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A GENERALIZED PROJECTION METHOD FOR THE DYNAMIC ANALYSIS OF CONSTRAINED MECHANICAL SYSTEMS

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The essence of the reported method lies in the partition of system configuration space into the orthogonal and tangent subspaces, defined relative to the constraint hypersurface. The projection of the initial (*constraint reaction–containing*) dynamical equations into the tangent subspace gives the constraint reaction–free (or canonical) equations of motion, whereas the orthogonal projection determines the associated constraint reactions. The proposed matrix/tensor/linear algebra mathematical formulation is suitable for the analysis carried out in generalized coordinates and/or quasi–velocities, and for systems subject to holonomic and/or nonholonomic constraints. Simplifications due to the use of independent coordinates/velocities are also discussed. An example illustrating these concepts is included.

1. Introduction

In [24] an alternative technique, called the *projection method*, for deriving the differential equations of motion for certain problems in classical mechanics is proposed. The crux of the technique is the projection of Newton's laws into the tangent and orthogonal directions relative the constraint manifolds. The tangent projection gives the equations of motion, whereas the orthogonal projection determines the constraint reactions. The technique is compared with Lagrange's equations and the advantages of the method, simplicity of derivation, an intuitive nature, the physical insight it gives, and the fact that it is somewhat more general than Lagrange's equations, are emphasised. The method has then been discussed by Storch and Gates [27], and the equivalence of the approach and Kane's method (cf [9,10]) was demonstrated. As will be shown in this paper, the projection technique is a variation of many other methods which have been around for many years in classical mechanics, e.g. Appell's equations or Maggi's equations (cf [4,12,15,19÷21]), and is comparable to the recent techniques dealing with

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the formulation of *constraint reaction-free* equations of motion, e.g. the orthogonal complement method (cf [5,6,8,23]), the coordinate partitioning methods (cf [1,11,13,14,25,28]), and some other methods discussed elswhere (cf [13,16÷18,29]).

In this paper a generalized projection method is proposed which gives a unified approach to the dynamic analysis of systems subject to any particular set of independent ideal constraints, including geometric (position), first-order kinematic (velocity), and/or second-order kinematic (acceleration) constraints (these linear in accelerations only). The formulation allows one to carry out the analysis in quasi-velocities and/or generalized coordinates. As a result of the analysis, constraint reaction-free equations of motion are obtained, as well as the formulation which enables one to determine the associated constraint reactions as functions of the current state of system motion. Some simplifications in the analysis following the choice of independent coordinates (for holonomic systems) and independent quasi-velocities (for systems subject to linear velocity constraints) are also demonstrated.

No virtual formalism is used throughtout the paper. Instead, tensor algebra analysis (in matrix notation) and vector space analysis are applied. The analysis is carried out in the n-dimensional space of the system configuration and intuitively appeals to a generalized particle motion on a *smooth* constraint hypersurface. The essence of the proposed approach lies in the partition of the system configuration space into an orthogonal subspace, spanned by the so-called *constraint vectors* [8], and a tangent subspace, which complements the orthogonal subspace in the n-space. The projection of the constraint reaction-containing equations of motion into the tangent subspace gives the reaction-free equations of motion, whereas the orthogonal projection serves for determination of the associated constraint reactions. A compact mathematical formulation, generality, and an intuitive appeal as a generalization of methods used in simple dynamic problems, are recommendable advantages of the approach.

2. Definitions and background

For the purpose of facilitating the reader to follow the subsequent mathematical formulation, let us review first some essential observations concerning the vector space and tensor algebra formalism used in the paper, see also Appendix.

- The analysis is carried out in the n-dimensional system configuration space. Many of the following mathematical transformations refer to the base transformations in the space.

- Matrix notation is used. Vectors are represented by column matrices of their components, and are denoted by lower case letters. Matrices are denoted by capital letters.

- In order to distinguish between the contravariant and covariant vector representations and base vectors, the covariant vector representations and the contravariant base vectors are denoted by the superscript (*).
- All the position, velocity and acceleration vectors are represented by contravariant components.
- All the force vectors, as well as the dynamic equations (in matrix notation) are described by covariant components.

The starting point of the analysis is the dynamics of an n–degree–of–freedom *unconstrained* system whose position is determined by a vector $\boldsymbol{x} = \mathbf{x}^\mathsf{T}\mathbf{e}_x$, where $\mathbf{x} = [x_1, ..., x_n]^\mathsf{T}$ are the system generalized coordinates, and $\mathbf{e}_x = [\boldsymbol{e}_{x1}, ..., \boldsymbol{e}_{xn}]^\mathsf{T}$ denotes a column matrix representation of the inertial base vectors. Let us introduce then a vector of system quasi–velocities $\boldsymbol{v} = \mathbf{v}^\mathsf{T}\mathbf{e}_v$, where $\mathbf{v} = [v_1, ..., v_n]^\mathsf{T}$, and $\mathbf{e}_v = [\boldsymbol{e}_{v1}, ..., \boldsymbol{e}_{vn}]^\mathsf{T}$ represents the base vectors of the frame (usually noninertial) of the system quasi–velocities. The transformation between the generalized velocity vector $\dot{\boldsymbol{x}} = \dot{\mathbf{x}}^\mathsf{T}\mathbf{e}_x$, $\dot{\mathbf{x}} = [\dot{x}_1, ..., \dot{x}_n]^\mathsf{T}$, and the quasi–velocity vector $\boldsymbol{v}$ is described in the usual linear (matrix) form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{v} + \mathbf{a}_0 \tag{2.1}$$

where $\mathbf{A}(\mathbf{x}, t)$ is an $n \times n$ invertible matrix, and $\mathbf{a}_0(\mathbf{x}, t)$ is an $n \times 1$ matrix. The matrix $\mathbf{A}$ can be interpreted as the transformation matrix of a generalized rotation between $\mathbf{e}_v$ and $\mathbf{e}_x$ bases, $\mathbf{e}_v = \mathbf{A}^\mathsf{T}\mathbf{e}_x$, and the column matrix $\mathbf{a}_0$ represents the velocity of the origin of the base $\mathbf{e}_v$ relative to the base $\mathbf{e}_x$ and expressed in the latter base.

To be strict, $v_1, ..., v_n$ should actually be called *kinematical parameters*, for they may be either new generalized velocities or quasi–velocities. As is well known, v_i is a new generalized velocity if the corresponding ith component of the relation

$$\mathbf{v} = \mathbf{B}\dot{\mathbf{x}} + \mathbf{b}_0 \tag{2.2}$$

where $\mathbf{B} = \mathbf{A}^{-1}$, and $\mathbf{b}_0 = -\mathbf{A}^{-1}\mathbf{a}_0$, is *integrable*. Otherwise, v_i is a quasi–velocity. In particular, for $b_{0i} = 0$, the kinematic parameter $v_i = \mathbf{b}_i^\mathsf{T}\dot{\mathbf{x}}$, where $\mathbf{b}_i$ denotes the ith column of $\mathbf{B}^\mathsf{T}$, is a new generalized velocity if the Jacobian $\partial\mathbf{b}_i/\partial\mathbf{x}$ is a symmetric matrix (cf [3,4,15]). It is worth noting, however, that the formulation of this paper allows one to pay no special attention to the problem whether $\mathbf{v}$ containts new generalized velocities or/and quasi–velocities. The relation (2.2) comprises all possible cases, and the problem is referred only to the base transformation. For brevity of the following, $v_1, ..., v_n$ will be called *quasi–velocities*.

Let us introduce now the dynamic equations of motion of the *unconstrained* system in the following compact form

$$\mathbf{M}\dot{\mathbf{v}} = \mathbf{h}^* \tag{2.3}$$

where $\mathbf{M}(\mathbf{x},t)$ is an $n \times n$ symmetric positive–definite matrix of inertia, $\dot{\mathbf{v}} = [\dot{v}_1, ..., \dot{v}_n]^\top$ are the system quasi–accelerations, and $\mathbf{h}^*(\mathbf{v},\mathbf{x},t)$ is a column matrix representation of applied forces and gyroscopic terms. The dynamic equations (2.3) are expressed in the contravariant base $\mathbf{e}_v^*$, and $\mathbf{M}$ is the metric tensor matrix of the base $\mathbf{e}_v$, $\mathbf{M} = \mathbf{e}_v\mathbf{e}_v^\top$ and $\mathbf{e}_v = \mathbf{M}\mathbf{e}_v^*$ (see Appendix). In this paper, Eqs (2.3) refer to the dynamics of an *unconstrained* system not only in the usual meaning of this word; that is, a system consisted of *unbounded* particles and/or bodies. The equations comprise also the dynamics of any internally constrained system (interconnected–body system, for instance) dynamics of which has been formulated previously in independent generalized coordinates and/or quasi–velocities. The corresponding dynamic equations can always be manipulated to the form (2.3). On the other hand, Eqs (2.3) may be considered as a generalized Newton's formula in the n–dimensional space, $(-\mathbf{M}\dot{\mathbf{v}} + \mathbf{h}^*)^\top\mathbf{e}_v^* = (-\dot{\mathbf{v}}^* + \mathbf{h}^*)^\top\mathbf{e}_v^* = \boldsymbol{s} + \boldsymbol{h}$, where $\boldsymbol{s} = (-\mathbf{M}\dot{\mathbf{v}})^\top\mathbf{e}_v^* = -\dot{\mathbf{v}}^{*\top}\mathbf{e}_v^*$ is an inertial force vector.

Let us assume now that a set of m $(m < n)$ independent constraints is imposed on the system, and introduce the constraint equations in the second–order kinematic (acceleration) form

$$\mathbf{C}\dot{\mathbf{v}} + \mathbf{c}_0^* = \mathbf{0} \tag{2.4}$$

where $\mathbf{C}(\mathbf{v},\mathbf{x},t)$ is an $m \times n$ constraint matrix (see [8]) of maximal rank, and $\mathbf{c}_0^*(\mathbf{v},\mathbf{x},t)$ is an $m \times 1$ matrix. If geometric (position) constraints, $f(\mathbf{x},t) = 0$, and/or first–order kinematic (velocity) constraints, $\varphi(\mathbf{v},\mathbf{x},t) = 0$, are considered, they have to be transformed to the acceleration form (2.4) by differentiating with respect to time twice and once, respectively. In these cases, $\mathbf{C}$ and $\mathbf{c}_0^*$ can be written as

$$\mathbf{C} = \begin{cases} \dfrac{\partial f}{\partial \mathbf{x}}\mathbf{A} & \text{for} f(\mathbf{x},t) = 0 \\[2ex] \dfrac{\partial \varphi}{\partial \mathbf{v}} & \text{for} \varphi(\mathbf{v},\mathbf{x},t) = 0 \end{cases} \tag{2.5}$$

$$\mathbf{c}_0^* = \begin{cases} \Big(\dfrac{\partial f}{\partial \mathbf{x}}\mathbf{A}\Big)^{\cdot}\mathbf{v} + \Big(\dfrac{\partial f}{\partial \mathbf{x}}\mathbf{a}_0 + \dfrac{\partial f}{\partial t}\Big)^{\cdot} & \text{for} f(\mathbf{x},t) = 0 \\[2ex] \dfrac{\partial \varphi}{\partial \mathbf{x}}\Big(\mathbf{A}\mathbf{v} + \mathbf{a}_0\Big) + \dfrac{\partial \varphi}{\partial t} & \text{for} \varphi(\mathbf{v},\mathbf{x},t) = 0 \end{cases} \tag{2.6}$$

Obviously, appropriate initial conditions must be assured, i.e. $f(x_0,t_0) = 0$, $\dot{f}(v_0,x_0,t_0) = 0$, and $\varphi(v_0,x_0,t_0) = 0$.

With regard to (2.4), m independent constraint vectors $\mathbf{c}_1, ..., \mathbf{c}_m$ can be defined in the n–space of system configuration [8], which are represented by covariant components, relative to the contravariant base $\mathbf{e}_v^*$, contained as columns in $\mathbf{C}^\top$, $\boldsymbol{c}_i = \mathbf{c}_i^{*\top}\mathbf{e}_v^*$ $(i = 1, ..., m)$, where $\mathbf{c}_i^*$ denotes the ith column of $\mathbf{C}^\top$. Using this definition, $\mathbf{C}\dot{\mathbf{v}}$ expresses dot products of the constraint vectors $\boldsymbol{c}_1, ..., \boldsymbol{c}_m$ and the vector $\dot{\boldsymbol{v}}$ of system acceleration, $\boldsymbol{c}_i \circ \dot{\boldsymbol{v}} = \mathbf{c}_i^{*\top}\dot{\mathbf{v}}$ $(i = 1, ..., m)$. In particular, for a time–independent position constraint $f_i(\mathbf{x}) = 0$, and on the assumption that $\mathbf{v} = \dot{\mathbf{x}}$, the corresponding constraint vector $\boldsymbol{c}_i$ is simply the constraint gradient, $\boldsymbol{c}_i = (\partial f_i/\partial \mathbf{x})\mathbf{e}_v^*$. Therefore, in this case, $\boldsymbol{c}_i$ can be interpreted as being orthogonal to the constraint manifold $f_i(\mathbf{x}) = 0$. Though, for the time–dependent position constraints, and the constraints of higher–order, as well as for the case of analysis being caried out in quasi–velocities, such an intuitive appeal as a direct generalization of simple dynamics problems cannot be strictly undertaken, an m-dimensional subspace spanned by the constraint vectors $\boldsymbol{c}_1, ..., \boldsymbol{c}_m$ will be called here an *orthogonal* subspace.

The analysis of this paper considers only the so–called *ideal* constraints. In the mathematical sense, this means that the constraint–induced forces (constraint reactions) on the system are postulated as collinear with the corresponding constraint vectors [8], $\boldsymbol{r}_i = \boldsymbol{c}_i\lambda_i$ $(i = 1, ..., m)$, where $\boldsymbol{r}_i$ is the ith constraint reaction vector and λ_i is the associated Lagrange multiplier. Obviously, for a specific state of system motion, the value of a constraint reaction may achieve zero when the applied and gyroscopic forces on the system in the direction orthogonal to the corresponding constraint are in balance and no additional constraint-induced reaction is required to assure the constraint realization. This is regulated by the current value of corresponding Lagrange multiplier. Using matrix notation, the vector of constraining forces on the system, $\boldsymbol{r} = \sum_{i=1}^{m} \boldsymbol{r}_i$, can be expressed in the base $\mathbf{e}_v^*$ as follows

$$\mathbf{r}^* = \sum_{i=1}^{m} \mathbf{r}_i^* = \sum_{i=1}^{m} \mathbf{c}_i^*\lambda_i = \mathbf{C}^\top\boldsymbol{\lambda} \tag{2.7}$$

where $\mathbf{r}^*$ and $\mathbf{r}_i^*$ are column matrix representations of $\boldsymbol{r}$ and $\boldsymbol{r}_i$ in the base $\mathbf{e}_v^*$, and $\boldsymbol{\lambda} = [\lambda_1, ..., \lambda_m]^\top$ are Lagrange multipliers.

The constraint reactions are not known á priori ($\boldsymbol{\lambda}$ are to be treated as a set of m new variables), and during the constrained motion they take the values such that to assure the system consistency with all the constraint conditions up to the acceleration form (2.4). Hence, the final (constraint reaction–containing) governing equations of the constrained motion can be written in the following form of differential–algebraic equations (DAEs)

$$\begin{aligned} &a) \qquad \mathbf{M}\dot{\mathbf{v}} = \mathbf{h}^* + \mathbf{C}^\top\boldsymbol{\lambda} \\ &b) \qquad \dot{\mathbf{x}} = \mathbf{A}\mathbf{v} + \mathbf{a}_0 \\ &c) \qquad \mathbf{C}\dot{\mathbf{v}} + \mathbf{c}_0^* = \mathbf{0} \end{aligned} \tag{2.8}$$

where the differential variables are $\mathbf{v}$ and $\mathbf{x}$, and the algebraic variables are $\boldsymbol{\lambda}$. The dimension of the DAE system is $2n+m$, and appropriate initial conditions must be assured if Eqs (2.8c) are the differentiated forms of lower-order constraints. Note also that Eqs (2.8a) can also be interpreted as d'Alambert's principle generalized to the n-dimensional space

$$(-\mathbf{M}\dot{\mathbf{v}} + \mathbf{h}^* + \mathbf{r}^*)^\top \mathbf{e}_v^* = (-\dot{\mathbf{v}} + \mathbf{h}^* + \mathbf{r}^*)^\top \mathbf{e}_v^* = \boldsymbol{s} + \boldsymbol{h} + \boldsymbol{r} = \mathbf{0} \tag{2.9}$$

The DAEs (2.8) can be treated as a generalized form of Lagrange's equations of the first-order [4], and this form of governing equations of constrained systems is encoutered frequently in many applications such as robotics, dynamic simulation of vehicles, and analysis of mechanisms, see e.g. [6÷11,13÷21,23,25,28,29]. However, the numerical treatment of *maximal-rank equations* (2.8) is often time-consuming and dealing with DAEs leads to additional inconveniences. Thus, many methods aimed at eliminating the constraining forces from the analysis and reducing the problem dimension have been applied and described in the aforementioned publications. The reported projection method approach is another contribution to these methods.

3. Projection method formulation

As stated in Section 2, the constraint vectors $\boldsymbol{c}_1, ..., \boldsymbol{c}_m$ defined in the base $\mathbf{e}_v^*$ by $\mathbf{c}_i^*$ (columns of $\mathbf{C}^\top$), $i = 1, ..., m$, are independent in principle (rank($\mathbf{C}$) = max = m). Hence, the vectors spaning an m-dimensional subspace in the n-space, called an *orthogonal* subspace lateron, can be considered as the base vectors of the subspace, denoted as $\mathbf{e}_c = [\boldsymbol{c}_1, ..., \boldsymbol{c}_m]^\top$. Then, k ($k = n - m$) additional independent vectors $\boldsymbol{d}_1, ..., \boldsymbol{d}_k$, assumed to be orthogonal to $\mathbf{e}_c$, can be chosen. Let us call the k-dimensional subspace spanned by these vectors a *tangent* subspace, and $\mathbf{e}_d = [\boldsymbol{d}_1, ..., \boldsymbol{d}_m]^\top$ is the base of the subspace. The orthogonality condition, $\mathbf{e}_d\mathbf{e}_c^\top = 0$, means that the orthogonal and tangent subspaces are complementary in the n-space. Denoting that the contravariant components of $\boldsymbol{d}_1, ..., \boldsymbol{d}_k$ in the base $\mathbf{e}_v$ are gathered as coluns in a matrix $\mathbf{D}^\top$ ($\mathbf{D}(\mathbf{v},\mathbf{x},t)$ is a $k \times n$ matrix of maximal rank), this can be written as follows

$$\mathbf{D}\mathbf{C}^\top = \mathbf{0} \tag{3.1}$$

In other words, $\mathbf{D}$ is an orthogonal complement of $\mathbf{C}$ in the n-space (refer also to e.g. [5,6÷8,18])

Since the vectors $\boldsymbol{c}_1, ..., \boldsymbol{c}_m, \boldsymbol{d}_1, ..., \boldsymbol{d}_k$ are linearly independent, they form a new base in the n-space, $\mathbf{e}_t = [\mathbf{e}_c^\top, \mathbf{e}_d^\top]^\top$. Then, considering that $\mathbf{e}_c = \mathbf{C}\mathbf{e}_v^* = \mathbf{C}\mathbf{M}^{-1}\mathbf{e}_v$, the following transformation formula can be written

$$\mathbf{e}_t = \begin{bmatrix} \mathbf{e}_c \\ \mathbf{e}_d \end{bmatrix} = \begin{bmatrix} \mathbf{C}\mathbf{M}^{-1} \\ \mathbf{D} \end{bmatrix} \mathbf{e}_v = \mathbf{T}^\top \mathbf{e}_v \tag{3.2}$$

and the metric tensor matrix $\mathbf{M}_t$ of the base $\mathbf{e}_t$ can be expressed as

$$\mathbf{M}_t = \mathbf{T}^\top \mathbf{M} \mathbf{T} = \begin{bmatrix} \mathbf{C}\mathbf{M}^{-1}\mathbf{C}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{D}\mathbf{M}\mathbf{D}^\top \end{bmatrix} = \begin{bmatrix} \mathbf{M}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_d \end{bmatrix} \tag{3.3}$$

where $\mathbf{M}_c$ and $\mathbf{M}_d$ are the metric tensor matrices of the orthogonal and tangent subspaces, respectively.

As the dynamical equations (2.8a) are represented in the base $\mathbf{e}_v^*$, their representation in the base $\mathbf{e}_t^*$ is equivalent to the left–sided multiplication of Eqs (2.8a) by the transformation matrix $\mathbf{T}^\top$ defined in (3.2) (see Appendix). This yields the following decomposition of the dynamical equations (projection into the orthogonal and tangent subspaces)

$$\begin{aligned} &a) \qquad \mathbf{C}\dot{\mathbf{v}} = \mathbf{C}\mathbf{M}^{-1}\mathbf{h}^* + \mathbf{M}_c\boldsymbol{\lambda} \\ &b) \qquad \mathbf{D}\mathbf{M}\dot{\mathbf{v}} = \mathbf{D}\mathbf{h}^* \end{aligned} \tag{3.4}$$

Using the tangential projection (3.4b), and (2.8c) and (2.8b), the following constraint *reaction–free* governing equations can be obtained

$$\begin{aligned} &a) \qquad \mathbf{T}^\top\mathbf{M}\dot{\mathbf{v}} = \mathbf{h}_t^* \\ &b) \qquad \dot{\mathbf{x}} = \mathbf{A}\mathbf{v} + \mathbf{a}_0 \end{aligned} \tag{3.5}$$

where $\mathbf{h}_t^* = [-\mathbf{c}_0^{*\top}, (\mathbf{D}\mathbf{h}^*)^\top]^\top$, and the dimension of Eqs (3.5) is reduced to $2n$ as compared with the dimension $2n+m$ of Eqs (2.8). Note also that Eqs (3.5) are conceptually equivalent to the results obtained by using the orthogonal complement method [5,8].

The orthogonal projection (3.4a) may then serve for explicit determination of $\boldsymbol{\lambda}$ as functions in $\mathbf{v}$, $\mathbf{x}$ and t, i.e. for the determination of the constraint reactions. Namely, considering (2.8c), it can be shown that

$$\boldsymbol{\lambda} = -\mathbf{M}_c^{-1}(\mathbf{c}_0^* + \mathbf{C}\mathbf{M}^{-1}\mathbf{h}^*) = \boldsymbol{\lambda}(\mathbf{v},\mathbf{x},t) \tag{3.6}$$

Following (2.7), the reaction of the ith constraint can be expressed as

$$\mathbf{r}_i^* = \mathbf{c}_i^*\lambda_i = \mathbf{r}_i^*(\mathbf{v},\mathbf{x},t) \tag{3.7}$$

which is a representation of the ith constraint reaction vector $\mathbf{r}_i$ in the base $\mathbf{e}_v^*$.

4. Projection method – holonomic case

In many applications, and in particular for holonomic systems, the analysis carried out in the initial (dependent) generalized coordinates/quasi–velocities may be inconvenient. Thus, very often, the analysis in *independent* variables is undertaken. There are at least two reasons for doing this. Firstly, the dimension of the problem reduces to the number of degrees of freedom of the system, which may greatly simplify computational analysis in problems of large systems with many constraints (inter–connected multibody systems, for instance). Secondly, the direct integration of the governing equations (3.5), when the constraint equations (3.4) (the first m equations of (3.5a)) are differentiated forms of lower–order constraints, may yield violation of the constraints due to the numerical errors of integration. Though there are different techniques based on monitoring the constraint violation at every step of integration and aimed at reducing the violation value, see e.g. [2,17], the appropriate choice of independent coordinates/velocities releases the analysis from this inconvenience. Herein, these aspects will be discussed from the point of view of the projection method. It will be shown that the results obtained using this technique are equivalent to Kane's form of Appell's equations [9,10,15], often referred as Kane's method.

The analysis of this section begins again with the governing equations (2.8) expressed in the initial (dependent) quasi–velocities $\mathbf{v}$ and generalized coordinates $\mathbf{x}$. In the case at hand, however, the m constraints imposed on the system are geometric (position, holonomic) constraints

$$f(\mathbf{x},t)=0 \tag{4.1}$$

The number of degrees of freedom is thus reduced to $k=n-m$, and the configuration of the system can be described by using k *independent* coordinates, denoted as $\mathbf{q}=[q_1,...,q_k]^\top$. The independent coordinates are usually chosen á priori for a particular system and, very often, without introducing the constraint equations (4.1) at all as being satisfied in principle in $\mathbf{q}$ (the constraint equations must be introduced, however, if the constraint reactions are to be determined). The interdependence between the initial and the independent coordinates can be written as follows

$$\mathbf{x}=\mathbf{g}(\mathbf{q},t) \tag{4.2}$$

For the purpose of generality, let us introduce also k independent quasi–velocities, $\mathbf{u}=[u_1,...,u_k]^\top$, and define a relation analogous to (2.1)

$$\dot{\mathbf{q}}=\tilde{\mathbf{A}}\mathbf{u}+\tilde{\mathbf{a}}_0 \tag{4.3}$$

where $\tilde{\mathbf{A}}(\mathbf{q},t)$ is a $k\times k$ invertible (transformation) matrix, and $\tilde{\mathbf{a}}_0(\mathbf{q},t)$ is a $k\times 1$ matrix. As before, $\mathbf{u}$ may contain quasi–velocities and/or generalized velocities.

Differentiating (4.2) and introducing (2.1) and (4.3), it is easy to find that

$$\mathbf{v}=\mathbf{A}^{-1}\mathbf{J}\tilde{\mathbf{A}}\mathbf{u}+\mathbf{v}_u \tag{4.4}$$

$$\dot{\mathbf{v}}=\mathbf{A}^{-1}\mathbf{J}\tilde{\mathbf{A}}\dot{\mathbf{u}}+\mathbf{a}_u \tag{4.5}$$

where $\mathsf{J}(\mathbf{q},t) = \partial\mathbf{g}/\partial\mathbf{q}$ is the $n \times k$ Jacobian, $\mathbf{v}_u(\mathbf{q},t) = \mathsf{A}^{-1}(\mathsf{J}\tilde{\mathbf{a}}_0 + \mathbf{g}_t - \mathbf{a}_0)$, $\mathbf{a}_u(\mathbf{u},\mathbf{q},t) = (\mathsf{A}^{-1}\mathsf{J}\tilde{\mathsf{A}})^{\cdot}\mathbf{u} + [\mathsf{A}^{-1}(\mathsf{J}\tilde{\mathbf{a}}_0 + \mathbf{g}_t - \mathbf{a}_0)]^{\cdot}$, and $\mathbf{g}_t = \partial\mathbf{g}/\partial t$. As $\mathbf{q}$, $\mathbf{u}$ and $\dot{\mathbf{u}}$ are independent coordinates, quasi–velocities and quasi–accelerations, respectively, all the constraint conditions, $f = 0$, $\dot{f} = 0$ and $\ddot{f} = 0$, expressed in the independent variables, are satisfied in principle. In particular, the acceleration form (2.4) of (4.1) leads to

$$\mathsf{C}\mathsf{A}^{-1}\mathsf{J}\tilde{\mathsf{A}}\dot{\mathbf{u}} + \mathsf{C}\mathbf{a}_u + \mathbf{c}_0^* \equiv \mathbf{0} \tag{4.6}$$

where C and $\mathbf{c}_0^*$ are defined by (2.5) and (2.6), and according to (4.2) and (4.4) are functions of $\mathbf{u}$, $\mathbf{q}$ and t. Since the values of $\dot{\mathbf{u}}$ are not restricted by the constraints, the identity condition (4.6) requires that

$$\mathsf{C}\mathsf{A}^{-1}\mathsf{J}\tilde{\mathsf{A}} = \mathbf{0} \tag{4.7}$$

$$\mathsf{C}\mathbf{a}_u + \mathbf{c}_0^* = \mathbf{0} \tag{4.8}$$

Note that (4.7) is usually obtained using the formulae

$$(\mathsf{C}\mathsf{A}^{-1}\mathsf{J}\tilde{\mathsf{A}})\delta\mathbf{u} = \mathbf{0} \qquad \text{or} \qquad (\mathsf{C}\mathsf{A}^{-1}\mathsf{J}\tilde{\mathsf{A}})\delta\dot{\mathbf{u}} = \mathbf{0}$$

where $\delta\mathbf{u}$ and $\delta\dot{\mathbf{u}}$ are column matrix representations of variations of the system quasi–velocities and quasi–accelerations, respectively.

The critical observation following from (4.7) is that $\mathsf{A}^{-1}\mathsf{J}\tilde{\mathsf{A}}$ contains the contravariant components, referred to the base $\mathbf{e}_v$, of the vectors which span the tangent subspace with regard to the constraints (4.1), that is

$$\mathsf{D}^{\mathsf{T}} = \mathsf{A}^{-1}\mathsf{J}\tilde{\mathsf{A}} \tag{4.9}$$

Now, substituting (4.9) and (4.5) into (3.5a), the first m equations leads to the identity (4.8), whereas the remaining k equations form the canonical (reduced–dimension) dynamic equations

$$\mathsf{M}_d\dot{\mathbf{u}} = \mathsf{D}(\mathbf{h}^* - \mathsf{M}\mathbf{a}_u) = \mathbf{h}_u^* \tag{4.10}$$

where $\mathsf{M}_d(\mathbf{q},t) = \mathsf{D}\mathsf{M}\mathsf{D}^{\mathsf{T}} = (\mathsf{A}^{-1}\mathsf{J}\tilde{\mathsf{A}})^{\mathsf{T}}\mathsf{M}(\mathsf{A}^{-1}\mathsf{J}\tilde{\mathsf{A}})$ is the metric tensor matrix of the base $\mathbf{e}_u$ of the tangent subspace, $\mathbf{e}_u = [d_1, ..., d_k]^{\mathsf{T}}$, $d_i = \mathbf{d}_i^{\mathsf{T}}\mathbf{e}_v$ $(i = 1, ..., k)$, and $\mathbf{d}_i$ is the ith column of $\mathsf{D}^{\mathsf{T}} = \mathsf{A}^{-1}\mathsf{J}\tilde{\mathsf{A}}$. Since $\mathbf{h}_u^* = \mathsf{D}(\mathbf{h}^* - \mathbf{a}_u)$ can be expressed as a function of $\mathbf{u}$, $\mathbf{q}$ ant t, the dynamic equations (4.10) are conceptually equivalent to (2.3).

The dynamic equations (4.10), combined with the kinematic equations (4.3), form the the governing equations of the problem at hand, and the dimension of the equations is $2k$. Note also that the constraint equations (4.1) do not have to be introduced explicitly for derivation of (4.10); an appropriated choice of $\mathbf{q}$ and $\mathbf{u}$, and the definition of (4.2), are sufficient. The constraint equations have to be formulated explicitly, however, when the associated constraint reactions are to be determined. In this case, the substitution of (4.2) and (4.4) into (3.6) and (3.7) enables one to determine λ_i and $\mathbf{r}_i^*$ $(i = 1, ..., m)$ as functions of current values of $\mathbf{u}$, $\mathbf{q}$, t.

The dynamic equations (4.10) can also be treated as a modified form of Kane's equations [10], or one of the forms of Appell's equations (see Chap.III.8 of [15]), which, in fact, consist also in the projection of the *initial* equations of motion into the tangent subspace. From (4.4) and (4.5) it comes evidently that

$$\mathbf{D}^{\top} = \mathbf{A}^{-1}\mathbf{J}\tilde{\mathbf{A}} = \frac{\partial \mathbf{v}}{\partial \mathbf{u}} = \frac{\partial \dot{\mathbf{v}}}{\partial \dot{\mathbf{u}}} = ... \tag{4.11}$$

hence, $\mathbf{D}$ is a matrix of so-called *partial velocities* [9,10]. Now, the projection of the *initial* dynamic equations (2.3) (or (2.8a)) into the tangent subspace leads to the following matrix form of Kane's equations

$$\mathbf{D}(-\mathbf{M}\dot{\mathbf{v}} + \mathbf{h}^*) = \mathbf{D}(-\dot{\mathbf{v}}^* + \mathbf{h}^*) \tag{4.12}$$

where $-\mathbf{v}^*$ denote the inertial forces. Introducing (4.2), (4.4) and (4.5), the Eqs (4.12) can easily be manipulated to the form (4.10).

5. Projection method – nonholonomic case

Assume now that the system is subject to m first-order kinematic (velocity or nonholonomic) constraints, and let start the analysis on the linear case of the constraints

$$\boldsymbol{\varphi} = \mathbf{C}\mathbf{v} + \boldsymbol{\eta} \tag{5.1}$$

where $\mathbf{C}(\mathbf{x}, t)$ is an $m \times n$ matrix of maximal rank, and $\boldsymbol{\eta}(\dot{\mathbf{x}}, t)$ is an $m \times 1$ matrix. As opposed to the differentiated form of geometric constraints, Eqs (5.1) are supposed to be nonintegrable (nonholonomic), for details refer to the discussion following (2.2). It is worth noting also that the nonholonomic constraints do not reduce the number of degrees of freedom of the system. The position of the system is still described by the *initial* generalized coordinates $\mathbf{x} = [x_1, ..., x_n]^{\top}$. The restrictions are imposed, however, on the *initial* quasi-velocities $\mathbf{v} = [v_1, ..., v_n]^{\top}$, which are now *dependent*. In other words, only $k = n - m$ *independent* quasi-velocities exist, denoted as $\mathbf{u} = [u_1, ..., u_k]^{\top}$.

Since Eqs (5.1) can be considered as a set of m new quasi-velocities which, due to the constraints imposed, remain zeros at every instant of the constrained motion, the independent quasi-velocities can conveniently by defined as being represented only in the tangent subspace. To this end, the following transformation formula of quasi-velocities can be proposed to define $\mathbf{u}$

$$\mathbf{M}_t \begin{bmatrix} \mathbf{0} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}_d\mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{C}\mathbf{v} + \boldsymbol{\eta} \\ \mathbf{D}\mathbf{M}\mathbf{v} \end{bmatrix} = \mathbf{T}^{\top}\mathbf{M}\mathbf{v} + \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{0} \end{bmatrix} \tag{5.2}$$

where $\mathbf{M}_t$, $\mathbf{M}_d$, $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{T}$ refer to the notation used in Section 3, and are functions of $\mathbf{x}$ and t. From Eqs (5.2), after some manipulations, the following inverse relation can be found

$$\mathbf{v} = \mathbf{D}^\top \mathbf{u} + \mathbf{v}_u \tag{5.3}$$

where $\mathbf{v}_u = -\mathbf{M}^{-1}\mathbf{C}^\top\mathbf{M}_c^{-1}\boldsymbol{\eta}$. The differentiation of Eqs (5.3) with respect to time leads to

$$\mathbf{M}_t\begin{bmatrix} \mathbf{0} \\ \dot{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}_d\dot{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{C}\dot{\mathbf{v}} + \mathbf{c}_0^* \\ \mathbf{D}\mathbf{M}\dot{\mathbf{v}} + \mathbf{d}_0^* \end{bmatrix} = \mathbf{T}^\top\mathbf{M}\dot{\mathbf{v}} + \begin{bmatrix} \mathbf{c}_0^* \\ \mathbf{d}_0^* \end{bmatrix} \tag{5.4}$$

where $\mathbf{c}_0^*(\mathbf{u},\mathbf{x},t) = \dot{\mathbf{C}}(\mathbf{D}^\top\mathbf{u} + \mathbf{v}_u) + \boldsymbol{\eta}$, and $\mathbf{d}_0^*(\mathbf{u},\mathbf{x},t) = (\mathbf{D}\mathbf{M})^{\cdot}(\mathbf{D}^\top\mathbf{u} + \mathbf{v}_u) + \dot{\mathbf{M}}_d\mathbf{u}$. Now, $\dot{\mathbf{v}}$ can be determined from Eqs (5.4) as

$$\dot{\mathbf{v}} = \mathbf{D}^\top\dot{\mathbf{u}} + \mathbf{a}_u \tag{5.5}$$

where $\mathbf{a}_u(\mathbf{u},\mathbf{x},t) = -\Big[(\mathbf{M}^{-1}\mathbf{C}^\top\mathbf{M}_c^{-1}\mathbf{c}_0^*)^\top, (\mathbf{D}^\top\mathbf{M}_d^{-1}\mathbf{d}_0^*)^\top\Big]^\top$.

The substitution of (5.5) into (3.5a) yields that the first m of these equations transform to the identity $\mathbf{c}_0^* = \mathbf{c}_0^*$, and the remaining k equations become the minimal-dimension dynamic equations expressed in the independent quasi-velocities

$$\mathbf{M}_d\dot{\mathbf{u}} = \mathbf{D}\mathbf{h}^* + \mathbf{d}_0^* = \mathbf{h}_u^* \tag{5.6}$$

Though Eqs (5.6) look in their symbolical form like Eqs (4.10) obtained for the system subject to holonomic constraints, in this case $\mathbf{M}_d(\mathbf{x},t)$ and $\mathbf{h}_u^*(\mathbf{u},\mathbf{x},t)$. Moreover, in order to form the governing equations of the motion, Eqs (5.6) are to be combined with the differential kinematic equations in the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{D}^\top\mathbf{u} + \boldsymbol{\xi} \tag{5.7}$$

where $\boldsymbol{\xi}(\mathbf{x},t) = \mathbf{a}_0 + \mathbf{A}\mathbf{v}_u = \mathbf{a}_0 - \mathbf{A}\mathbf{M}^{-1}\mathbf{C}^\top\mathbf{M}_c^{-1}\boldsymbol{\eta}$. The dimension of the governing equations is $n + k$.

As in the previous section, the reactions of the constraints (5.1) can be determined from Eqs (3.6) and (3.7). After substituting (5.3), λ_i and $\mathbf{r}_i^*$ $(i = 1, ..., m)$ will be functions of current values of $\mathbf{u}$, $\mathbf{x}$ and t. One can easy deduce, also, that the presented projection method approach to the dynamic analysis of systems subject to linear velocity constraints (5.1) is conceptually equivalent to Maggi's approach (cf [4,12,15,19,21]). As compared with the previous formulations, however, the present formulation seems to be more intuitive as well as compact. The geometrical insight into the problems concerned is also commendable.

Unfortunately, for nonlinear velocity constraints, $\boldsymbol{\varphi}(\mathbf{v},\mathbf{x},t) = 0$, the definition of independent quasi-velocities u is not so evident, if feasible in practice at all. The constraint equations $\boldsymbol{\varphi} = 0$ may still be treated as *zero* quasi-velocities and the following transformation between independent and dependent quasi–velocities may be proposed

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varphi}(\mathbf{v},\mathbf{x},t) \\ \boldsymbol{\gamma}(\mathbf{v},\mathbf{x},t) \end{bmatrix} \tag{5.8}$$

where rank $\left((\partial\boldsymbol{\varphi}/\partial\mathbf{v})^{\top},(\partial\boldsymbol{\gamma}/\partial\mathbf{v})^{\top}\right) = \max = n$ is required. However, the à priori determination of the functions $\boldsymbol{\gamma}$ is most often unpracticable. The same refers usually to an à priori determination, basing only on the equations $\boldsymbol{\varphi} = \mathbf{0}$, of an inverse to (5.8) relation (referring to Eqs (4.2)), that is

$$\mathbf{v} = \mathbf{g}(\mathbf{u},\mathbf{x},t) \tag{5.9}$$

Hence, in the case of nonlinear constraint equations, $\boldsymbol{\varphi}(\mathbf{v},\mathbf{x},t) = 0$, a general projection method approach reported in Section 3 is recommended.

6. Example

To illustarate the concepts introduced in this paper, consider the classical problem of dynamic analysis of a homogeneous disk of radius r and mass m that rolls without sliding on the horizontal plane (see Fig.1). Let the generalized coordinates of the *unconstrained* disk be as follows; x_P, y_P, z_P – the coordinates of the point P in the inertial reference frame $0x_Iy_Iz_I$, ϕ – the angle between the contact tangent and the positive $0x_I$ axis, θ – the angle of inclination of the disk to the vertical, and ψ – the angle of rolling. The condition of pure rolling of the disk on the plane $0x_Iy_I$ leads to the following constraint conditions, see also [4,15]

$$\begin{aligned} &a) \qquad \dot{x}_P + r\dot{\psi}\cos\phi = 0 \\ &b) \qquad \dot{y}_P + r\dot{\psi}\sin\phi = 0 \\ &c) \qquad z_P = 0 \end{aligned} \tag{6.1}$$

The constraints (6.1a) and (6.1b) are linear nonholonomic (velocity) constraints, and (6.1c) is a holonomic (geometric) constraint.

Let us define as initial quasi–velocities the components v_1, v_2, v_3 of the linear velocity of G in the $G\xi\eta\zeta$ coordinate system, and the components v_4, v_5, v_6 of the disk angular velocity referred to the same axes. The $G\xi\eta\zeta$ reference frame is chosen in such a way that the origin of the system is at the center of the disk, the $G\xi\eta$ plane is that of the disk, and $G\xi$ axis remains parallel to the $0x_Iy_I$ plane. According to the definitions, the kinematic differential equations referring to Eqs

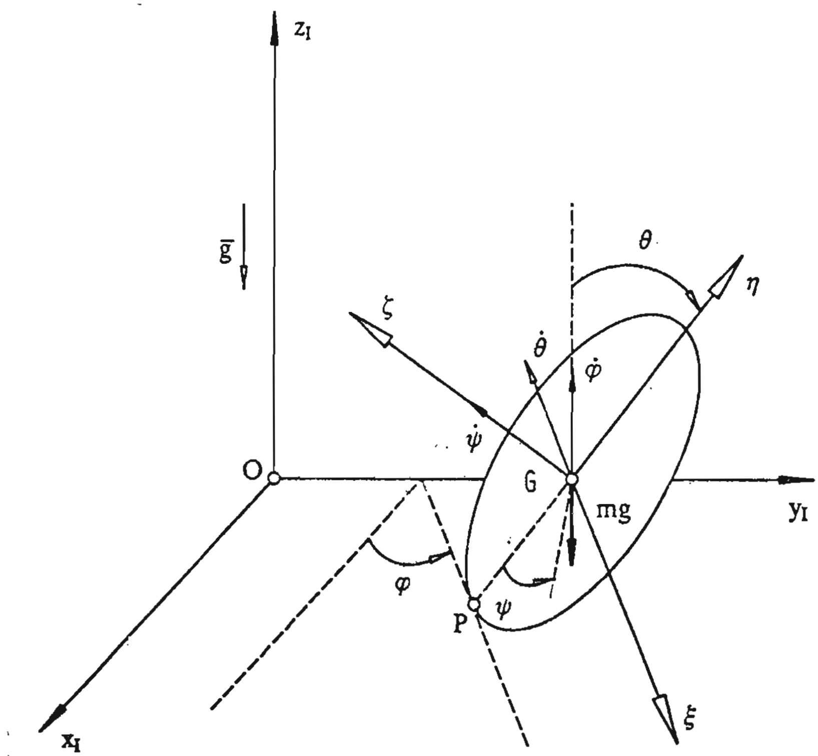

Fig. 1. A rolling disk

(2.1) and (2.2) can be written as

$$\dot{\mathbf{x}} = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix} \mathbf{v} = \mathbf{A}\mathbf{v} \tag{6.2}$$

$$\mathbf{v} = \begin{bmatrix} B_1 & B_3 \\ 0 & B_2 \end{bmatrix} \dot{\mathbf{x}} = \mathbf{B}\dot{\mathbf{x}} \tag{6.3}$$

where $\dot{\mathbf{x}} = [\dot{x}_P, \dot{y}_P, \dot{z}_P, \dot{\phi}, \dot{\theta}, \dot{\psi}]^\top$, $\mathbf{v} = [v_1, v_2, v_3, v_4, v_5, v_6]^\top$, and

$$\mathbf{A}_1 = \begin{bmatrix} \cos\phi & -\sin\theta\sin\phi & \cos\theta\sin\phi \\ \sin\phi & \sin\theta\cos\phi & -\cos\theta\cos\phi \\ 0 & \cos\theta & \sin\theta \end{bmatrix} \qquad \mathbf{A}_2 = \begin{bmatrix} 0 & \cos^{-1}\theta & 0 \\ -1 & 0 & 0 \\ 0 & \tan\theta & 1 \end{bmatrix}$$

$$\mathbf{A}_3 = \begin{bmatrix} -r\cos\theta\sin\phi & r\tan\theta\cos\phi & 0 \\ r\cos\theta\cos\phi & r\tan\theta\sin\phi & 0 \\ -r\sin\phi & 0 & 0 \end{bmatrix} \qquad \mathbf{B}_1 = \mathbf{A}_1^{-1} = \mathbf{A}_1^\top$$

$$\mathbf{B}_2 = \begin{bmatrix} 0 & -1 & 0 \\ \cos\theta & 0 & 0 \\ \sin\theta & 0 & 1 \end{bmatrix} \qquad \mathbf{B}_3 = -\mathbf{B}_1\mathbf{A}_3\mathbf{B}_2 = \begin{bmatrix} -r\sin\theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -r & 0 \end{bmatrix}$$

One can easy ascertain that the transformation of (6.1) to the dynamic form (2.4) leads to rather complicated relations. Therefore, the constraints (6.1) are often reformulated in a more convenient form. Namely, assuming that $z_P(t_0) = 0$, the constraint (6.1c) can be replaced by its differentiated form

$$\dot{z}_P = 0 \tag{6.4}$$

Now, Eqs (6.1a), (6.1b) and (6.4) represent the condition that the vectors of rim velocity at the point P and the velocity of the contact point motion on the plane $0x_Iy_I$ are equal and have opposite directions, and the condition is expressed in the $0x_Iy_Iz_I$ reference frame. Expressing the condition in the $G\xi\eta\zeta$ reference system, that is left–sided premultiplying (6.1a), (6.1b) and (6.4) by $\mathbf{B}_1$, and introducing (6.2), the constraint equations become

$$\begin{aligned} &a) \qquad v_1 + rv_6 = 0 \\ &b) \qquad v_2 = 0 \\ &c) \qquad v_3 - rv_4 = 0 \end{aligned} \tag{6.5}$$

The differentiation with respect to time of (6.5) leads to the following convenient dynamic form of the constraints

$$\begin{aligned} &a) \qquad \dot{v}_1 + r\dot{v}_6 = 0 \\ &b) \qquad \dot{v}_2 = 0 \\ &c) \qquad \dot{v}_3 - r\dot{v}_4 = 0 \end{aligned} \tag{6.6}$$

Physically, Eqs (6.5) express the condition of null velocity of the point P of the disk. The transformation from (6.5) to (6.6) requires that $v_1(t_0)$, $v_2(t_0)$, $v_3(t_0)$, $v_4(t_0)$ and $v_6(t_0)$ must satisfy Eqs (6.5).

It is easy to find that for the case at hand

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & r \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -r & 0 & 0 \end{bmatrix} \tag{6.7}$$

and the orthogonal complement matrix $\mathbf{D}$ can be chosen as

$$\mathbf{D} = \begin{bmatrix} -r & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & r & 1 & 0 & 0 \end{bmatrix} \tag{6.8}$$

Introducing the dynamic equations of the unconstrained disk in the form (2.3), where for the case at hand $\mathbf{M}$ and $\mathbf{h}^*$ are

$$\mathbf{M} = \operatorname{diag}(m, m, m, J, J, 2J) \tag{6.9}$$

$$\mathbf{h}^* = \begin{bmatrix} mv_5(v_2 \tan\theta - v_3) \\ m(v_4 v_3 - v_5 v_1 \tan\theta - g\cos\theta) \\ m(v_5 v_1 - v_4 v_2 - g\sin\theta) \\ Jv_5(v_5\tan\theta - 2v_6) \\ Jv_4(2v_6 - v_5\tan\theta) \\ 0 \end{bmatrix} \tag{6.10}$$

where $\mathbf{J} = \mathbf{J}_{\xi\xi} = \mathbf{J}_{\eta\eta} = \frac{1}{2}\mathbf{J}_{\zeta\zeta}$, and g is the acceleration due to gravity, the tangential projection of the equations (referring to (3.4b)) are

$$\begin{aligned} -mr\dot{v}_1 + 2J\dot{v}_6 &= -mrv_5(v_2\tan\theta - v_3) \\ J\dot{v}_5 &= Jv_4(2v_6 - v_5\tan\theta) \\ mr\dot{v}_3 + J\dot{v}_4 &= mr(v_5v_1 - v_4v_2 - gr\sin\theta) + Jv_5(v_5\tan\theta - 2v_6) \end{aligned} \tag{6.11}$$

The set of Eqs (6.6), (6.11) and (6.2) refers to the reaction-free governing equations (3.5), and the associated constraint reactions can be determined through the following relations

$$\boldsymbol{\lambda} = \begin{bmatrix} \mu_1 v_5(v_2\tan\theta - v_3) \\ m(v_4v_3 - v_5v_1\tan\theta - g\cos\theta) \\ \mu_2\Big(v_5v_1 - v_4v_2 - g\sin\theta - \frac{v_5}{r}(v_5\tan\theta - 2v_6)\Big) \end{bmatrix} \tag{6.12}$$

$$\mathbf{r}_1^* = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ r \end{bmatrix}\lambda_1 \qquad \mathbf{r}_2^* = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\lambda_2 \qquad \mathbf{r}_3^* = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -r \\ 0 \\ 0 \end{bmatrix}\lambda_3 \tag{6.13}$$

where $\mu_1 = 2mJ/(mr^2 + 2J)$, and $\mu_2 = mJ/(mr^2 + J)$. Is is worth noting that $\mathbf{r}_1^*$, $\mathbf{r}_2^*$ and $\mathbf{r}_3^*$ defined in Eqs (6.13) are represented in the base $\mathbf{e}_v^*$. Since $\boldsymbol{e}_{v1}$, $\boldsymbol{e}_{v2}$ and $\boldsymbol{e}_{v3}$ are the unit vectors of the orthonormal $G\xi\eta\zeta$ reference frame, λ_1, λ_2 and λ_3 are, for the case considered, the values of the reaction forces applied at the point P.

Let us illustrate now the concepts discussed in Sections 4 and 5. In order to attain this, consider again the constraint equations in the form (6.1), and first release the analysis from the geometric constraint (6.1c). It can be done by choosing the independent generalized coordinates $\mathbf{q} = [x_P, y_P, \phi, \theta, \psi]^{\mathsf{T}}$. Defining

$\mathbf{u} = \dot{\mathbf{q}} = [\dot{x}_P, \dot{y}_P, \dot{\phi}, \dot{\theta}, \dot{\psi}]^\top$, the matrix $\mathbf{D}$ referring to the definition (4.10) comes from

$$\mathbf{v} = \mathbf{B}\dot{\mathbf{x}} = \tilde{\mathbf{B}}\dot{\mathbf{q}} = \tilde{\mathbf{B}}\mathbf{u} = \mathbf{D}^\top\mathbf{u} \tag{6.14}$$

where $\tilde{\mathbf{B}}$ is the 6×5 matrix created from $\mathbf{B}$ (defined in Eqs (6.3)) by omitting the 3th column. After some manipulations, the dynamic equations in the form (2.8a), where $\mathbf{M}$ and $\mathbf{h}^*$ are defined by Eqs (6.9) and (6.10), and $\mathbf{C}$ and c_0^* are

$$\mathbf{C} = [0, \cos\theta, \sin\theta, -r\sin\theta, 0, 0] \tag{6.15}$$

$$\mathbf{c}_0^* = \dot{\theta}(\dot{x}_P \sin\phi - \dot{y}_P\cos\phi) \tag{6.16}$$

can be transformed to the form (4.10), where $\mathbf{M}_d = \tilde{\mathbf{B}}^\top\mathbf{M}\tilde{\mathbf{B}}$ and $\mathbf{h}_u^* = \tilde{\mathbf{B}}^\top\mathbf{h}^* - \tilde{\mathbf{B}}^\top\mathbf{M}(\tilde{\mathbf{B}})^{\cdot}\mathbf{u}$ are

$$\mathbf{M}_d = \begin{bmatrix} m & 0 & -mr\sin\theta\cos\phi & -mr\cos\theta\sin\phi & 0 \\ * & m & -mr\sin\theta\sin\phi & mr\cos\theta\cos\phi & 0 \\ * & * & (mr^2+J)\sin^2\theta + J & 0 & 2J\sin\theta \\ * & * & * & mr^2+J & 0 \\ * & * & * & * & 2J \end{bmatrix} \tag{6.17}$$

$$\mathbf{h}_u^* = \begin{bmatrix} -mr[(\dot{\phi}^2+\dot{\theta}^2)\sin\theta\sin\phi - 2\dot{\phi}\dot{\theta}\cos\theta\cos\phi] \\ mr[(\dot{\phi}^2+\dot{\theta}^2)\sin\theta\cos\phi + 2\dot{\phi}\dot{\theta}\cos\theta\sin\phi] \\ -2(mr^2+J)\dot{\phi}\dot{\theta}\sin\theta\cos\theta - 2J\dot{\psi}\dot{\theta}\cos\theta \\ (mr^2-J)\dot{\phi}^2\sin\theta\cos\theta + 2J\dot{\phi}(\dot{\psi}+\dot{\phi}\sin\theta)\sin\theta + mg\cos\theta \\ -2J\dot{\phi}\dot{\theta}\cos\theta \end{bmatrix} \tag{6.18}$$

and $(*)$ in Eq (6.17) denote the symmetric entries. Finally, following Eqs (3.6) and (3.7), the reaction of the constraint (6.1c) is

$$\lambda = -mr\dot{\phi}(\dot{\psi}\cos\theta + \dot{\phi}\sin\theta\cos\theta)\sin\theta + mg \tag{6.19}$$

$$\mathbf{r}^* = \mathbf{C}^\top\lambda \tag{6.20}$$

where $\mathbf{C}$ is defined in Eq (6.15).

After the exclusion of the holonomic constraint (6.1c), the starting point of the following analysis are the equations

$$\begin{aligned} &a) \qquad \mathbf{M}_d\dot{\mathbf{u}} = \mathbf{h}_u^* + \mathbf{C}^\top\lambda \\ &b) \qquad \dot{\mathbf{q}} = \mathbf{u} \\ &c) \qquad \mathbf{C}\dot{\mathbf{u}} + \mathbf{c}_0^* = \mathbf{0} \end{aligned} \tag{6.21}$$

where $\mathbf{q} = [x_P, y_P, \phi, \theta, \psi]^\top$, $\mathbf{u} = [\dot{x}_P, \dot{y}_P, \dot{\phi}, \dot{\theta}, \dot{\psi}]^\top$, $\mathbf{M}_d$ and $\mathbf{h}_u^*$ are defined in Eqs (6.17) and (6.18), and

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & r\cos\phi \\ 0 & 1 & 0 & 0 & r\sin\phi \end{bmatrix} \tag{6.22}$$

$$\mathbf{c}_0^* = \begin{bmatrix} -r\dot{\phi}\dot{\psi}\sin\phi \\ r\dot{\phi}\dot{\psi}\cos\phi \end{bmatrix} \tag{6.23}$$

Following the formulation given in Section 5, the orthogonal complement matrix $\mathbf{D}$ can be constructed as

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -r\cos\phi & -r\sin\phi & 0 & 0 & 1 \end{bmatrix} \tag{6.24}$$

and the relationship between the dependent quasi-velocities $\mathbf{u}$ and the independent quasi velocities $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]^\top$ is

$$\mathbf{u} = \mathbf{D}^\top \boldsymbol{\omega} \tag{6.25}$$

Then, the final dynamic equations in the independent quasi-velocities $\boldsymbol{\omega}$, referring to Eqs (5.6), are

$$\mathbf{M}_\omega \dot{\omega} = \mathbf{h}_\omega^* \tag{6.26}$$

where

$$\mathbf{M}_\omega = \begin{bmatrix} (mr^2 + J)\sin^2\theta + J & 0 & (mr^2 + J)\sin\theta \\ 0 & mr^2 + J & 0 \\ (mr^2 + J)\sin\theta & 0 & mr^2 + 2J \end{bmatrix} \tag{6.27}$$

$$\mathbf{h}_\omega^* = \begin{bmatrix} mr^2\omega_2\omega_3\cos\theta \\ (mr^2 + J)\omega_1^2\sin\theta\cos\theta + (mr^2 + 2J)\omega_1\omega_1\cos\theta + mg\cos\theta \\ -mr^2\omega_2\omega_1\cos\theta \end{bmatrix} \tag{6.28}$$

The Eqs (6.26) should be completed with the differential kinematic equations, referring to Eqs (5.7), as follows

$$\dot{\mathbf{x}} = \mathbf{D}^\top \omega \tag{6.29}$$

Finally, the reactions of the constraints (6.1a) and (6.1b) can be found using the relations (3.6) and (3.7). However, since the manipulations heed the inversion of $\mathbf{M}_d$ defined in Eqs (6.21), which is rather a laborious task for analytical calculations, the formulae for determination of the constraint reactions will not be reported here.

7. Concluding remarks

The advantages of the reported projection method approach to the dynamic analysis of constrained systems can be summarized as follows

- Compact mathematical formulation – tensor/matrix notation.
- Geometrical insight into the problems solved and its intuitive appeal as a direct generalization of methods used in simple dynamics problems.
- Unified treatment of systems subject to holonomic and/or nonholonomic constraints.
- The formulation enables one to carry out analyses in generalized coordinates and/or quasi–velocities, without paying any attention to distinguishing these cases.
- Constraint reaction–free equations of motion as well as formulae for determination of the associated constraint reactions are obtained.
- The method comprises many other well known approaches to the constrained dynamic analysis, and seems to be more general.
- No scalar function of velocity energy (acceleration energy, Hamiltonian function,...) needs to be introduced and then differentiated, which is often a laborious task.

An evident drawback in applications of the method is, however, the necessity of determination of the orthogonal complement matrix $\mathbf{D}$ (only for holonomic systems, when Eqs (4.2) is formulated à priori, $\mathbf{D}$ can be obtained by simple mathematical transformations). As shown in Section 6, for small systems $\mathbf{D}$ can be guessed. For large systems, however, the problem may be more complicated. This problem will not be discussed here.

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Appendix

Consider an n–dimensional metric space. A vector $\boldsymbol{a}$ can be expressed by its contravariant components $\mathbf{a} = [a_1, ..., a_n]^\top$ in the covariant base of this space $\mathbf{e} = [\boldsymbol{e}_1, ..., \boldsymbol{e}_n]^\top$, or by its covariant components $\mathbf{a}^* = [a_1^*, ..., a_n^*]^\top$ in the contravariant base $\mathbf{e}^* = [\boldsymbol{e}_1^*, ..., \boldsymbol{e}_n^*]^\top$ (see e.g. [22,26]), i.e.

$$\boldsymbol{a} = \mathbf{a}^\top \mathbf{e} = \mathbf{a}^{*\top} \mathbf{e}^* \tag{A.1}$$

With the use of the metric tensor matrix $\mathbf{M}$ of the space,

$$\mathbf{M} = \mathbf{e}\mathbf{e}^\top \tag{A.2}$$

the interdependences between the contravariant and covariant vector components and base vectors are as follows

$$\mathbf{e} = \mathbf{M}\mathbf{e}^* \qquad \mathbf{a}^* = \mathbf{M}\mathbf{a} \tag{A.3}$$

A dot product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written in four possible ways

$$\boldsymbol{a} \circ \boldsymbol{b} = \mathbf{a}^\top \mathbf{M}\mathbf{b} = \mathbf{a}^\top \mathbf{b} = \mathbf{a}^{*\top} \mathbf{M}^{-1}\mathbf{b}^* = \mathbf{a}^{*\top}\mathbf{b} \tag{A.4}$$

and the orthogonality condition is defined as $\boldsymbol{a} \circ \boldsymbol{b} = 0$. When the reference frame changes from a given one to another, denoted by $\hat{()}$, the transformation formulae are

$$\hat{\mathbf{e}} = \mathbf{B}^\top \mathbf{e} \qquad \mathbf{e}^* = \mathbf{B}\hat{\mathbf{e}}^*$$
$$\hat{\mathbf{a}}^* = \mathbf{B}^\top \mathbf{a}^* \qquad \mathbf{a} = \mathbf{B}\hat{\mathbf{a}} \tag{A.5}$$

and the metric tensor matrix of the base $\hat{\mathbf{e}}$ is

$$\hat{\mathbf{M}} = \mathbf{B}^\top \mathbf{M}\mathbf{B} \tag{A.6}$$

where $\mathbf{B}$ is the $n \times n$ transformation matrix.

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Uogólniona metoda rzutowania dla dynamicznej analizy nieswobodnych układów mechanicznych

Streszczenie

Istota prezentowanej metody polega na rozdzieleniu przestrzeni konfiguracji układu na podprzestrzenie styczną i ortogonalną, zdefiniowane względem hiperpowierzchni więzów. Rzut wyjściowych (zależnych od reakcji więzów) dynamicznych równań ruchu do podprzestrzeni stycznej prowadzi do uwolnionych od reakcji więzów (kanonicznych) równań ruchu, natomiast rzut ortogonalny pozwala na wyznaczenie reakcji więzów. Zaproponowane macierzowo/tensorowo/wektorowe sformułowanie matematyczne metody przystosowano do prowadzenia analizy we współrzędnych uogólnionych lub/oraz quasi-prędkościach oraz dla układów skrępowanych więzami holonomicznymi lub/oraz nieholonomicznymi. Dyskutuje się uproszczenia w analizie wynikające z użycia niezależnych współrzędnych/prędkości. Rozważania teoretyczne zilustrowano przykładem.

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