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PLASTICITY OF A ROTATING HYPERBOLIC DISK

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The problem of elastoplastic disks of hyperbolic shape under centrifugal forces has been considered in the present paper. The angular velocities delimiting the initial yielding and complete plasticity material state, respectively are determined due to the Mises and Tresca criteria. The procedure including work hardening is also presented.

1. Introducing

Assuming plane stress $(\sigma_z = 0)$ in a disk of hyperbolic shape with thickness

$$t = t_0\Big(\frac{a}{r}\Big)^n \qquad a \le r \le b \qquad n > 0 \tag{1.1}$$

the equilibrium equation under centrifugal load is

$$\sigma_\phi = r\frac{d\sigma_r}{dr} + (1-n)\sigma_r + \rho\omega^2 r^2 \tag{1.2}$$

For the elastic part using Hookean law a second order differential equation is obtained [1]

$$\frac{d^2\sigma_r}{dr^2} + \frac{3-n}{r}\frac{d\sigma_r}{dr} - \frac{n(1+\nu)}{r^2}\sigma_r + (3+\nu)\rho\omega^2 = 0 \tag{1.3}$$

the solution to which is

$$\sigma_r = Ar^p + Br^q - \frac{3+\nu}{(2-q)(2-p)}\rho\omega^2 r^2 \tag{1.4}$$

where

$$p,q = \frac{n}{2} - 1 \pm \sqrt{\Big(\frac{n}{2}\Big)^2 + \nu n + 1} \tag{1.5}$$

while the circumferential (hoop) stress component is due to equation (1.2)

$$\sigma_\phi = A(1+p-n)r^p + B(1+q-n)r^q - \rho\omega^2 r^2\Big[\frac{(3-n)(3+\nu)}{(2-q)(2-p)} - 1\Big] \tag{1.6}$$

Applying the boundary conditions of a hollow ring, free of any surface constraints

$$\sigma_r(r=a) = 0 \qquad \sigma_r(r=b) = 0 \tag{1.7}$$

the integration constants become

$$A = \frac{(3+\nu)\rho\omega^2}{(2-q)(2-p)}\frac{a^2b^q - b^2a^q}{a^pb^q - b^pa^q} \tag{1.8}$$

$$B = \frac{(3+\nu)\rho\omega^2}{(2-q)(2-p)}\frac{a^2b^p - b^2a^p}{a^qb^p - b^qa^p} \tag{1.9}$$

so that the stress distributions in radial and hoop directions are given by

$$\sigma_r = \frac{(3+\nu)\rho\omega^2}{(2-q)(2-p)}\Big[\frac{(a^2b^q - b^2a^q)r^p - (a^2b^p - b^2a^p)r^q}{a^pb^q - b^pa^q} - r^2\Big] \tag{1.10}$$

$$\sigma_\phi = \frac{(3+\nu)\rho\omega^2}{(2-q)(2-p)} \cdot \tag{1.11}$$

$$\cdot\Big[\frac{(a^2b^q - b^2a^q)(1+p-n)r^p - (a^2b^p - b^2a^p)(1+q-n)r^q}{a^pb^q - b^pa^q} - \frac{1+3\nu}{3+\nu}r^2\Big]$$

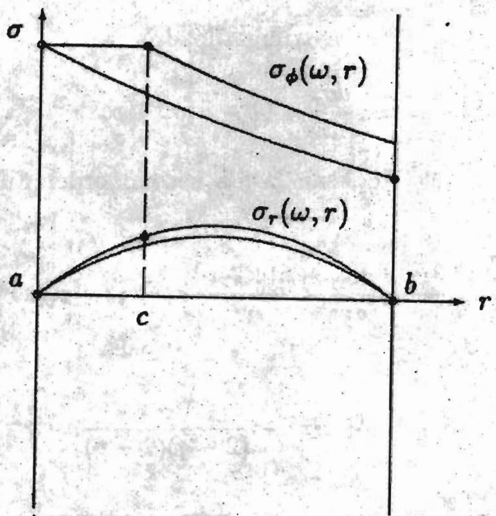

Fig. 1. Centrifugal stresses

The hoop stress has its greatest value occuring at $r = a$ (Fig.1). Yield will start at $\sigma_\phi = 2k$. Hence the angular velocity at the initial yielding is given by

$$\frac{\rho\omega_e^2}{2k} = \frac{(2-q)(2-p)}{(3+\nu)\left[\frac{(a^2b^q-b^2a^q)(1+p-n)a^p-(a^2b^p-b^2a^p)(1+q-n)a^q}{a^pb^q-b^pa^q} - \frac{1+3\nu}{3+\nu}a^2\right]} \tag{1.12}$$

If $n = 0$, known value is obtained [2]

$$\frac{\rho\omega_e^2}{8k} = \frac{1}{(3+\nu)b^2 + (1-\nu)a^2} \tag{1.13}$$

2. Tresca yield with ideal plasticity

The yield criterion due to Tresca (if $\sigma_\phi\sigma_r < 0$) is $(k = \sigma_Y/2)$

$$|\sigma_\phi - \sigma_r| = 2k \tag{2.1}$$

while in the case considered (since $\sigma_\phi\sigma_r > 0$)

$$|\sigma_\phi| = 2k \tag{2.2}$$

In conjunction with the equilibrium equation (1.2) a first order differential equation is obtained

$$r\frac{d\sigma_r}{dr} + (1-n)\sigma_r = 2k - \rho\omega^2r^2 \tag{2.3}$$

the solution of which reads as follows

$$\sigma_r = Br^{n-1} - \frac{2k}{n-1} - \frac{\rho\omega^2r^2}{3-n} \tag{2.4}$$

Due to boundary conditions $\sigma_r(r=a) = 0$ in the inner plastic zone $(a \le r \le c \le b)$

$$\sigma_r = \frac{2k}{n-1}\left[\left(\frac{r}{a}\right)^{n-1} - 1\right] + \frac{\rho\omega^2r^2}{3-n}\left[\left(\frac{a}{r}\right)^{3-n} - 1\right] \tag{2.5}$$

while in the outer elastic zone $(a \le c \le r \le b)$ stresses are given by equations (1.4) and (1.6). In view of the boundary condition $\sigma_r(r=b) = 0$ these expressions may be rewritten in a more convenient form

$$\sigma_r = B(r^q - r^pb^{q-p}) + \frac{(3+\nu)\rho\omega^2b^2}{(2-q)(2-p)}\left[\left(\frac{r}{b}\right)^p - \left(\frac{r}{b}\right)^2\right] \tag{2.6}$$

$$\begin{aligned}\sigma_\phi &= B\left[(1+q-n)r^q - (1+p-n)r^pb^{q-p}\right] + \\ &+ \rho\omega^2b^2\left\{\frac{(3+\nu)(1+p-n)}{(2-q)(2-p)}\left(\frac{r}{b}\right)^p - \left(\frac{r}{b}\right)^2\left[\frac{(3-\nu)(3+\nu)}{(2-q)(2-p)} - 1\right]\right\}\end{aligned} \tag{2.7}$$

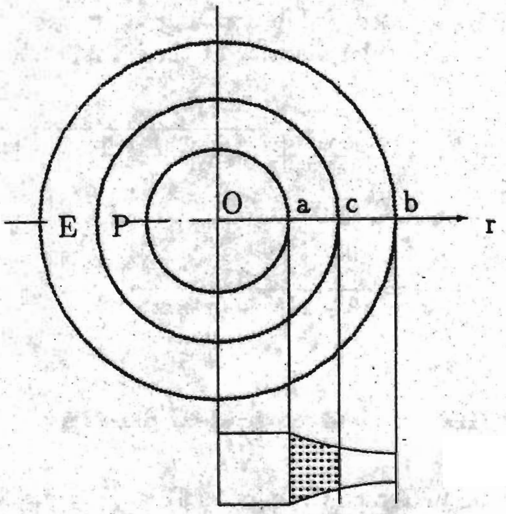

Fig. 2. Disk plastification

For a given position (Fig.2) of the elastic/plastic interface, B and ω can be determined from the continuity of stresses accross $r = c$ assumption. From equations (2.4) and (2.6) & (2.2) and (2.7), respectively, values obtained are

$$B = \frac{\frac{2k}{n-1}\left[\left(\frac{c}{a}\right)^{n-1} - 1\right] + \rho\omega^2\left\{\frac{a^2}{3-n}\left[\left(\frac{c}{a}\right)^{n-1} - \left(\frac{c}{a}\right)^2\right] - \frac{(3+\nu)b^2}{(2-q)(2-p)}\left[\left(\frac{c}{b}\right)^p - \left(\frac{c}{b}\right)^2\right]\right\}}{c^q - c^p b^{q-p}} \tag{2.8}$$

and

$$B = \frac{2k - \rho\omega^2\left\{\frac{(3+\nu)(1+p-n)}{(2-q)(2-p)} b^{2-p} c^p - c^2\left[\frac{(3-\nu)(3+\nu)}{(2-q)(2-p)} - 1\right]\right\}}{(1+q-n)c^q - (1+p-n)c^p b^{q-p}} \tag{2.9}$$

Hence the angular velocity is

$$\frac{\rho\omega^2 b^2}{2k} = \frac{c^q - c^p b^{q-p} - \frac{1}{n-1}\left[\left(\frac{c}{a}\right)^{n-1} - 1\right]\left[(1+q-n)c^q - (1+p-n)c^p b^{q-p}\right]}{M\left[(1+q-n)c^q - (1+p-n)c^p b^{q-p}\right] + N(c^q - c^p b^{q-p})} \tag{2.10}$$

where

$$M = \frac{1}{3-n}\left[\left(\frac{c}{a}\right)^{n-1} - \left(\frac{c}{a}\right)^2\right]\left(\frac{a}{b}\right)^2 - \frac{3+\nu}{(2-q)(2-p)}\left[\left(\frac{c}{b}\right)^p - \left(\frac{c}{b}\right)^2\right] \tag{2.11}$$

and

$$N = \frac{(3+\nu)(1+p-n)}{(2-q)(2-p)}\left(\frac{c}{b}\right)^p - \left(\frac{c}{b}\right)^2\left[\frac{(3-\nu)(3+\nu)}{(2-q)(2-p)} - 1\right] \tag{2.12}$$

For complete plasticity $(c = b)$ angular speed becomes

$$\frac{\rho\omega_0^2 b^2}{2k} = \frac{3-n}{1-n}\frac{\left(\frac{b}{a}\right)^{n-1} - 1}{\left(\frac{b}{a}\right)^{n-3} - 1} \tag{2.13}$$

If $n = 0$, know value is obtained [2]

$$\frac{\rho\omega_0^2 b^2}{6k} = \frac{1}{1 + \frac{a}{b} + \left(\frac{a}{b}\right)^2} \tag{2.14}$$

Considering data $(n = 1/2,\ b/a = 3,\ \nu = 0.3)$ the fully plastic speed of the disk is found to be 1.284 times as large as that at the initial yielding. With $n = 1/4$ it is 1.304, while for $n = 0$ this factor is 1.324 as given by [2]. Disk endurance increases with n due to more adequate shape with respect to centrifugal forces (centroid of the meridional cross-section closer to the rotation axis).

3. Mises yield with ideal plasticity

The yield criterion due to Mises $(k = \sigma_Y/\sqrt{3})$ is given by

$$\sigma_r^2 - \sigma_r\sigma_\phi + \sigma_\phi^2 = 3k^2 \tag{3.1}$$

which can be solved for hoop stress

$$\sigma_\phi = \frac{1}{2}\sigma_r + \frac{\sqrt{3}}{2}\sqrt{4k^2 - \sigma_r^2} \tag{3.2}$$

When this expression is substituted into equilibrium equation (1.2), a first order differential equation is obtained

$$r\frac{d\sigma_r}{dr} = \left(n - \frac{1}{2}\right)\sigma_r + \frac{\sqrt{3}}{2}\sqrt{4k^2 - \sigma_r^2} - \rho\omega^2 r^2 \tag{3.3}$$

which may be solved numerically in the inner plastic zone $(a \le r \le c \le b)$, while in the outher elastic zone $(a \le c \le r \le b)$ stresses are given by equations (2.6) and (2.7). For a position on the elastic/plastic interface, B and ω can be determined from the continuity of stresses accross $r = c$ assumption, i.e.

$$B = \frac{\sigma_r(c) - \rho\omega^2 b^2 \frac{3+\nu}{(2-q)(2-p)}\left[\left(\frac{c}{b}\right)^p - \left(\frac{c}{b}\right)^2\right]}{c^q - c^p b^{q-p}} \tag{3.4}$$

and

$$B = \frac{\sigma_\phi(c) - \rho\omega^2 b^2 \left\{ \frac{(3+\nu)(1+p-n)}{(2-q)(2-p)} \left(\frac{c}{b}\right)^p - \left(\frac{c}{b}\right)^2 \left[\frac{(3-n)(3+\nu)}{(2-q)(2-p)} - 1 \right] \right\}}{(1+q-n)c^q - (1+p-n)c^p b^{q-p}} \tag{3.5}$$

Hence the angular velocity is given by

$$\rho\omega^2 b^2 = \frac{\sigma_r(c)\left[(1+q-n)c^q - (1+p-n)c^p b^{q-p}\right] - \sigma_\phi(c)(c^q - c^p b^{q-p})}{M\left(\frac{c}{b}\right)^p + N\left(\frac{c}{b}\right)^2} \tag{3.6}$$

where

$$M = \frac{(3+\nu)(q-p)}{(2-q)(2-p)} c^q \tag{3.7}$$

and

$$N = \frac{1+p+\nu}{2-p} c^q - \frac{1+q+\nu}{2-q} c^p b^{q-p} \tag{3.8}$$

The solution to equation (3.6) is possible by numerical means only. For the complete plasticity $(c = b)$ with data $(b/a = 3,\ \nu = 0.33)$ the ratio of fully plastic speed versus that at the initial yielding was found to be 1.337 $(n = 1/2)$, 1.357 $(n = 1/4)$ and 1.377 $(n = 0)$, respectively.

4. Tresca yield with work hardening

Assuming plane stress $(\sigma_z = 0)$ and $\varepsilon_r^p + \varepsilon_\phi^p = 0$, the following two equations may be written

$$\frac{du}{dr} = \varepsilon_\phi^p + \frac{d\varepsilon_\phi^p}{dr} + \frac{1}{E}(\sigma_Y - \nu\sigma_r) + \frac{r}{E}\left(\frac{d\sigma_Y}{d\varepsilon_\phi^p}\frac{d\varepsilon_\phi^p}{dr} - \nu\frac{d\sigma_r}{dr}\right) \tag{4.1}$$

$$\frac{du}{dr} = \frac{1}{E}(\sigma_r - \nu\sigma_Y)\varepsilon_\phi^p \tag{4.2}$$

and with the flow rule [3] (Fig.3)

$$H = \frac{d\sigma_Y}{d\varepsilon_\phi^p} \tag{4.3}$$

bearing in mind equation (2.3), the following differential equation for hoop plastic strain in the rotating disk of hyperbolic shape is obtained

$$2\varepsilon_\phi^p + r\left(1 + \frac{H}{E}\right)\frac{d\varepsilon_\phi^p}{dr} + \frac{1}{E}\left[\sigma_Y - (1+n\nu)\sigma_r + \nu\rho\omega^2 r^2\right] = 0 \tag{4.4}$$

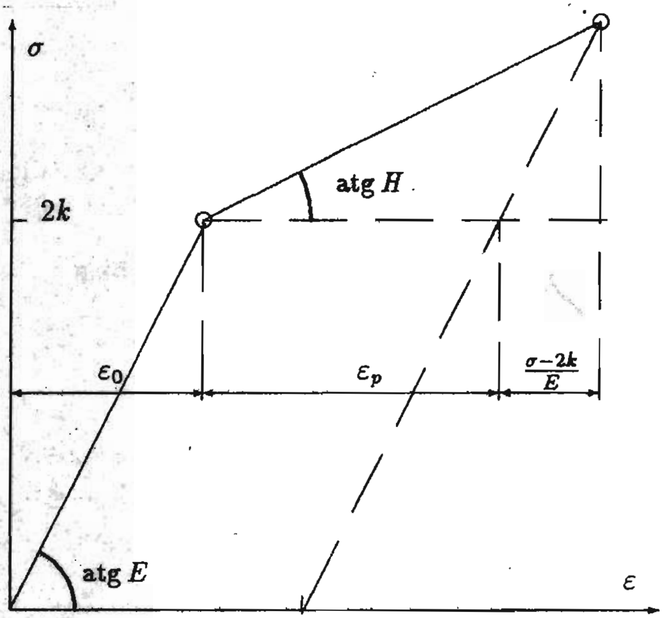

Fig. 3. Linear hardening

which subject to equation (2.5) has the solution

$$\begin{aligned}\varepsilon_\phi^p &= \frac{C}{r^{\frac{2}{1+H/E}}} - \frac{1}{E}\Big\{\frac{\sigma_Y}{n-1}\Big[\frac{n(1+\nu)}{2} - \frac{1+n\nu}{n+1+(n-1)\frac{H}{E}}\Big(\frac{r}{a}\Big)^{n-1}\Big] + \\ &+ \frac{\rho\omega^2a^2}{3-n}\Big[\frac{1+3\nu}{2\big(2+\frac{H}{E}\big)}\Big(\frac{r}{a}\Big)^2 - \frac{1+n\nu}{n+1+(n-1)\frac{H}{E}}\Big(\frac{r}{a}\Big)^{n-1}\Big]\Big\}\end{aligned} \tag{4.5}$$

Integration constant C may be determined subject to the boundary condition $\varepsilon_\phi^p(r=c)=0$, while $\sigma_Y(\varepsilon_\phi^p=0)=2k$, rendering

$$\begin{aligned}\varepsilon_\phi^p &= \frac{1}{E}\Big(\Big\{\frac{2k}{n-1}\Big[\frac{n(1+\nu)}{2} - \frac{1+n\nu}{n+1+(n-1)\frac{H}{E}}\Big(\frac{c}{a}\Big)^{n-1}\Big] + \\ &+ \frac{\rho\omega^2a^2}{3-n}\Big[\frac{1+3\nu}{2\big(2+\frac{H}{E}\big)}\Big(\frac{c}{a}\Big)^2 - \frac{1+n\nu}{n+1+(n-1)\frac{H}{E}}\Big(\frac{c}{a}\Big)^{n-1}\Big]\Big\}\Big(\frac{c}{a}\Big)^{\frac{2}{1+H/E}} - \\ &- \frac{\sigma_Y}{n-1}\Big[\frac{n(1+\nu)}{2} - \frac{1+n\nu}{n+1+(n-1)\frac{H}{E}}\Big(\frac{r}{a}\Big)^{n-1}\Big] - \\ &- \frac{\rho\omega^2a^2}{3-n}\Big[\frac{1+3\nu}{2\big(2+\frac{H}{E}\big)}\Big(\frac{r}{a}\Big)^2 - \frac{1+n\nu}{n+1+(n-1)\frac{H}{E}}\Big(\frac{r}{a}\Big)^{n-1}\Big]\Big)\end{aligned} \tag{4.6}$$

From the equilibrium (1.2) and Tresca criterion $(\sigma_\phi \sigma_r > 0)$

$$r\frac{d\sigma_r}{dr} = \sigma_Y + (n-1)\sigma_r - \rho\omega^2 r^2 \tag{4.7}$$

equation (4.4) gives the differential equation (eliminating r as a variable in favour of ε^p_ϕ)

$$\frac{d\sigma_r}{d\varepsilon^p_\phi} + \frac{\left[\sigma_Y + (n-1)\sigma_r - \rho\omega^2 r^2\right](E+H)}{\sigma_Y - (1+n\nu)\sigma_r + \nu\rho\omega^2 r^2 + 2E\varepsilon^p_\phi} = 0 \tag{4.8}$$

which may be solved formally $[\sigma_r(r=c) = -p_c]$ as

$$\sigma_r + p_c + \int_0^{\varepsilon^p_\phi} \frac{\left[\sigma_Y + (n-1)\sigma_r - \rho\omega^2 r^2\right](E+H)}{\sigma_Y - (1+n\nu)\sigma_r + \nu\rho\omega^2 r^2 + 2E\varepsilon^p_\phi} d\varepsilon^p_\phi = 0 \tag{4.9}$$

Due to $\sigma_r(r=a) = 0$ and $\varepsilon^p_\phi(r=a) = \varepsilon^p_0$ the following equation is obtained

$$\sigma_r + \int_{\varepsilon^p_0}^{\varepsilon^p_\phi} \frac{\left[\sigma_Y + (n-1)\sigma_r - \rho\omega^2 r^2\right](E+H)}{\sigma_Y - (1+n\nu)\sigma_r + \nu\rho\omega^2 r^2 + 2E\varepsilon^p_\phi} d\varepsilon^p_\phi = 0 \tag{4.10}$$

to be solved numerically by iteration.

With $n = 0$ and $\omega = 0$ the procedure reduces to [3].

The Mises yield case with work hardening can not be given in explicit manner since it requires a double numerical procedure.

Additional results on the subject are published elsewhere (cf [4,5]).

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Plastyczność wirującej tarczy o zmiennej grubości

Streszczenie

W przedstawionej pracy rozpatruje się zagadnienie sprężysto–plastycznych cienkich tarcz o zmiennej grubości poddanych siłom odśrodkowym. Na podstawie kryteriów Treski i Misesa określono prędkości obrotowe wyznaczające początek i koniec płynięcia materiału. Oprócz warunków idealnej sztywności podano procedurę uwzględniającą wzmocnienie materiału.

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