MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
3, 30, 1992

A NEW LINEAR ELASTODYNAMIC SOLUTION TO BOUNDARY EIGENVALUE PROBLEM OF FLEXURAL VIBRATION OF VISCOELASTIC LAYERED AND HOMOGENEOUS BANDS

STANISŁAW KARCZMARZYK

Warsaw University of Technology

A new linear elastodynamics formulation together with a solution to the eigenvalue problem for viscoelastic bands both clamped and simply supported and consisting of any number of high–strength, fibrous and stiffness–comparable layers have been developed. New boundary clamping conditions have been introduced. On the basis of numerical results, obtained after solving a non–conventional numerical problem resulting from the formulation, for both simply supported (three- and two–layer) and clamped (homogeneous) bands, an accuracy and usefulness of the solution developed is shown.

1. Introduction

Viscoelastic materials such as plastics, resins and rubbers can be components of layered structure elements such as bands, beams or plates. However viscoelasticity of the materials is the main reason of the structure elements vibration damping. Therefore the analysis of vibration damping of the structures is one of the most important practical problems. In order to calculate the damping parameters such as logarithmic decrement, loss factor or damping capacity accurately one ought to formulate the considered problem as exactly as possible. First of all one should include the transverse shear deformations in each layer into the formulation (cf [20,22]). It is also important to take into account complete material characteristics (i.e., stiffness matrices) of layers (cf [16]).

As the viscoelastic material characteristics are dependent on frequency then in the case of free vibration damping analysis one has first to calculate eigenfrequency of a whole structure without taking into account material loss factors of its layers. The parameter (i.e. eigenfrequency) ought to be evaluated accurately as it is necessary for the proper choice of the material loss factors which will afterwards be the input data for calculations of any damping parameter of the structure. Since an eigenfrequency value depends on the kind of supports it is desirable to define boundary conditions at the ends of the band or beam as precisely as possible.

Let us note that we have plane strain within a band and plane stress within a beam. Inspite of the difference occuring in formulations of the eigenvalue problems (for band and beam) one obtains, for the same values of material and geometrical parameters of layers in each case, identical values of damping parameters of the structures (cf [14,15]). Thus it is justifiable to compare the approach given here with the theories presented in the papers on vibration damping analysis of layered beams.

Most of papers on transverse vibration damping analysis of layered beams referes to the sandwich structures (cf [1 ÷ 10]). Few authors only deal with the problem of two-layer beams (cf [14,15]). Also filamentary composite beams have been rarely investigated [16]. A short survey of the works published is given below from the point of view of clear presentation of differences between the method of vibration damping analysis proposed in the present paper and those published elsewhere. As the published, analytic formulations of the beam problems have been derived by using so-called semi-inverse techniques then a special attention is paid to the kinematical assumptions, the boundary conditions at the ends of the beams and the simplifications of material characteristics which have been used in the papers cited.

The analysis of the vibrations of sandwich beams has been carried out by many authors (cf [3 ÷ 8,10]). In such beams the material of the faces was considered to be much more stiff and heavy than the material of the core. Due to the foregoing restrictions it was justifiable to omit in formulation of the eigenvalue problem both the normal cross-sectional stresses and the material damping resulting from dilatation of the core. The energy dissipation (both shearing and dilatational) in the faces likewise anisotropy of the layers were seldom taken into account. In order to derive equations of motion the Kirchhoff hypothesis of flat cross-sections was generally applied (cf [1 ÷ 10]). Behaviour of the sandwich clamped-clamped beams has been modeled in the papers considered by application of the classical clamping conditions. Majority of the aforementioned simplifications were also applied in formulations of the eigenvalue problem of two-layer beams (cf [14,15]).

Due to the foregoing simplifications the theories are not useful for the vibration damping analysis of both multilayered beams/bands and two/three-layer beams/bands composed of the anisotropic stiffness-comparable layers. However investigation of such beams/bands is important due to the development of high-strength plastics - applicable for making of multilayered structures.

Boundary value problem for filamentary viscoelastic composite beam/band can be formulated in the same way as for the laminated composite plate (cf [17,20]). However in this paper an original work by Ni and Adams [16] on evaluating damping capacity of symmetrically laminated beam is discussed. Ni and Adams have solved the problem considered in three stages. At first a compliance matrix of the beam consisting of fiber–reinforced and stiffness–comparable layers is obtained. Elements of the matrix depend on the number of layers, the material and geometrical parameters of the layers and the arrangement of fibres within the beam. At the second stage the strain energy stored within homogeneous beam, characterized by the evaluated compliance matrix and undergone to the bending moments, is obtained. Let us note that relationships for the strains–moment used by the authors are the same as those predicted by the Saint–Venant linear elasticity solution for the pure static bending of isotropic homogeneous beam. At the third stage both the energy dissipation and the damping capacity of the structure is obtained. The parameters are calculated as the sums of three products. One of the coefficients in each product is so–called specific damping capacity evaluated experimentally. The final expression on damping capacity of the laminated beam is not dependent on bending moment.

It can easily be noticed that Ni and Adams's formulation of the problem is in fact not a dynamic one. By using this formulation the authors obtained a good agreement between theoretical and experimental results for slender beams i.e., 200 mm long, 12 mm wide and 1.6 mm thick. It seems to be rather doubtful to acquire accurate values of damping capacity by using the non–dynamic approach given by Ni and Adams in the case of non–slender laminated beam. It should be noted finally that the boundary conditions at the ends of the beam have not been considered in the foregoing paper.

In the present paper a new linear elastodynamics solution of the flexural, free vibration damping problem of layered band, consisting of any number of anisotropic, viscoelastic stripes, is reported. An idea of the solution is an application of the modified Levinson kinematic assumptions (cf [11]) and the linear elasticity equations of motion together with the constitutive equations of viscoelastic material for the analytic formulation of the eigenvalue problem for both a clamped and a simply supported band. It is assumed that within the band we have the plane strain. Due to the approach developed both the shearing and dilatational deformations and the resulting material damping of vibration in each layer are taken into account. New boundary conditions (supports) for clamped band have been introduced. Thus a new, original method for calculating both the eigenfrequencies and the logarithmic decrement of the layered bands is presented.

Formulation of the problem for the simply supported, anisotropic, viscoelastic, multilayered band is in fact a direct extension of the Levinson's work (cf [11]). However both the formulation and the solution of the eigenvalue problem for clamped band in the way described in section 3 of the present paper are essentially quite new. According to the best of the author's knowledge for the first time the eigenvalue problem has been formulated analytically and exactly within the linear theory of (visco)elasticity. The final form of the formulation depends on kind of supports. Thus we obtain one transcendental complex equation in the case of simply supported band and a set of three transcendental coupled equations in the case of clamped band. As a consequence of the approach reported one obtains a new type of the displacement boundary conditions (supports) for clamped bands. The conditions are different and in fact more realistic than those discussed by Valisetty et al. [19]. A comment on the foregoing conditions is given in section 3 of this paper.

Solutions to the algebraic problems derived have been obtained by means of computer calculations according to Fortran programmes written by author of this paper. The programmes have been prepared in double precision. Standard library routines for evaluating roots of transcendental, nonlinear, complex equations according to the Muller method with deflation have been used. Numerical problems resulting from the formulations described above have been new (in respect of the complexity and number of final algebraic equations) for both the simply supported (layered) and the clamped (homogeneous and layered) bands.

The problems considered in this paper have been formulated for the multilayered bands composed of fibre–reinforced, viscoelastic layers while the numerical results have been given only for bands consisting of the isotropic layers. The results however are sufficient for the main purpose assumed i.e., for the presentation of both the accuracy and usefulness of the approach developed. Thus one can find in this paper several values of the logarithmic decrement for both the simply supported (two–layer) and the clamped (homogeneous) bands. Additionally for both the three–layer and the homogeneous bands one can observe differences between eigenfrequency values obtained according to the presented method and those predicted by the classical theories based on the Kirchhoff assumption and some other simplifications.

2. Case of simply supported band

Contents of this section is a foundation for understanding of the next one so it is written widely and as clear as possible. In order to formulate the boundary value problem the following Levinson kinematic assumptions (cf [11]) are applied

$$\begin{aligned} u_{x_j} &= -g_j(z)\frac{dW(x)}{dx}\exp(\mathrm{i}\omega_m t) \\ u_{y_j} &= 0 \\ u_{z_j} &= f_j(z)W(x)\exp(\mathrm{i}\omega_m t) \end{aligned} \tag{2.1}$$

where $\mathrm{i}^2 = -1$, $j = 1,2,3,...$ denotes a number of the layer, variable z is the coordinate in direction of the band deflection, symbol t stands for time and ω_m for the eigenfrequency of mth mode of vibration. Functions $g_j(z)$, $f_j(z)$ are unknown however

$$W(x) = W_m \sin\Big(\frac{m\pi}{L}x\Big) \qquad m = 1,2,3,\dots \tag{2.2}$$

where L is the length of the band. Let us note that Levinson introduced his original kinematic assumptions in order to solve the eigenvalue problem of an isotropic simply supported plate thus in his paper the displacement u_y has not been put equal to zero. Applying the assumptions (2.1), (2.2) one obtains a plane strain. Besides the stresses $\big(\sigma_{yz}\big)_j \equiv \big(\sigma_{23}\big)_j$, $\big(\sigma_{xy}\big)_j \equiv \big(\sigma_{12}\big)_j$ are equal to zero.

It has been assumed for further considerations that the fibres of each layer are parallel to the longitudinal axis of the band. Such arrangement of the fibres is most desirable considering bending stiffness of the band (cf [16]). Let us assume additionally that the material properties of any layer are isotropic within each cross-section of the layer. If the two foregoing assumptions are fulfiled we will have so-called hexagonally anisotropic layer.

Constitutive equation of fibrous, hexagonally anisotropic, viscoelastic material in case of the plane strain can be written in the form

$$\boldsymbol{\sigma}_j = \mathbf{D}_j\boldsymbol{\varepsilon}_j \tag{2.3}$$

where $\boldsymbol{\sigma}_j$ denotes stress vector, $\boldsymbol{\varepsilon}_j$ is strain vector and $\mathbf{D}_j$ is stiffness matrix of jth layer. The matrices are defined as follows

$$\begin{aligned} \boldsymbol{\sigma}_j &\equiv [\sigma_{xx}, \sigma_{zz}, \sigma_{zx}] \\ \mathbf{D}_j &\equiv \begin{vmatrix} b_j & a_j & 0 \\ a_j & q_j0 & \\ 0 & 0 & 2\mu'_j \end{vmatrix} \\ \boldsymbol{\varepsilon}_j &\equiv [\varepsilon_{xx}, \varepsilon_{zz}, \varepsilon_{zx}] \end{aligned} \tag{2.4}$$

In expressions (2.4) symbol μ'_j denotes complex Kirhchoff modulus of jth fibrous viscoelastic layer in x–z plane however the quantities a_j, b_j, q_j depend on four remaining independent, complex, material parameters of a viscoelastic hexagonally anisotropic material [26].

By using the formulas (2.1) $\div$ (2.4) one can transform the linear elasticity equations of motion

$$\sigma_{kl,k} = \rho\frac{\partial^2 u_l}{\partial t^2} \tag{2.5}$$

to the following form

$$\begin{aligned} -\mu'_j\frac{d^2g_j}{dz^2} + (b_j\alpha_m^2 - \rho_j\omega_m^2)g_j + (a_j + \mu'_j)\frac{df_j}{dz} &= 0 \\ -q_j\frac{d^2f_j}{dz^2} - (\mu'_j\alpha_m^2 - \rho_j\omega_m^2)f_j + \alpha_m^2(a_j + \mu'_j)\frac{dg_j}{dz} &= 0 \end{aligned} \tag{2.6}$$

where

$$\alpha_m = \frac{m\pi}{L} \qquad \text{for} \qquad m = 1,2,3,... \tag{2.7}$$

For isotropic material the quantities a_j, b_j, q_j are defined as follows $a_j = \lambda_j$, $b_j \equiv q_j = \lambda_j + 2\mu_j$, $\mu'_j = \mu_j$ where λ_j, μ_j denote the complex, frequency–dependent Lame parameters. In this case the constitutive equation (2.3) can be written in the well known form

$$\left(\sigma_{kl}\right)_j = 2\mu_j\left(\varepsilon_{kl}\right)_j + \delta_{kl}\lambda_j\left(\varepsilon_{rr}\right)_j \qquad k,l,r = 1,3 \tag{2.8}$$

however equations of motion will be of the same form as those given by Levinson [11] when α_m is replaced with the coefficient $M\pi$ used by Levinson i.e.

$$\begin{aligned}
&-\mu_j\frac{d^2 g_j}{dz^2} + \Big[(\lambda_j + 2\mu_j)\alpha_m^2 - \rho_j\omega_m^2\Big]g_j + (\lambda_j + \mu_j)\frac{df_j}{dz} = 0\\
&(\lambda_j + 2\mu_j)\frac{d^2 f_j}{dz^2} - (\mu_j\alpha_m^2 - \rho_j\omega_m^2)f_j + \alpha_m^2(\lambda_j + \mu_j)\frac{dg_j}{dz} = 0
\end{aligned} \tag{2.9}$$

After solving the set of equations of motion one obtains functions $f_j(z)$, $g_j(z)$. A form of the functions in the case of elastic layer depends on quantitative relationships between the geometrical and the material parameters appearing Eqs (2.6) or (2.9). The aforementioned problem has been discussed in reference by Levinson for the elastic, isotropic plate thus it is not discussed here. In the case of a viscoelastic layer the functions $f_j(z)$, $g_j(z)$ are complex. Taking into account the correspondence principle one can write the functions for isotropic layer in the following form

$$\begin{aligned}
f_j(z) &= X_{1_j}\cosh(z\beta_{1j}) + X_{2_j}\sinh(z\beta_{1j}) + X_{3_j}\cosh(z\beta_{2_j}) + X_{4_j}\sinh(z\beta_{2_j})\\
g_j(z) &= X'_{1_j}\cosh(z\beta_{1j}) + X'_{2_j}\sinh(z\beta_{1j}) + X'_{3_j}\cosh(z\beta_{2_j}) + X'_{4_j}\sinh(z\beta_{2_j})
\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}
\beta_{1j}^2 &= \alpha_m^2 - \frac{\rho_j\omega_m^2}{\mu_j}\\
\beta_{2j}^2 &= \alpha_m^2 - \frac{\rho_j\omega_m^2}{\lambda_j + 2\mu_j}
\end{aligned} \tag{2.11}$$

The vector $\boldsymbol{X}'_j$ is dependent on vector $\boldsymbol{X}_j$ thus there are only five unknown values in Eqs (2.10) i.e., the vector $\boldsymbol{X}_j$ and natural frequency ω_m. For any layer denoted by $n \neq j$ subscript we have another unknown vector $\boldsymbol{X}_n$. Thus for the mth vibration mode of a band consisting of p layers one obtains $4p+1$ unknown parameters while one of them is the eigenfrequency ω_m.

After substitution of the functions (2.10) into the expressions (2.1) one obtains the displacement field within the jth layer. However by using the displacement field functions and constitutive equations one can derive the field stress. Taking into account the homogeneous stress boundary conditions on the band free surfaces (2.12) as well as the continuity conditions of stresses (2.13) and displacements (2.14) between adjoining layers

$$\begin{aligned}\Big(\sigma_{zz}(x,0)\Big)_1 &= \Big(\sigma_{zz}(x,h)\Big)_p = 0\\ \Big(\sigma_{zx}(x,0)\Big)_1 &= \Big(\sigma_{zx}(x,h)\Big)_p = 0\end{aligned} \tag{2.12}$$

$$\begin{aligned}\Big(\sigma_{zz}(x,h_j)\Big)_j - \Big(\sigma_{zz}(x,h_j)\Big)_{j+1} &= 0\\ \Big(\sigma_{zx}(x,h_j)\Big)_j - \Big(\sigma_{zx}(x,h_j)\Big)_{j+1} &= 0\end{aligned} \tag{2.13}$$

$$\begin{aligned}\Big(u_x(x,h_j)\Big)_j - \Big(u_x(x,h_j)\Big)_{j+1} &= 0\\ \Big(u_z(x_j,h_j)\Big)_j - \Big(u_z(x_j,h_j)\Big)_{j+1} &= 0\end{aligned} \tag{2.14}$$

where $h = \sum\limits_1^p h_j$ one can transform the eigenvalue problem to the form of algebraic, homogeneous, matrix equation

$$\mathbf{A}\cdot\mathbf{X} = \mathbf{0} \tag{2.15}$$

Symbol h_j denotes thickness of the jth layer, while h stands for the total thickness of the band. $\mathbf{A}$ is the $4p \times 4p$ square matrix where p denotes the number of band layers. ω_m is obtained from the equation

$$\det(\mathbf{A}) = 0 \tag{2.16}$$

If any layer of the band is a viscoelastic one then ω_m consists of real and imaginary part

$$\omega_m = \omega_{mR} + \mathrm{i}\omega_{mF} \qquad \mathrm{i}^2 = -1 \tag{2.17}$$

and a periodic logarithmic decrement is defined as follows

$$\delta_T = 2\pi\frac{\omega_{mF}}{\omega_{mR}} \tag{2.18}$$

After calculating δ_T one can obtain both the loss factor η and the damping capacity Ψ according to the formulas (cf [4])

$$\eta = \frac{\delta_T}{\pi} \qquad \Psi = 1 - \exp(-2\delta_T) \cong 2\delta_T \tag{2.19}$$

The formulation (2.15) can be easily obtained, for a band consisting of any number of stripes, by using of a computer. However computation of the eigenfrequencies is somewhat more difficult. Matrix elements depend on hyperbolic and trigonometric functions of eigenfrequency i.e.

$$A_{kl} = A_{kl}\Big(\sin(\omega_m,...), \cos(\omega_m,...), \sinh(\omega_m,...), \cosh(\omega_m,...),...\Big) \tag{2.20}$$

and because of this Eq (2.16) cannot be transformed to the following one

$$\det\Big(\mathbf{B}_1 - \omega_m^2\mathbf{B}_2\Big) = 0 \tag{2.21}$$

where $\mathbf{B}_1$, $\mathbf{B}_2$ are given matrices. Eq (2.21) can be solved by using the standard software modules.

To obtain solution of the eigenproblem (2.16) the following procedure has been proposed (cf [12,13,18]) – at first a function $F(\omega_m) \equiv \det(\mathbf{A})$ is derived, then an eigenfrequency of undamped (elastic) band is estimated, finally a complex eigenfrequency of a damped (viscoelastic) system is computed from the equation $F(\omega_m) = 0$. All steps of this procedure were realized by the author using of IBM personal computer. The third step only was carried out by using the standard subroutine for evaluating roots of an algebraic, nonlinear, complex equation according to the Muller method with deflation. The eigenfrequency of undamped system was useful as an approximative value of the complex solution (eigenfrequency) i.e., as one of the input parameters required by the standard subroutine. It was verified that Fortran code necessary for the calculations has to be prepared in double precision.

3. Case of clamped band – new elastodynamic solution

The formulation of the eigenvalue problem in this case has been derived by the author as a direct development (extension) of the one for the simply supported band. Therefore this section is more brief than the previous one. On the other hand the approach presented in this section is essentially quite new as the final form of the eigenvalue problem derived consists of three coupled transcendental equations.

Let us assume that the origin of the coordinate system lies in the cross-section, of the band, equally distant from the supports. For convenience the symmetric and the antisymmetric modes of vibration are considered separately. For both symmetric and antisymmetric vibration modes, it is assumed that the displacement field in the band is of the form

$$\begin{aligned} u_{x_j} &= -\Big[\Big(G_1(z)\Big)_j \frac{dW_1(x)}{dx} + \Big(G_2(z)\Big)_j \frac{dW_2(x)}{dx}\Big] \exp(\mathrm{i}\omega_m t) \\ u_{y_j} &= 0 \\ u_{z_j} &= \Big[\Big(F_1(z)\Big)_j W_1(x) + \Big(F_2(z)\Big)_j W_2(x)\Big] \exp(\mathrm{i}\omega_m t) \end{aligned} \tag{3.1}$$

where $i^2 = -1$, subscript j denotes a number of the layer and functions $\Big(G_1(z)\Big)_j$, $\Big(G_2(z)\Big)_j$, $\Big(F_1(z)\Big)_j$, $\Big(F_2(z)\Big)_j$, are unknown. It is assumed additionally that in the case of symmetric modes of vibration

$$W_1(x) = \cos(\beta x) \qquad\qquad W_2(x) = \cosh(\gamma x) \tag{3.2}$$

where β, γ are unknown constants. As the displacements u_x, u_z consist of two parts thus formulae for strains and stresses are also composed of two terms – one dependent on βx and the other dependent on γx.

By using the expressions (3.1), (3.2), constitutive equations of viscoelastic material and functions $\Big(G_1(z)\Big)_j$, $\Big(G_2(z)\Big)_j$, $\Big(F_1(z)\Big)_j$, $\Big(F_2(z)\Big)_j$ (resulting from equations of motion) one can transform the conditions (2.12) $\div$ (2.14) to the form of two complex matrix equations, analogous to Eq (2.15)

$$\mathbf{A}_1\mathbf{X}_1 = \mathbf{0} \qquad\qquad \mathbf{A}_2\mathbf{X}_2 = \mathbf{0} \tag{3.3}$$

where both square matrices $\mathbf{A}_1$, $\mathbf{A}_2$, as in the case of simply supported band, are of order $4p$ (p denotes the number of layers). Let us assume that by using the left–hand side equation of the set (3.2) one obtains, by applying the procedure described in section 2, the left–hand side equation in the set (3.3). The same rule refers to the right–hand side equations of the two sets. Thus the elements of the matrix $\mathbf{A}_1$ depend on β and ω_m while the elements of the matrix $\mathbf{A}_2$ depend on γ and ω_m. For nontrivial solutions to the matrix equations one can write

$$\begin{aligned} \det(\mathbf{A}_1) &\equiv F_1(\beta,\omega_m) = 0 \\ \det(\mathbf{A}_2) &\equiv F_2(\gamma,\omega_m) = 0 \end{aligned} \tag{3.4}$$

In the case of clamped technical supports shown in Fig.1 the following conditions have to be fulfiled

$$\begin{aligned} u_z\Big(x = -\frac{L}{2}, z = z^*, t\Big) &= u_z\Big(x = \frac{L}{2}, z = z^*, t\Big) = 0 \\ u_{z,x}\Big(x = -\frac{L}{2}, z = z^*, t\Big) &= u_{z,x}\Big(x = \frac{L}{2}, z = z^*, t\Big) = 0 \end{aligned} \tag{3.5}$$

where z^* is a coordinate of any plane $\Phi(x,y)$ within the band. For instance $\Phi(x,y)$ can be an outside surface of a band as in Fig.1b. The technical supports shown in Fig.1 are of great practical importance. However it is possible to find out some other supports modeled by the conditions (3.5).

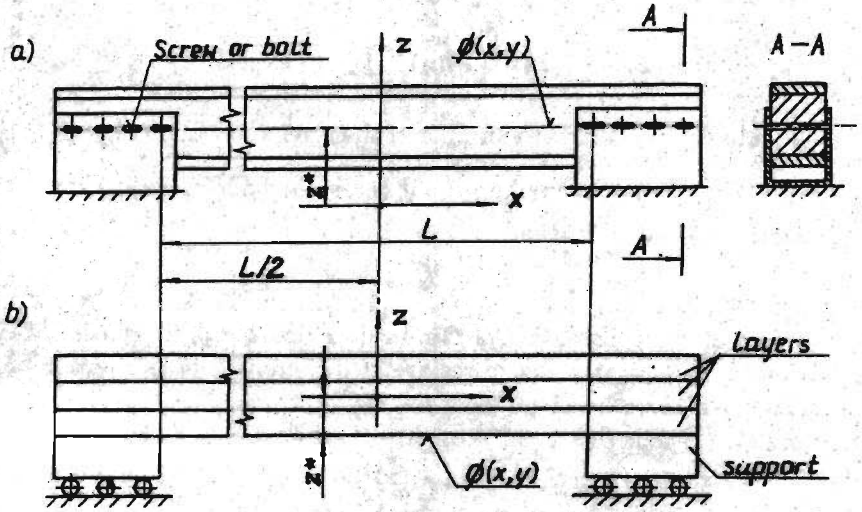

Fig. 1. Clamped technical boundary conditions introduced in the paper: a) the band is bolted, b) the band is stuck to the supports

It can be verified easily that the conditions (3.5) are equivalent to the following equation

$$\beta \tan\Big(\beta\frac{L}{2}\Big) = -\gamma \tanh\Big(\gamma\frac{L}{2}\Big) \tag{3.6}$$

Eqs (3.4), (3.6) are nonlinear, transcendental and coupled. After solving the set one can obtain three parameters i.e., constants β, γ and eigenfrequency ω_m, respectively. To calculate the eigenvectors $\mathbf{X}_1, \mathbf{X}_2$ both the equations (3.4) and one of the equations (3.5) have to be used. One of the eigenvectors is fully dependent on the other. Eqs (3.5) establish a relationship between eigenvectors $\mathbf{X}_1$, $\mathbf{X}_2$.

A difference between formulations of the eigenvalue problem for symmetric and antisymmetric modes of vibration results from the different forms of functions W_1, W_2 in each case. In the case of antisymmetric vibration modes one ought to assume W_1, W_2 in the form

$$W_1(x) = \sin(\beta x) \qquad\qquad W_2(x) = \sinh(\gamma x) \tag{3.7}$$

and then the equation corresponding to Eq (3.6) is of the form

$$\beta \cot\Big(\beta\frac{L}{2}\Big) = \gamma \coth\Big(\gamma\frac{L}{2}\Big) \tag{3.8}$$

As far as the author knows, this formulation of the eigenvalue problem for clamped band is essentially quite new. The Eqs (3.1), (3.5), (3.2)/(3.7), introduced by the author indicate the formulation novelty. Due to applying the foregoing equations the problem considered has been formulated for the first time exactly within the linear theory of (visco)elasticity. Thus both the transverse shearing deformations and complete material damping characteristics of each layer have been taken into account.

As a consequence of the approach reported one obtains a new type of displacement boundary conditions (described by Eqs (3.5)) for clamped bands. The conditions are different and in fact more realistic than those discussed by Valisetty et al. [19] and can be especially useful for investigation of laminated bands as in this case a unique definition of clamping is not available (cf [19]). One of the main features of the conditions can be expressed as follows: values of an eigenvector of the eigenvalue problem (thus values of displacements of the clamped band) depend on the way of fastening the band to the supports however values of eigenfrequencies do not depend on the coupling way. Two identical bands, one of them supported in the way shown in Fig.1a and the other supported in the way shown in Fig.1b, vibrate after simultaneous identical short press on the same eigenfrequency however displaying different displacements in corresponding cross-sections.

The approach presented here has also been numerically new, within the area of clamped (both layered and homogeneous) bands, in respect of both the complexity and the number of final algebraic equations.

Solution to the algebraic problem consisting of three coupled, complex equations (3.4), (3.6) has been obtained on the IBM personal computer using a Fortran programme written by the author. The programme has been prepared in double precision. Standard library routines for evaluating roots of algebraic, nonlinear, complex equations according to Muller method with deflation have been used. To solve the problem considered one has to evaluate the approximative values of three unknown parameters ω_m, β, γ and then to use them as the input data required by the standard routine used for solving the set of algebraic equations. To obtain the approximative values for a homogeneous band presented in section 4 the following procedure has been applied: at first eigenfrequency ω_k of the clamped band has been evaluated according to the characteristic equation of the classical theory, then (by using ω_k) the parameter β_k has been calculated from the first equation of the set (3.4), then (by using β_k) the parameter γ_k has been obtained from Eq (3.6).

4. Numerical results and discussion

The main purpose of the numerical results presentation is to show the usefulness and accuracy of the solution developed in the present paper. Mathematical correctness is shown in sections 2, 3 while below it is additionally confirmed by comparing the eigenfrequency values (for three-layer and homogeneous bands) and the damping parameters (for layered and homogeneous bands) calculated according to the method developed with those obtained according to the theories (cf [5,10,14,15]) based on Kirchhoff's assumption. The analysis is limited to the 1-st and the 3-rd vibration mode of homogeneous band and the 1-st mode of vibration of layered bands consisting of isotropic layers.

In order to show the significance of the problem formulation for the quality (accuracy) of results and to present more carefully the method proposed in this paper several values of eigenfrequency of the first vibration mode for simply supported bands/beams consisting of three isotropic layers are given in Table 1. For each value of the length L the band and the beam have the same corresponding material and geometrical parameters. Symbol ω_{A1} denotes eigenfrequency of three-layer band. The parameter is calculated by using the method proposed in this paper. ω_{MZ} is the eigenfrequency of three-layer beam. This parameter is calculated according to the following formula, published by Ziemiański [10] and obtainable from [5],

$$\omega_m^2 = \alpha_m^2 \frac{K_S}{M} \frac{1 + \frac{K_Z}{K_M} + \alpha_m^2 \frac{K_Z}{K_S}}{1 + \frac{K_S}{K_M \alpha_m^2}} \tag{4.1}$$

where

$$K_S = \frac{G_2(a_1 + a_3)^2}{h_2} \qquad K_Z = \frac{E_1 h_1^3 + E_3 h_3^3}{12}$$
$$K_M = E_1 h_1 a_1^2 + E_3 h_3 a_3^2 \qquad M = \rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3 \tag{4.2}$$

α_m is defined by Eq (2.7), while E_j, ρ_j, h_j denote the Young modulus, the mass density and the thickness of the jth layer, respectively G_2 is the Kirchhoff modulus of the middle layer (core), symbols a_1 ,a_3 stand for the distances between the centre of gravity of the cross section of the core and analogous points on the faces. We note that under two assumptions, first $h_1 = h_3$ and second $E_1 = E_3$ one can derive the formula (4.1) directly from the equation of motion given by Mead and Markus [5] where notations used by the authors i.e., m, D_t, g, Y can be expressed as follows

$$m = M \qquad D_t = K_Z \qquad g = \frac{K_S}{K_M} \qquad Y = \frac{K_M}{K_Z} \tag{4.3}$$

One should remember that we have the plane strain within a band and the plane stress within a beam. Inspite of the difference occuring in formulations of the eigenvalue problems (for band and beam) one obtains, for the same values of material constants and geometrical parameters of corresponding layers of the structures, the identical values of damping parameters for the structures. The conclusion is drawn from the works by Oberst [14] and Baumgarten & Pearce [15]. On the other hand an eigenfrequency of a band will be higher than a corresponding eigenfrequency of a beam when both the material and geometical parameters of the corresponding layers of the structures are the same. Because of this both ω_{A1} and ω_{MZ}, given in Table 1, have been calculated for two types of layered bands/beams. In the case (**a**) we have the structures composed of stiffness–comparable layers however in the case (**b**) we have typically sandwich bands and beams.

Table 1. Comparison of the 1–st mode eigenfrequency values for three–layer simply supported band/beam: ω_{A1} – obtained by using author method and ω_{MZ} – calculated according to the formula (4.1) published in [10] and obtainable from [5]

case (**a**)			case (**b**)		
thickness of facings $h_1 = h_3 = 2.5$ [mm]		thickness of core $h_2 = 120$ [mm]	thickness of facings $h_1 = h_3 = 3.9$ [mm]		thickness of core $h_2 = 3.2$ [mm]
Results			Results		
L [mm]	ω_{A1} [rad/s]	ω_{MZ} [rad/s]	L [mm]	ω_{A1} [rad/s]	ω_{MZ} [rad/s]
500	4664.2	2908.0	410	354.40	344.1
750	2201.2	1315.0	460	284.40	276.3
1000	1267.5	744.6	510	233.90	227.4
1500	573.4	332.4	560	197.00	191.0
2000	324.6	187.3	610	167.69	163.1
Material parameters of layers: $E_1 = E_3 = 68.9 \cdot 10^3$ [mPa] $E_2 = 16 \cdot 10^3$ [mPa] $\nu_1 = \nu_3 = 0.276$ $\rho_1 = \rho_3 = 2680$ [kg/m^3] $\rho_2 = 1750$ [kg/m^3], $\nu_2 = 0.3$			Material parameters of layers: $E_1 = E_3 = 20.7 \cdot 10^4$ [mPa] $E_2 = 2$ [mPa] $\nu_1 = \nu_3 = 0.25$ $\rho_1 = \rho_3 = 7860$ [kg/m^3] $\rho_2 = 1180$ [kg/m^3], $\nu_2 = 0.4$		

As can be observed in the case (**a**) the values ω_{A1} are almost two times higher than the values of ω_{MZ} however in the case (**b**) the eigenfreguencies ω_{MZ} are close to ω_{A1}. The inequality $\omega_{A1} \gg \omega_{MZ}$ in the case (**a**) appears due to neglecting in the eigenvalue problem formulation given by Ziemiański [10] both the longitudinal stresses and the strains within the middle layer (core) of the beam. Thus the values of ω_{MZ} in the case (**a**) are false.

On the ground of the results given in Table 1 one can come (among the others) to the following conclusions: – if the neighbouring layers of the band are stiffness–comparable then dilatational deformations of the core and the material damping resulting from the dilatation should not be omitted. – the solution to the eigenvalue problem for a layered band, developed here, can be applied to the analysis of vibration damping of both classical sandwich bands and layered bands consisting of stiffeness–comparable layers.

In Table 2 we have shown several values of the loss factor η for two–layer simply supported band. η_{II} has been calculated according to the method developed in this paper however η_I has been read of charts given elsewhere (cf [14,15]). The factor ε_1 in the Table 2 is defined as follows

$$\varepsilon_1 = \frac{(\eta_{II} - \eta_I)100}{\eta_I} \tag{4.4}$$

As stated by Oberst [14] and Baumgarten & Pearce [15] the loss factor η of a two–layer structure depends on three non–dimensional parameters i.e.,

$$\eta \equiv \frac{\delta_T}{\pi} = \eta\Big(\frac{h_2}{h_1}, \frac{E_{21}}{E_{11}}, \frac{E_{22}}{E_{21}}\Big) \tag{4.5}$$

where h_2, h_1 denote the thickness of viscoelastic layer and elastic stripe, respectively, however E_{j1}, E_{j2} are real and imaginary parts of the complex Young modulus of jth layer. For the elastic stripe (when subscript $j = 1$) we have $E_{12} = 0$.

We notice that the formula (4.5) has been derived by applying the well known Kirchhoff hypothesis of flat cross–sections. Thus the transverse shear deformations have been neglected [15]. However neglecting the shear effects in formulation of the eigenvalue problem of layered beam or band is incorrect. The conclusion has been confirmed by Oberst during the experimental investigation [14]. Eight experimental results, for dimensionless ratio $h_2/h_1 \equiv d_2/d_1$ varying within the range from 1 to 3, presented by Oberst in Fig.10 (cf [14]) have been higher than those calculated according to the theory based on the Kirchhoff assumption. It seems to be unquestionable that the fact revealed experimentally has been confirmed in Table 2. We notice that within the range from 1 to 6 the values of η_{II} are considerably higher than the values of η_I. It means that the approach given here is more exact than the theory presented by Oberst and Baumgarten & Pearce [14,15].

Table 2. The loss factors values of the 1–st vibration mode for two–layer simply supported band: η_I given in papers [14,15] and η_{II} – calculated, for $\nu_{22} = 0$ and $\nu_{21} = 0.475$, according to the method presented in section 2. Dimensionless parameters $E_{21}/E_{11} = 0.1$, $E_{22}/E_{21} = 1$

No.	1	2	3	4	5	6	7	8
h_2/h_1	0.2	0.4	0.6	0.8	1.0	2.0	4.0	6.0
η_I	0.07	0.15	0.24	0.32	0.40	0.52	0.52	0.50
η_{II}	0.0897	0.2021	0.3122	0.4027	0.4689	0.5779	0.5521	0.5208
ε_1	28.2	34.7	30.1	25.8	17.2	11.1	6.2	4.1

Let us note finally that the approach presented here enables us to include into computations the complete material characteristics (i.e., stiffness matrices) of each viscoelastic layer (cf [12,13,18]). For instance in the case of isotropic layers we can take as input data the following complex parameters

$$E_j = E_{1j}(1 + i\eta_{Ej}) \qquad\qquad \nu_j = \nu_{1j}(1 + i\eta_{nj}) \tag{4.6}$$

where $\eta_{Ej} = E_{2j}/E_{1j}$, $\eta_{nj} = \nu_{2j}/\nu_{1j}$ while E_j denotes the complex Young modulus, ν_j is the Poisson ratio and the subscript j is the number of the layer.

To verify the essentially new formulation of the eigenvalue problem for clamped band, several eigenfrequencies were calculated and compared with those predicted by the classical theory based on the Kirchhoff assumption. The comparison has been limited to the first and third vibration modes of homogeneous band.

Table 3. Values of flexural eigenfrequency ω_k obtained according to Eq (4.7) and parameters β, γ, $\omega_{A2} \equiv \omega_m$ fulfiling the Eqs (3.4), (3.6) for the 1-st vibration mode of homogeneous (steel) clamped bands of thickness $h = 20$ [mm]. Material parameters: $E = 0.2 \cdot 10^{12}$ [Pa], $\nu = 0.25$, $\eta_E = 0$, $\eta_n = 0$

No.	L [mm]	$\beta\, L/2$ [rad]	$\gamma\, L/2$ [rad]	ε_2 [%]	ω_k [rad/s]	ω_{A2} [rad/s]	ε_3 [%]	ε_4 [%]
1	340	2.37196	2.34126	1.31	5821.4	5780.2	0.713	1.8396
2	300	2.37392	2.33461	1.68	7477.3	7409.7	0.912	1.8457
3	260	2.37684	2.32473	2.24	9955.0	9836.2	1.208	1.8559
4	220	2.38146	2.30918	3.13	13904	13675	1.675	1.8692
5	180	2.38943	2.28269	4.68	20770	20271	2.462	1.8996
6	140	2.40488	2.23222	7.74	34335	33023	3.973	1.9469

The numerical results relating to clamped band and obtained here are given in Tables 3 and 4, respectively. In columns 3,4,7 of Table 3 one can see the parameters β, γ (multiplied by $L/2$) and the eigenfrequencies $\omega_m \equiv \omega_{A2}$ satisfying Eqs (3.4) and (3.6). ω_k denotes the eigenfrequency obtained after solving the following characteristic equation

$$\tan\Big(k\frac{L}{2}\Big) + \tanh\Big(k\frac{L}{2}\Big) = 0 \equiv \cos(kL)\cosh(kL) - 1 = 0 \tag{4.7}$$

where L is the length of the band and k is defined as follows

$$k^2 = \begin{cases} \omega_k \dfrac{2}{h}\sqrt{\dfrac{3\rho(1-\nu^2)}{E}} & \text{-- for band} \\ \omega_k \dfrac{2}{h}\sqrt{\dfrac{3\rho}{E}} & \text{-- for beam} \end{cases} \tag{4.8}$$

while h, ρ, ν, E denote the thickness, the mass density, the Poisson ratio and the Young modulus of the homogeneous band/beam, respectively. Eq (4.7) is derived within the so-called classical theory which is based on the Kirchhoff assumption of flat cross-sections. The parameters ε_2, ε_3, ε_4 given in Table 3 are defined as follows

$$\varepsilon_2 = \frac{(\beta-\gamma)100}{\gamma} \qquad \varepsilon_3 = \frac{(\omega_{A2}-\omega_k)100}{\omega_k} \qquad \varepsilon_4 = \frac{\varepsilon_2}{\varepsilon_3} \tag{4.9}$$

On the basis of the values of ω_k, ω_{A2} and ε_3 one can see that the eigenfrequencies obtained in the present analysis (given in section 3) are lower than those calculated from characteristic equation (4.7) of the classical theory. As the classical theory overestimates the eigenfrequency values, in particular for non-slender bands/beams [22], thus it has been shown that the new approach presented is much more accurate than that based on the Kirchhoff assumption. It is interesting to notice that the percentage difference ε_2 between parameters β, γ varies with the length L of the band in the same way as the difference ε_3 between the eigenfrequencies ω_k and ω_{A2}. As can be seen the factor ε_4 is almost constant. Thus for slender bands/beams the relationship $\beta \cong \gamma$ is valid and Eq (3.6) has the same form as the left-hand side of the expression (4.7).

Table 4. Logarithmic decrement values of the 1-st vibration mode for homogeneous (steel) clamped bands of thickness $h = 20$ [mm]: δ_{TA} – obtained according to author's method described in sections 2,3 and δ_{Tk} – calculated according to equation (4.7). Material parameters $E = 0.2\cdot10^{12}$ [Pa], $\nu = 0.25$, $\eta_E = 0.02$, $\eta_n = 0.1$

No.	1	2	3	4	5	6
L [mm]	340	300	260	220	180	140
δ_{Tk}	0.1047	0.1047	0.1047	0.1047	0.1047	0.1047
δ_{TA}	0.1029	0.1025	0.1018	0.1007	0.0988	0.0955

In Table 4 there are given a few values of the logarithmic decrement for homogeneous (steel) clamped bands of thickness $h = 20$ [mm] and different lengths L. Symbols η_E and η_n are defined in expressions (4.6), δ_{Tk} is the logarithmic decrement calculated from Eqs (4.7), (2.18) while δ_{TA} denotes the value of the same parameter obtained after solving Eqs (3.4), (3.6). One should remember that both η_E and η_n are not equal to zero. As can be expected values of the logarithmic decrement δ_{Tk} obtained according to the classical theory do not vary with the length of the band. However the values of δ_{TA} calculated from Eqs (3.4), (3.6) are slightly dependent on the geometrical parameter. Besides one has the relationship $\delta_{Tk} > \delta_{TA}$. Also it is noticed that as the slenderness ratio of the band is lower so difference $\Delta = \delta_{Tk} - \delta_{TA}$ is higher. For $L = 140$ [mm] the value of the decrement evaluated according to the new method is about 10% lower than that predicted by the classical theory. Taking into account that the parameters ω_k, δ_{Tk} have been calculated according to Eq (4.7), however the parameters ω_A,

δ_{TA} calculated from Eqs (3.4), (3.6) one concludes, from the results given both in Tables 3,4 and by Huang [22], that the new method predicts (for homogeneous isotropic clamped band) more accurate and lower values of the logarithmic decrement than the classical theory. This is due to the dependence of the Poisson ratio on frequency included in the formulation given in section 3.

Table 5. Values of the flexural eigenfrequency ω_k obtained according to Eq (4.7) and parameters β, γ, $\omega_{A2} \equiv \omega_m$ fulfiling Eqs (3.4), (3.6) for the 3–rd vibration mode of homogeneous (steel) clamped bands of thickness $h = 20$ [mm]. Material parameters: $E = 0.2 \cdot 10^{12}$ [Pa], $\nu = 0.25$, $\eta_E = 0$, $\eta_n = 0$

No.	L	$\beta L/2$	$\gamma L/2$	ε_2	ω_k	ω_{A2}	ε_3	ε_4
	[mm]	[rad]	[rad]	[%]	[rad/s]	[rad/s]	[%]	[%]
1	340	5.53143	5.17155	6.96	31458.3	29824.7	5.193	1.3410
2	300	5.54042	5.08749	8.90	40406.5	37786.7	6.484	1.3735
3	260	5.55345	4.96762	11.79	53795.6	49336.0	8.290	1.4222
4	220	5.57337	4.78928	16.37	75136.0	66943.4	10.900	1.5018
5	180	5.60590	4.50925	24.32	112240	95596.4	14.828	1.6399
6	140	5.66440	4.03597	40.35	185540	146656	20.957	1.9251

Table 5 contains the eigenfrequencies and the parameters β, γ (multiplied by $L/2$) for the 3–rd mode of vibration of the homogeneous clamped bands. One can notice that dependence of each of the factors ε_2, ε_3, ε_4 on the length L of the band is similar to that one observed in the case of 1–st mode of vibration. However the factors given in Table 5 are generally much greater than those presented in Table 3. The fact, as regards the factor ε_3, is very well consistent with Huang's predictions [22]. By looking at the results given in Table 6 one can conclude that the logarithmic decrement calculated for $\eta_E \neq 0$ and $\eta_n = 0$ is not dependent on L. We stress that the numerical results were obtained for some exemplary values of complex parameters of the constitutive equation (2.8).

Table 6. Logarithmic decrement values of the 3–rd vibration mode for homogeneous (steel) clamped bands of thickness $h = 20$ [mm]: δ_{TA} – obtained according to author's method described in sections 2,3 and δ_{Tk} – calculated according to equation (4.7). Material parameters $E = 0.2 \cdot 10^{12}$ [Pa], $\nu = 0.25$, $\eta_E = 0.02$, $\eta_n = 0.0$

No.	1	2	3	4	5	6
L [mm]	340	300	260	220	180	140
δ_{Tk}	0.0628	0.0628	0.0628	0.0628	0.0628	0.0628
δ_{TA}	0.0628	0.0628	0.0628	0.0628	0.0628	0.0628

5. Concluding remarks

New linear elastodynamics solution for the free vibration problem of layered, viscoelastic bands, both simply and technically supported, has been developed. Both the shearing and the longitudinal deformations as well as the complete material characteristics of each isotropic layer have been taken into account in formulation of the problem. New type of the displacement boundary conditions at the ends of the band has been introduced. The approach can easily be extended in order to calculate the parameters (stresses, displacements and damping capacity) of any layered band under a sinusoidally varying with time load.

Due to the perfect linear elasticity formulation of the problem considered the solution proposed can be also applied in the analysis of the free vibrations of classical sandwich bands. Numerical results for such bands obtained by means of this method closely approximate those calculated according to the other theories (cf [5,10]).

Both the eigenfrequencies and the logarithmic decrement values of the homogeneous technically supported band obtained according to the new solution presented in sections 2,3 are lower (thus more accurate) than those predicted by the classical theory. As the slenderness ratio of the band is higher, the better is agreement between predictions of the present solution and the classical theory.

References

1. YU Y.Y., 1959, *A new theory of elastic sandwich plates – one dimensional case*, ASME J.Appl.Mech., September, 415-421
2. RAVILLE M.E., UENG E-S., LEI M-M., 1961, *Natural frequencies of vibration of fixed-fixed sandwich beams*, ASME J.Appl.Mech., September, 367-371
3. DITARANTO R.A., 1965, *Theory of vibratory bending for elastic and viscoelastic layered finite-length beams*, ASME J.Appl.Mech., December, 881-886
4. BERT C.W., WILKINS D.J. JR., CRISMAN W.C., 1967, *Damping in sandwich beams with shear-flexible cores*, ASME J.Appl.Mech., November, 662-670
5. MEAD D.J., MARKUS S., 1970, *Loss factors and resonant frequencies of encastre damped sandwich beams*, J.Sound Vib., 12, 99-112
6. YAN M.J., DOWELL E.H., 1972, *Governing equations for vibrating constrained-layer damping sandwich plates and beams*, ASME J.Appl.Mech., December, 1041-1046
7. AHMED K.M., 1972, *Dynamic analysis of sandwich beams*, J.Sound Vib., 21, 263-276
8. LEONE S.G., PERLMAN A.B., *A numerical study of damping in viscoelastic sandwich beams*, ASME publication 73-DET-73 1-8

9. RAO D.K., 1976, *Transverse vibrations of pre-twisted sandwich beams*, J.Sound Vib., 44, 159-168

10. ZIEMIAŃSKI L., 1984, *Analysis of eigenfrequencies spectrum of three-layered beams*, Proc.of VII Symp.on Vib.Techn.and Vibroacoustic, Cracow 1984, (in Polish)

11. LEVINSON M., 1985, *Free vibrations of a simply supported, rectangular plate. an exact elasticity solution*, J.Sound Vib., 98, 289-298

12. KARCZMARZYK S., 1989, *Analysis of vibration damping by internal friction in sandwich beams*, PhD dissertation, Warsaw University of Technology (in Polish)

13. KARCZMARZYK S., OSIŃSKI Z., 1990, *A new approach to vibration damping analysis of simply supported sandwich beams*, Proceedings of the 28-th Polish Solid Mechanics Conference, Kozubnik 1990, 112-115

14. OBERST H., 1952, *Über die dampfung der biegeschwingungen dunner bleche durch fest haftende belage*, Acustica 4, 181-194

15. BAUMGARTEN J.R., PEARCE B.K., 1971, *The damping effects of viscoelastic materials, part 1/2 - transverse vibrations of beams/plates with viscoelastic coatings*, Journal of Engineering for Industry, May, 645-655

16. NI R.G., ADAMS R.D., 1984, *The damping and dynamic moduli of symmetric laminated composite beams - theoretical and experimental results*, Journal of Composite Materials, 18, 104-21

17. ALAM N., ASNANI N.T., 1986, *Vibration and damping analysis of fibre reinforced composite material plates*, Journal of Composite Materials, 20, 2-18

18. KARCZMARZYK S., 1991, *A new approach to flexural free vibration damping analysis of a two-layer viscoelastic plate*, Machine Dynamics Problems, 6, 5-16

19. VALISETTY R.R., MURTHY P.L.N., REHFIELD L.W., 1988, *An ideal clamping analysis for a cross-ply laminate*, Journal of Composite Materials, 22, 136-153

20. XIAO C.Z., LIN D.X., JU F., 1988, *Finite element analysis on modal parameters of anisotropic laminated plates*, Journal of Vibration, Acoustics, Stress, and Reliability in Design, 110, 473-477

21. CHANG F.K., PEREZ J.L., CHANG K.Y., 1990, *Analysis of thick laminated composites*, Journal of Composite Materials, 24, 801-821

22. HUANG T.C., 1961, *The effect of rotatory inertia and of shear deformation on the frequency and normal mode equations of uniform beams with simple end conditions*, Journal of Applied Mechanics, December, 579-584

23. SUN C.T., SIJIAN LI., 1989, *Three-dimensinal effective elastic constants for thick laminates*, J.of Comp.Mat., 22

24. LEE C.Y., PFEIFER M., THOMPSON B.S., GANDHI V., 1989, *The characterization of elastic moduli and damping capacities of graphite/epoxy composite laminated beams in hygrothermal enviroments*, Journal of Composite Materials, 23, 819-845

25. SRINIVAS S., JOGA RAO C.V., RAO A.K., 1970, *An exact analysis for vibration of simply-supported homogeneous and laminated thick rectangular plates*, J.Sound Vib., 12(2), 187-199

26. AMBARTJUMIAN C.A., 1987, *Theory of anisotropic plates*, Nauka Moscow (in Russian)

Nowe liniowe, elastodynamiczne rozwiązanie brzegowego problemu własnego drgań giętnych lepkosprężystego warstwowego lub jednorodnego pasma

Streszczenie

W pracy otrzymano nowe sformułowanie i rozwiązanie brzegowego problemu własnego dla lepkosprężystego pasma, zarówno obustronnie zamocowanego jak i swobodnie podpartego, składającego się z dowolnej liczby wysokowytrzymałych, włóknistych warstw o porównywalnej sztywności. Wprowadzono nowe warunki brzegowe (zamocowania). Na podstawie wyników obliczeniowych, zarówno dla pasma swobodnie podpartego (dwu- i trójwarstwowego) jak i obustronnie zamocowanego (jednorodnego), otrzymanych po rozwiązaniu niekonwencjonalnego zadania numerycznego wynikającego ze sformułowania zagadnienia, pokazano dokładność i użyteczność otrzymanego rozwiązania.

Manuscript received January 8, 1992; accepted for print February 21, 1992