary Element Methods in Engineering Science*, Mc Graw-Hill
6. BESKOS D.E., 1987, *Boundary Element Methods in Mechanics*, North-Holland
7. BŁOCKI J., 1991, *Numerical Solutions of Laplace's Equation in the Axisymmetrical Domain Obtained by the Boundary Element Solution*, IMP-PAN, Report 325/1279/91, Gdańsk, (in polish)
8. BREBBIA C.A., TELLES J.C.F., WROBEL L.C., 1984, *Boundary Element Techniques*, Springer-Verlag
9. CODY J., 1965, *Chebyshev Approximations for the Complete Elliptic Integrals K and E*, Math.of Comp., 19, 89, 105-112
10. CRUSE T.A., SNOW D.W., WILSON R.B., 1977, *Numerical Solutions in Axisymmetric Elasticity*, Comp. & Struct., 7, 445-451
11. DANSON D.J., 1981, *A Boundary Element Formulation of Problems in Linear Isotropic Elasticity with Body Forces*, Boundary Element Methods, Proc.of the Third Inter.Seminar, California, ed. C.A.Brebbia, Springer-Verlag
12. HARTMANN F., 1981, *Elastostatics*, Progress in Boundary Element Methods, 1, ed. C.A.Brebbia, Halsted Press, New York, 84-167
13. HELLEN T.K., GALLUZZO N.G., KFOURI A.P., 1977, *The Elastic-Plastic Analysis of a Thick Spherical Shell under Thermal Loading – A Comparison of Three Numerical Procedures*, Int.J.Mech.Sci., 19, 4, 209-221
14. KUPRADZE W.D., 1963, *Potential Method in Theory of Elasticity*, Gosud.Izdat. Fiz.Mat.Liter., Moskwa, (in russian)
15. LACHAT J.C., 1975, *A Further Development of the Boundary Integral Technique for Elastostatics*, Ph.D. Thesis University of Southampton
16. NOWACKI W., 1970, *Theory of Elasticity*, PWN, Warszawa, (in polish)
17. RIZZO F.J., SHIPPY D.J., 1986, *A Boundary Element Method for Axisymmetric Elastic Bodies*, Developments in Boundary Element Methods - 4, ed. P.K.Banerjee, J.O.Watson, Elsevier Applied Science Publishers, London
18. SLÁDEK V., SLÁDEK J., 1986, *Improved Computation of Stresses Using the Boundary Element Method*, Appl.Math.Modelling 10, 249-255
19. SLÁDEK V., SLÁDEK J., 1989, *Computation of Stresses in Axisymmetric Elastostatical Problems Using BEM*, Computational Mechanics 4, 59-72
20. TELLES J.C.F., 1987, *A Self-Adaptive Co-Ordinate Transformation for Efficient Numerical Evaluation of General Boundary Element Integrals*, Int. J.Num. Meth.Eng., 24, 959-973

Appendix

The components of A_{Ki} i B_{Ki} are equal to

$$
\begin{aligned}
A_{\rho\rho} &= \frac{\gamma}{\rho_Q\rho_P}(C_4a+\bar{z}^2) \qquad & B_{\rho\rho} &= \kappa \\
A_{\rho z} &= -\frac{\gamma\bar{z}}{\rho_P} & B_{\rho z} &= \varepsilon \\
A_{z\rho} &= \frac{\gamma\bar{z}}{\rho_Q} & B_{z\rho} &= B_{\rho z} \\
A_{zz} &= 2\gamma C_4 & B_{zz} &= -B_{\rho\rho}
\end{aligned}
\tag{A.1}
$$

$$(K,i=\rho,z)$$

where

$$\bar{z}=z_Q-z_P \qquad \bar{\rho}=\rho_Q-\rho_P \tag{A.2}$$

$$C_3=\frac{1}{16\pi^2G(1-\nu)} \qquad C_4=3-4\nu \tag{A.3}$$

$$\gamma=\frac{C_3}{\sqrt{a+b}} \qquad \kappa=-2\gamma\Big(C_4+\frac{\bar{z}^2}{r^2}\Big) \qquad \varepsilon=2\gamma\rho\frac{\bar{z}}{r^2} \tag{A.4}$$

The quantities a and b are defined by Eqs (2.46) and (2.47).

The components of the tensor $\bar{T}_{Ki}$ are equal to

$$
\begin{aligned}
\bar{T}_{\rho\beta} &= \Big[C_5\bar{U}_{\rho\beta,\rho}+C_6\Big(\bar{U}_{z\beta,z}+\frac{\bar{U}_{\rho\beta}}{\rho_Q}\Big)\Big]n_\rho+\mu t_\beta n_z \\
&\qquad\qquad (\beta=\rho,z) \\
\bar{T}_{z\beta} &= \Big[C_5\bar{U}_{z\beta,z}+C_6\Big(\bar{U}_{\rho\beta,\rho}+\frac{\bar{U}_{\rho\beta}}{\rho_Q}\Big)\Big]n_z+\mu t_\beta n_\rho
\end{aligned}
\tag{A.5}
$$

where

$$t_\beta=\bar{U}_{\rho\beta,z}+\bar{U}_{z\beta,\rho} \tag{A.6}$$

$$C_5=\frac{2G(1-\nu)}{(1-2\nu)} \qquad C_6=\frac{2G\nu}{(1-2\nu)} \qquad G=\frac{E}{2(1+\nu)} \tag{A.7}$$

in which a comma represents the partial derivative with respect to the coordinates of the point Q.

In the cylindrical system of coordinates the components of $\bar{V}_K$ i $\bar{W}_K$ are equal to

$$
\begin{aligned}
\bar{V}_\rho &= \alpha E\gamma\Big\{\Big(\frac{\rho_Q^2+\rho_P^2}{\rho_Q\rho_P}n_\rho + \frac{\bar{z}}{\rho_P}n_z\Big)[K(m)-E(m)] + 2\Big[\Big(1-\frac{\bar{z}^2}{r^2}\Big)n_\rho + \\
&+ \frac{\bar{z}}{\rho_P}\Big(1-\frac{\rho_Q\bar{\rho}}{r^2}\Big)n_z\Big]E(m)\Big\} \\
\bar{V}_z &= \alpha E\gamma\Big\{\Big(2n_z - \frac{\bar{z}}{\rho_Q}n_\rho\Big)[K(m)-E(m)] + 2\Big[\Big(1-\frac{\bar{z}^2}{r^2}\Big)n_z - \frac{\bar{\rho}\bar{z}}{r^2}n_\rho\Big]E(m)\Big\} \\
\bar{W}_\rho &= \frac{\alpha E\gamma}{\rho_P}\Big\{(a-2\rho_Q)[K(m)-E(m)] + 2(\rho_P\bar{\rho}_Q+\bar{z}^2)E(m)\Big\} \qquad (A.8) \\
\bar{W}_z &= 2\alpha E\gamma\bar{z}K(m)
\end{aligned}
$$

where a and b are defined by Eqs (2.46) and (2.47).

Metoda elementów brzegowych w osiowosymetrycznych problemach termosprężystości

Streszczenie

Stosując ogólny zapis tensorowy, wyprowadzono brzegowe równanie całkowe dla stacjonarnego problemu termosprężystości. Wskazano, że jedynymi wielkościami potrzebnymi do pełnego sformułowania problemu w dowolnym krzywoliniowym układzie współrzędnych są rozwiązanie podstawowe U_{Ki} oraz tensor $G_K{}^i$ nazywany tensorem Galerkina. W numerycznym algorytmie zastosowano paraboliczne elementy brzegowe oraz proste metody całkowania zarówno dla całek osobliwych jak i nieosobliwych. Wyprowadzono odpowiednie zależności umożliwiające określenie stanu naprężenia wewnątrz obszaru oraz na jego brzegu. Opisane wyniki obliczeń numerycznych uzyskane za pomocą programu komputerowego wskazują na poprawność proponowanego algorytmu.

Manuscript received April 2, 1992; accepted for print May 11, 1992