MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
3, 30, 1992

AN ANALYSIS OF CRACKS INFLUENCE ON THE EIGENFREQUENCIES OF THE TORSIONAL VIBRATIONS OF SHAFTS

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In this work, an analysis of the influence of the transverse cracks on the eigenfrequencies of the torsional vibrations of shaft is presented. A procedure of calculation of stiffness coefficients in cross-section weakned by the cracks has been given. A method of construction of the equations of motion for the system has been given. The equation have been derived from the general formula for the torsional vibrations of shafts taking into account the boundary conditions. The method has been illustrated with the help of examples; the first three eigenfrequencies of the torsional vibrations of a shaft with one and two cracks, respectively, were calculated.

1. Introduction

There are plenty of causes of the formation of cracks in shafts. One can have cracks caused by fatigue, which are formed in shafts running within the range of a limited fatigue strength. They may also occur due to mechanical damage. The other group of cracks form those inside the material. They are caused by he technological processing of the material, which is used to make the shaft.

Cracks create a risk factor for good operation of shafts. Most of breakdowns in modern machinery is due to the fatigue of the material. Thus, methods which enable early detection and localization of cracks are currently under investigation in many scientific centers. In the case of cracks which form first on the outer surfaces one can identify them by optical observations. For the case of internal cracks one can use X–rays or ultrasonic methods. In both cases one has to switch off the whole machinery, what in economic terms quite amounts to big losses.

A cracks in the shaft causes local changes in stiffness [1]. These changes, in turn, affect the dynamics of the system. Both natural frequencies and the amplitudes of forced vibrations are changed [2 ÷ 13]. Analysis of such changes enables one to identify the cracks during operation, withaut any need to switch off the machinery [14 ÷ 17].

In the present work we shall present an analysis of the influence of n cracks on the eigenfrequencies of the torsional vibrations of the shaft.

2. Calculation of the effective stiffness at the location of the crack

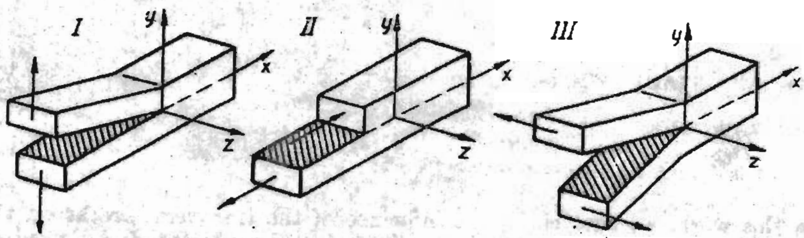

Fig. 1. Crack–tip deformation. (I) opening mode, (II) sliding mode, (III) tearing mode

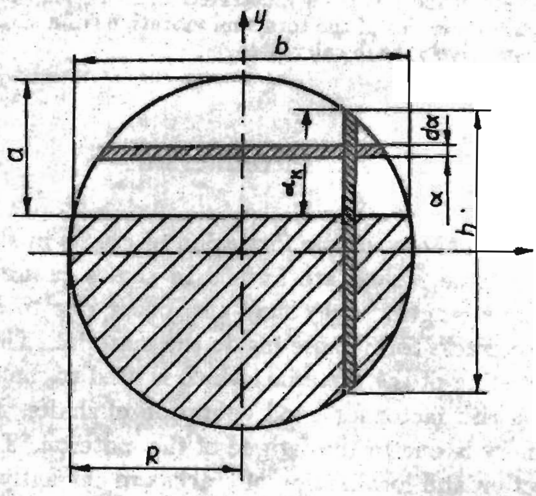

Fig. 2. Nomenclature for cracked cross–section of a shaft

We assume, that a crack was formed according to the second and third model of the development of the crack (Fig.1). The dimensions of a transverse edge crack are presented in Fig.2. The local flexibility of the crack is given by the relation (cf [10])

$$c_{ij} = \frac{\partial^2}{\partial P_i \partial P_j} \int_{-b}^{b} \int_{0}^{a} J(\alpha) d\alpha dx \tag{2.1}$$

where

$J(\alpha)$ – strain energy density function (SEDF),

P_i – corresponding force at the cross-section.

In a general form

$$J(\alpha) = \frac{1}{E_p}\Big[\Big(\sum_{i=1}^{6} K_{Ii}\Big)^2 + \Big(\sum_{i=1}^{6} K_{IIi}\Big)^2 + m\Big(\sum_{i=1}^{6} K_{IIIi}\Big)^2\Big] \tag{2.2}$$

where

$E_p = E$ for plane state of strain,

$E_p = \frac{E}{1-\nu^2}$ for plane state of strain,

E – Young modulus,

$m = 1 + \nu$,

ν – Poisson ratio,

K_{ij} – stress intensity factor (SIF), $i = I, II, III$, $j = 1, ..., 6$.

In the case under consideration we assume $i = II, III$ and $j = 6$, what is equivalent to the load of the shaft being caused only by the torsion moment. SIF are calculated with the use of the relations [10]

$$K_{II6} = \sigma_{6II}\sqrt{\Pi\alpha}\, F_{II}\Big(\frac{\alpha}{h}\Big) \tag{2.3}$$

and

$$K_{III6} = \sigma_{6III}\sqrt{\Pi\alpha}\, F_{III}\Big(\frac{\alpha}{h}\Big) \tag{2.4}$$

where

$$\sigma_{6II} = \frac{2P_6 x}{\Pi R^2} \tag{2.5}$$

$$\sigma_{6III} = 2P_6\frac{\sqrt{R^2 - x^2}}{\Pi R^4} \tag{2.6}$$

and also

$$F_{II}\Big(\frac{\alpha}{h}\Big) = \frac{1.122 - 0.561\big(\frac{\alpha}{h}\big) + 0.085\big(\frac{\alpha}{h}\big)^2 + 0.18\big(\frac{\alpha}{h}\big)^3}{\sqrt{1 - \big(\frac{\alpha}{h}\big)}} \tag{2.7}$$

$$F_{III}\Big(\frac{\alpha}{h}\Big) = \sqrt{\frac{\tan\lambda}{\lambda}} \tag{2.8}$$

with that: $\lambda = \Pi\alpha/(2h)$, $h = 2\sqrt{R^2 - x^2}$.

Denoting $\bar{x} = x/R$, $\bar{\alpha} = \alpha/R$, $\bar{h} = \alpha/h$ and assuming a plane state of the deformation, for the case of torsion the stiffness coefficient (2.1) is given by relation

$$c_{66} = \frac{1-\nu^2}{E}\frac{16}{\Pi R^3}\int_0^{\bar{b}}\int_0^{\bar{\alpha}_k}\Big[\bar{\alpha}\bar{x}^2F_{II}^2(\bar{h}) + m\bar{\alpha}(1-\bar{x}^2)F_{III}^2(\bar{h})\Big]d\bar{\alpha}d\bar{x} \tag{2.9}$$

Quite often, the stiffness coefficient is presented in the dimensionless form

$$\bar{c}_{66} = \frac{\Pi E R^3}{1-\nu^2}c_{66} \tag{2.10}$$

then

$$\bar{c}_{66} = 16\int_0^{\bar{b}}\int_0^{\bar{\alpha}_k}\Big[\bar{\alpha}\bar{x}^2F_{II}^2(\bar{h}) + m\bar{\alpha}(1-\bar{x}^2)F_{III}^2(\bar{h})\Big]d\bar{\alpha}d\bar{x} \tag{2.11}$$

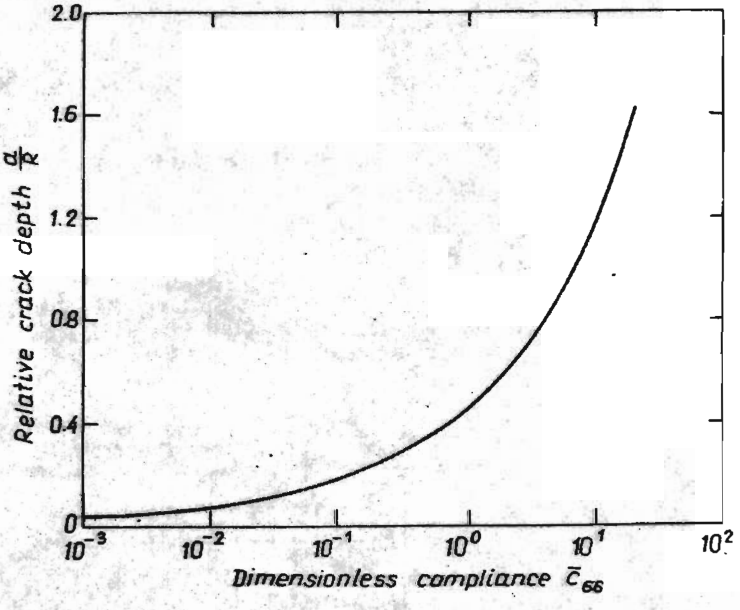

Fig. 3. Dimensionless form of the stiffness coefficient $\bar{c}_{66}$

The values of the coefficient have been calculated by the authors with the use of the standard procedure for numerical integration. The subroutine QSF of the IBM library of programs has been employed. The results are given in Fig.3.

3. Calculation of the natural frequencies of the torsional vibrations of the shaft with cracks

We assume the equation of the free vibrations of the shaft in the form

$$\frac{\partial^2 \Phi}{\partial t^2} - \frac{G}{\rho}\frac{\partial^2 \Phi}{\partial x^2} = 0 \tag{3.1}$$

where

G – Kirchhoff modulus,

ρ – density of the material.

Having the dimensionless quantity $\xi = x/L$ (where L is the length of the shaft), and the solution to the Eq (3.1) in the form of

$$\Phi(x,t) = \Phi(x)e^{i\Omega t} \tag{3.2}$$

we obtain

$$\frac{\partial^2 \Phi(\xi)}{\partial \xi^2} + \omega_0^2 \Phi(\xi) = 0 \tag{3.3}$$

where: $\omega_0^2 = \rho L^2 \Omega^2 / G$.

Introducing, in the places where the n cracks are located, elastic elements we get a system of $n+1$ segments of the shaft. The stiffness of the above mentioned elastic elements is calculated as the inverse of the flexibility c_{66}.

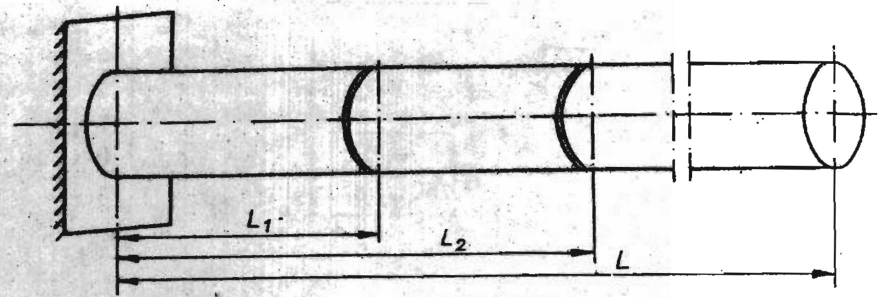

Fig. 4. The model of the shaft with n cracks

The model of the system is given in Fig.4. For every segment of the shaft the solution to Eq (3.3) is assumed to be of the form

$$\Phi_i(\xi) = A_i \sin(\omega_0 \xi) + B_i \cos(\omega_0 \xi) \qquad i = 1,...,n+1 \tag{3.4}$$

The following boundary conditions are assumed

1) angle of rotation in the place of fixing equals zero

$$\Phi_1(0) = 0 \tag{3.5}$$

2) torsion moments within cross-sections of the shaft have the same magnitude ($i = 1, ..., n$)

$$\Phi_i'(\beta_i) = \Phi_{i+1}'(\beta_i) \tag{3.6}$$

where: $\beta_i = L_i/L$

3) there is a discontinuous jump of the torsion angle at the location of a crack ($i = 1, ..., n$)

$$-\Phi_i(\beta_i) + \Phi_{i+1}(\beta_i) = k_i G J_0 \Phi_{i+1}'(\beta_i) \tag{3.7}$$

where: $k_i = 1/c_{i66}$

4) torsion moment vanishes over the end section

$$\Phi_{i+1}'(1) = 0 \tag{3.8}$$

Inserting Eq (3.4) into Eq (3.3), if one takes into account the boundary conditions (3.5) ÷ (3.8), one gets a set of equations in the form of

$$\begin{bmatrix} \boxed{0\ 1\ 0\ 0} & & & & & & \\ \boxed{\mathcal{S}_1} & & & & & & \\ & \boxed{\mathcal{S}_2} & & & & & \\ & & \boxed{\mathcal{S}_3} & & & & \\ & & & \ddots & & & \\ & & & & \boxed{\mathcal{S}_i} & & \\ & & & & & \ddots & \\ & & & & & & \boxed{\mathcal{S}_n} \\ & & & & & & \boxed{0\ 0\ S_p\ S_r} \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ \vdots \\ \vdots \\ \vdots \\ A_{n+1} \\ B_{n+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{3.9}$$

where: $S_p = \cos\omega_0$, $S_r = -\sin\omega_0$

$$\mathcal{S}_i = \begin{array}{|c|c|c|c|} \hline \cos\omega_0\beta_i & -\sin\omega_0\beta_i & -\cos\omega_0\beta_i & \sin\omega_0\beta_i \\ \hline -\sin\omega_0\beta_i & -\cos\omega_0\beta_i & \begin{array}{c}\sin\omega_0\beta_i - \\ -k_i G J_0 \omega_0 \cos\omega_0\beta_i\end{array} & \begin{array}{c}\cos\omega_0\beta_i + \\ +k_i G J_0 \omega_0 \sin\omega_0\beta_i\end{array} \\ \hline \end{array}$$

The determinant of the square matrix $\mathbf{S}$ of Eq (3.9) has characteristic roots ω_{0i}, which we can calculate by solving the characteristic equation

$$\det \mathbf{S} = 0 \tag{3.10}$$

Knowing the roots ω_{0i}, we calculate the eigenfrequencies of the torsional vibrations

$$\Omega_i = \frac{\omega_{0i}}{L}\sqrt{\frac{G}{\rho}} \qquad i = 1,...,2(n+1) \tag{3.11}$$

4. Example of calculations

In order to perform numerical calculations the authors have written a computer program named SHAFT, which can be used for calculations of the natural frequencies of the torsional vibrations of a shaft with constant transverse cross-section and with n cracks. The program has been written in Fortran 77, and has been performed by an IBM PC/AT microcomputer.

For the purpose of simplicity, without a loss of generality, assume that there is a simple shaft with two cracks located as shown in Fig.4. The following data are assumed: length $L = 1$ m; diameter 0.2 m; Young modulus $E = 2.1 \cdot 10^5$ MPa; Poisson ratio $\nu = 0.3$. The calculations were performed by using various quotients a/R and L_i/L $(i = 1,2)$, where a, R, L_i and L are described in Fig.2,4.

The results of calculations are presented in Fig.5 ÷ 10. Fig.5 shows the first characteristic root ω_{01} of the equation (3.10) as the function of the location and size of one crack. Fig.6 illustrates the effect of changing the first characteristic root as a function of crack depth a/R. In this case the shaft has two cracks – the size and location of the first crack, are described by $a/R = 0.3$ and $L_1/L = 0.1$, respectively. The size and location of the second crack are presented in the Fig.6. Fig.7 presents similar functions as those presented in Fig.6 – the size and location of the first crack are described by $a/R = 0.7$ and $L_1/L = 0.1$, respectively. Figures 8 ÷ 10 show the first three characteristic roots as the function of the size and location of the single crack. The plots depend on the mode of vibrations.

5. Conclusions

This paper essentially describes the model and an analysis of influence of the location and the size of one or two cracks on the natural frequencies of the torsional vibrations of a shaft fixed at one of its end. The numerical results are obtained by using the computer program SHAFT. The subroutine SIMUL [18] has been used in the iteration procedure designed for solving the characteristic Eq (3.10). The proposed iterative procedure is rapidly convergent. It is assumed that the difference between determinant and zero must be smaller than some small number ϵ established beforehand. If the above condition has not been fulfilled, iteration

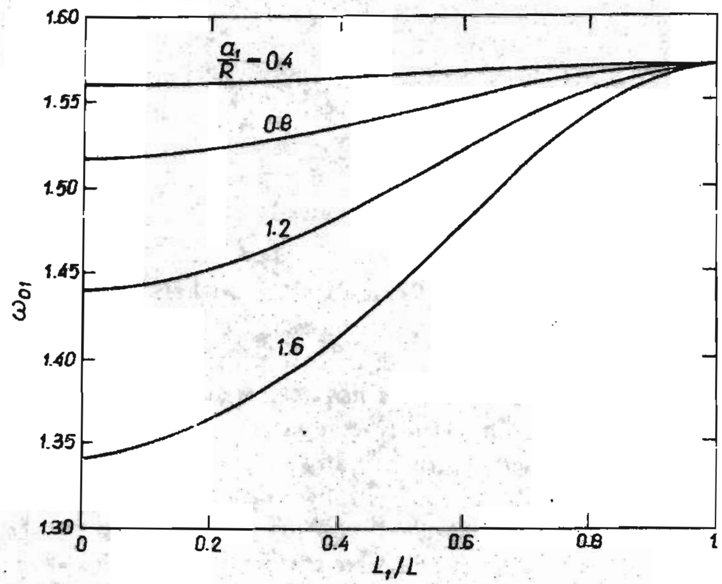

Fig. 5. First characteristic root ω_{01} as the function of the location L_1/L and the size a_1/R of one crack

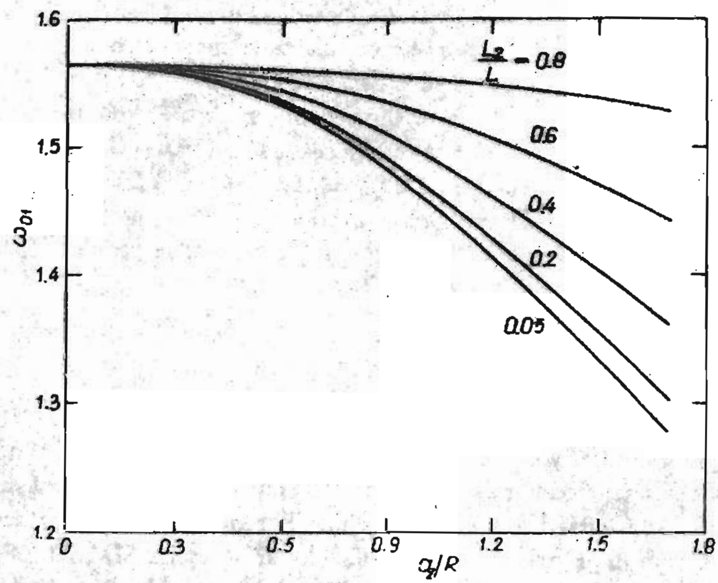

Fig. 6. First characteristic root ω_{01} as the function of the location L_2/L and the size a_2/R of the second crack, (the first crack parameters $L_1/L = 0.1$, $a_1/R = 0.3$)

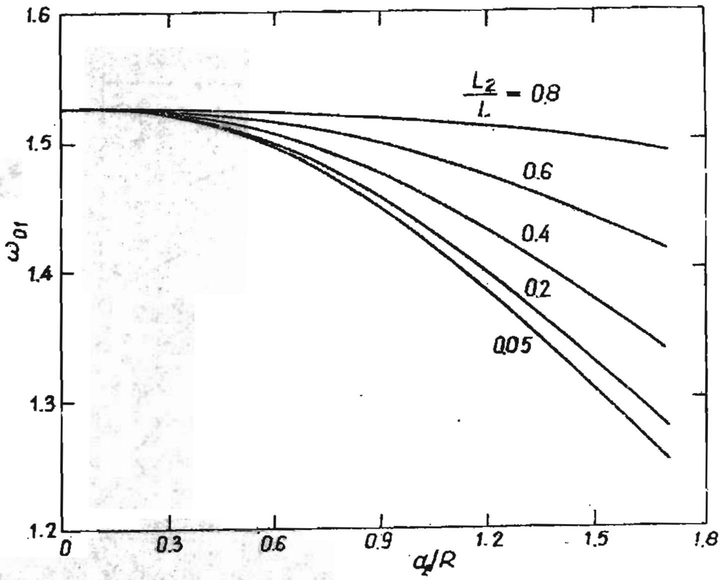

Fig. 7. First characteristic root ω_{01} as the function of the location L_2/L and the size a_2/R of the second crack, (the first crack parameters $L_1/L = 0.1$, $a_1/R = 0.7$)

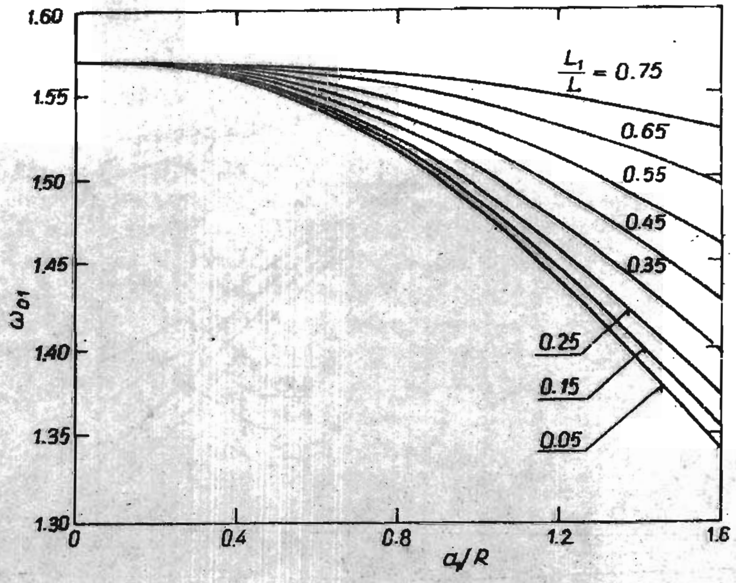

Fig. 8. First characteristic root ω_{01} as the function of the location L_1/L and the size a_1/R of a single crack

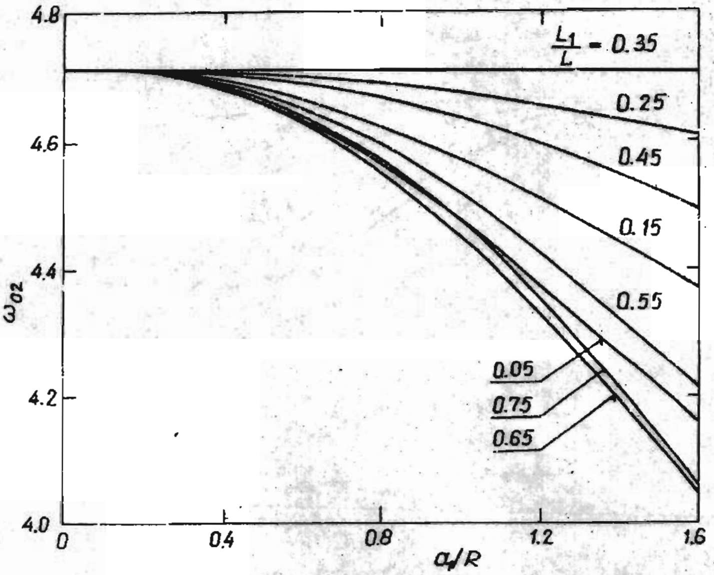

Fig. 9. Second characteristic root ω_{02} as the function of the location L_1/L and the size a_1/R of a single crack

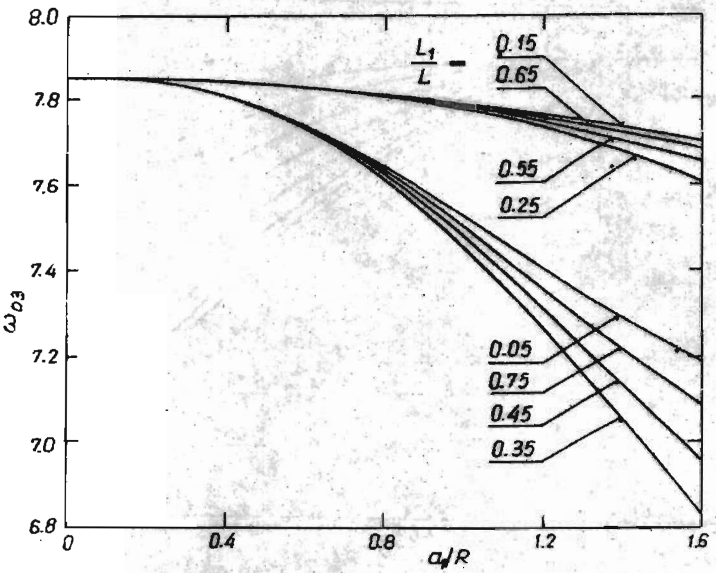

Fig. 10. Third characteristic root ω_{03} as the function of the location L_1/L and the size a_1/R of a single crack

is repeated by the program. The presented procedure can be used to identify cracks by linking the variations in service of the structural natural frequencies to structural changes due to the cracks.

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Analiza wpływu szczelin na częstości własne drgań skrętnych wałów

Streszczenie

W pracy przedstawiono analizę wpływu jednostronnych szczelin poprzecznych na częstości własne drgań skrętnych wałów. Omówiono metodę obliczania sztywności zastępczych wału w miejscu szczeliny, oraz metodę tworzenia równania charakterystycznego służącego do wyznaczania częstości skrętnych drgań własnych pękniętego wału. Przedstawiony w pracy model i algorytm zilustrowano przykładami obliczeń numerycznych wpływu położenia i głębokości jednej i dwóch szczelin na trzy pierwsze częstości skrętnych drgań własnych wału.

Manuscript received October 30, 1991; accepted for print February 20, 1992