MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
1, 30, 1992

ON A TWO-FIELD FINITE ELEMENT APPROXIMATION IN ELASTICITY

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In this paper relatively well-known two-field (mixed) finite element approximations in the mechanics of solids are studied. However, in the proposed approach a significant novelty (in the finite element methods) is introduced – the treatment of the boundary traction constraints as essential boundary conditions. Although multifield approximations are in no way limited to the realm of classical elasticity, to fix ideas we are considering the linear elastic domain, where the comparable analytical solutions are available.

1. Introduction

The main difference between the one-field (e.g. the classical, "displacement" type) and the two-field finite element approximations is the treatment of the boundary conditions and the inter-element continuity.

In the case of one-field approximation only the boundary conditions for this particular field can be treated as essential ones (e.g. in the displacement method – the displacement boundary conditions). The boundary conditions for the other fields in play, appear as natural ones.

A fulfillment of the boundary conditions on the element interfaces is a necessary, but not always a sufficient condition for the inter-element continuity. It should be noted however that, especially for the homogeneous fields, it is easier to satisfy the latter condition than the former one. Hence, in the known applications (e.g. [1], [2]), the mechanical inter-element conditions are treated as the natural ones.

From the computational point of view, it is relatively easy to maintain the stress continuity, if the stress and the displacement fields are independently interpolated. This approach leads straightforward to the mixed (two-field) finite element approximation. Obviously, to control the stress continuity, there should exist a common coordinate system for the part of a body, approximated by the mixed finite elements. It was shown, in [3], that an algebraic system, resulting from these interpolations (approximations), can be efficiently solved using the block factorization approach, a slight extension of the natural factor procedure, developed by Argyris et al. [4].

Furthermore, if it is needed, in addition to the displacement constraints, to treat also the stress constraints as essential boundary conditions, it is necessary to introduce the special coordinate systems, having coordinate surfaces coincident with the boundary surfaces (and interfaces) of a body. In this special case it is possible to determine some of the stress tensor components from the boundary tractions at the point of consideration.

As it can be seen in the paper of Cantin et al. [5], where the stress boundary conditions were satisfied in an iterative manner, such an approach can significantly improve the results, especially at the vicinity of a boundary or when the number of elements is small.

Although the authors of [1] and [2] were aware of this possibility, as it can be concluded by the careful reading of these papers, none of them used it practically, and in both papers the mechanical boundary conditions are treated as the natural ones.

Hence, the direct treatment of the stress constraints as essential boundary conditions remained an unsolved problem, although it is a legal procedure from the theory of Galerkin approximations point of view. Also, on the basis of the computational results [5], this approach is a very promising way towards improvement of the performances of mixed finite elements.

The present paper is devoted to the theoretical and practical aspects of introduction of the mechanical boundary conditions as the essential ones in the two-field finite element analysis.

2. Field equations of the linear elasticity

Let us consider a complete set of the field equations in the linear elasticity, where

$$\operatorname{div}\mathbf{t} + (\boldsymbol{f} - \rho\boldsymbol{a}) = 0 \qquad \text{in } \mathcal{B} \tag{2.1}$$

$$\boldsymbol{e} - \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathsf{T}}) = \boldsymbol{0} \qquad \text{in } \mathcal{B} \tag{2.2}$$

$$\mathbf{t} - \mathsf{E}\cdot\boldsymbol{e}^{-1} = \boldsymbol{0} \qquad \text{in } \mathcal{B} \tag{2.3}$$

$$\mathbf{t}\cdot\boldsymbol{n} - \boldsymbol{p} = \boldsymbol{0} \qquad \text{on } \partial\mathcal{B}_t \tag{2.4}$$

$$\boldsymbol{u} - \boldsymbol{w} = \boldsymbol{0} \qquad \text{on } \partial\mathcal{B}_u \tag{2.5}$$

are respectively the equations of motion, strain–displacement and stress–strain relationships, boundary traction conditions and geometric boundary conditions.

In these expressions, $\mathbf{t}$, ρ, $\boldsymbol{f}$ and a stand for the stress tensor, the mass density, the vector of the body forces, and acceleration vector, respectively. $\mathcal{B}$ describes an open, bounded domain of the elastic body, while $\boldsymbol{n}$ denotes the unit normal vector to the boundary $\partial\mathcal{B}$; $\partial\mathcal{B}_t$ and $\partial\mathcal{B}_u$ stand for the portions of $\partial\mathcal{B}$ where the stresses or the displacements are prescibed, respectively. Additionally, $\boldsymbol{p}$, $\boldsymbol{u}$, $\boldsymbol{w}$, $\boldsymbol{e}$, $\mathbf{E}$ denote the vector of the prescribed boundary tractions, the displacement vector, the vector of the prescibed displacements, the strain tensor, and the elasticity tensor, respectively.

3. The boundary stress constraints

Let us consider the boundary traction condition (2.4), and rewrite the stress tensor $\mathbf{t}$ in a dyadic form

$$t^{pq}\boldsymbol{g}_p \otimes \boldsymbol{g}_q \cdot \boldsymbol{n} = \boldsymbol{p} \tag{3.1}$$

where g_p denote the base vectors of the coordinates x^p. As it has been noted in the Introduction, the special common coordinates x^p should be chosen in such a way that the corresponding coordinate surfaces coincide with the boundary surfaces of a body. For instance, if $x^{(r)}$ = const is the equation of a boundary surface, then

$$\boldsymbol{g}_q \cdot \boldsymbol{n} = \begin{cases} 0 & \text{if} \quad q \neq r \\ n_{(r)} & \text{if} \quad q = r \end{cases} \tag{3.2}$$

Note that always

$$|n_{(r)}| > 0 \tag{3.3}$$

because

$$n_{(r)} = |\boldsymbol{g}_{(r)}||\boldsymbol{n}| \cos \vartheta \tag{3.4}$$

where ϑ denotes the angle measured from the out–of $x^{(r)}$–surface base vector $\boldsymbol{g}_{(r)}$ to $\boldsymbol{n}$. More specifically, if the coordinates x^p are the physical ones, i.e. $|\boldsymbol{g}_{(r)}| \equiv 1$, we can write that

$$1 \geq |n_{(r)}| > 0. \tag{3.5}$$

Finally, if x^p are the orthogonal physical (e.g. Cartesian) coordinates

$$|n_{(r)}| \equiv 1. \tag{3.6}$$

Anyhow, from (3.1) it follows that

$$t^{p(r)}\boldsymbol{g}_p n_{(r)} = \boldsymbol{p}$$

or, after the elimination of g^q and multiplication by $n^{(r)} = 1/n_{(r)}$

$$t^{q(r)} = p^q n^{(r)} \qquad \text{for} \quad x^{(r)} = \text{const.} \tag{3.7}$$

Hence, if the boundary tractions p^q are known, one can easily determine the corresponding stresses $t^{q(r)}$ on the boundary surface $x^{(r)} = \text{const.}$

4. Finite element approximations of the field equations

4.1. The equations of equilibrium

For the sake of clarity and simplicity let us consider only the homogeneous part of the Cauchy's equations (2.1), i.e. the equations of equilibrium

$$\text{div}\mathbf{t} = \mathbf{0} \qquad \text{in} \quad B. \tag{4.1}$$

Using the Galerkin procedure, one can seek for the weak solution of (4.1) from the scalar product

$$\int_B \mathbf{u} \cdot \text{div}\mathbf{t} dV = 0 \tag{4.2}$$

where the displacement vector u stands for a test function. The next step are the finite element interpolations (approximations) of the vector test functions $\mathbf{u}$, and the tensor trial functions $\mathbf{t}$

$$\mathbf{u} = P^K \mathbf{u}_K \qquad \mathbf{t} = S_L \mathbf{t}^L. \tag{4.3}$$

In these expressions, $\mathbf{u}_K$ and $\mathbf{t}^L$ denote the nodal values of the vector $\mathbf{u}$ and tensor $\mathbf{t}$, respectively. Accordingly, P^K and S_L express the corresponding values of the interpolation functions, connecting the displacements and stresses at an arbitrary point in $\mathcal{E}$ (the body of an element), and the nodal values of these quantities.

Although in principle, the possibility of use of the tensor interpolations of type (4.3) has been pointed out a relatively long time ago by Oden [6], scalar interpolation is still of common use.

Only recently it has been shown by Drašković [7], that, to maintain the finite element approximations invariant for the coordinate transformations, tensorial character of these approximations should be strictly holded.

Obviously, if the common coordinates are the Cartesian ones, there is no difference between the scalar and tensorial approximations. In the case of classical, displacement type one–field finite element approximation, it is usually sufficient to use the Cartesian common coordinates, and hence the scalar approximation.

However, as it has been shown in the Section 3 of this paper, it is, in general case, necessary to use the curvilinear coordinates to satisfy the boundary traction conditions as the essential ones. Consequently, the tensorial approximations are unavoidable.

Let us note further that, in a dyadic form, the nodal displacements and stresses can be written as

$$\mathbf{u}_K = u_{Kq}\mathbf{g}^{(K)q} \qquad \mathbf{t}^L = t^{Lst}\mathbf{g}_{(L)s} \otimes \mathbf{g}_{(L)t} \tag{4.4}$$

where the indices in parentheses denote the nodalvalues of the base vectors. The finite element interpolation (approximation) of the stress divergence deserves the special care. Note that, in the interior of an element

$$\mathrm{div}\mathbf{t} = \nabla S_L \mathbf{t}^L = S_{Lr}\mathbf{g}^r\mathbf{t}^L \tag{4.5}$$

where $S_{Lr} = \partial S_L/\partial x^r$.

Hence, the equation (4.2) can be rewritten as

$$u_{Kq}\sum_e \mathbf{g}^{(K)q}\int_{\mathcal{E}} p^K S_{Lr}\mathbf{g}^r t^{Lst}\mathbf{g}_{(L)s} \otimes \mathbf{g}_{(L)t} dV = 0.$$

However, since this expression should be valid for any value of the nodal displacements, and after the contraction of the base vectors, one can write that

$$\sum_e \mathbf{g}^{(K)q}_{(L)s}\int_{\mathcal{E}} p^K S_{Lr} g^r_{(L)t} dV t^{Lst} = 0 \tag{4.6}$$

where $g^{(K)q}_{(L)s}$ and $g^r_{(L)t}$ denote the Euclidean shifters, integration is performed over the each element of body $\mathcal{E}$, and the summation is perfomed for all the elements of a system. Obviously, one can rewrite the coefficients at t^{Lst} in a symbolic form A^{Kq}_{Lst} and hence

$$A^{Kq}_{Lst}t^{Lst} = 0. \tag{4.7}$$

4.2. The strain–displacement and the stress–strain relationships

Because the elastic constitutive equations (2.3) are invertible, one can write that

$$\mathbf{e} = \mathbf{C}\cdot\mathbf{t}^{-1}. \tag{4.8}$$

From the comparison of (2.2) and (4.8) it follows that

$$\mathbf{C}\cdot\mathbf{t}^{-1} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}). \tag{4.9}$$

Because of the symmetry of $\mathbf{t}$, the weak solution of (4.9) can be determined with the help of a relatively simple expression

$$\int_{\mathcal{B}} \mathbf{t}\cdot(\mathbf{C}\cdot\mathbf{t}^{-1} - \nabla\mathbf{u})dV = 0. \tag{4.10}$$

Like the divergence of a tensor, the gradient of a vector requires a special attention in the finite element interpolation (approximation) of a weak solution

$$\nabla\mathbf{u} = \nabla P^K u_K = P^K_{s}\mathbf{g}^s\mathbf{u}_K. \tag{4.11}$$

Now, the expression (4.10) can be rewritten as

$$t^{Lst}\sum_e \mathbf{g}_{(L)s}\mathbf{g}_{(L)t}\int_{\mathcal{E}} S_L\Big(C_{ijkl}\mathbf{g}^i\otimes\mathbf{g}^j\otimes\mathbf{g}^k\otimes\mathbf{g}^l S_M t^{Mmn}\mathbf{g}_{(M)m}\otimes\mathbf{g}_{(M)n} -$$

$$-P^K_s\mathbf{g}^s\otimes\mathbf{g}^{(K)q}u_{Kq}\Big)dV = 0.$$

Bearing in mind that the equation given above should be valid for any value of t^{Lst}, after performing the contraction of the corresponding base vectors one obtains, in a symbolical form

$$C_{LstMmn}t^{Mmn} - B^{Kq}_{Lst}u_{Kq} = 0. \tag{4.12}$$

In this expression

$$C_{LstMmn} = \sum_e\int_{\mathcal{E}} S_L g^i_{(L)s}g^j_{(L)t}C_{ijkl}g^k_{(M)m}g^l_{(M)n}S_M dV \tag{4.13}$$

$$B^{Kq}_{Lst} = \sum_e\int_{\mathcal{E}} S_L P^K_s g^s_{(L)s}dV g^{(K)q}_{(L)t}. \tag{4.14}$$

4.3. The matrix equations of a system

The equations (4.12) and (4.7) can be rewritten in a matrix form

$$\begin{aligned}\mathbf{Ct} - \mathbf{Bu} &= \mathbf{0}\\ \mathbf{At} &= \mathbf{0}.\end{aligned} \tag{4.15}$$

Note that, in accordance with (2.4) some boundary displacements, and, due to (2.4) also some stresses at the boundary of a body, are prescribed. Hence, one can decompose (4.15) into two parts

- for the unknown (variable) stresses $\mathbf{t}_v$ and displacements $\mathbf{u}_v$, and
- for the known (prescibed) ones ($\mathbf{t}_p$ and $\mathbf{u}_p$), to get

$$\begin{bmatrix} \mathbf{C}_{vv} & \mathbf{C}_{vp} \\ \mathbf{C}_{pv} & \mathbf{C}_{pp} \end{bmatrix} \begin{bmatrix} \mathbf{t}_v \\ \mathbf{t}_p \end{bmatrix} - \begin{bmatrix} \mathbf{B}_{vv} & \mathbf{B}_{vp} \\ \mathbf{B}_{pv} & \mathbf{B}_{pp} \end{bmatrix} \begin{bmatrix} \mathbf{u}_v \\ \mathbf{u}_p \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{A}_{vv} & \mathbf{A}_{vp} \\ \mathbf{A}_{pv} & \mathbf{A}_{pp} \end{bmatrix} \begin{bmatrix} \mathbf{t}_v \\ \mathbf{t}_p \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{4.16}$$

However, because $\mathbf{t}_p$ and $\mathbf{u}_p$ are the prescibed values, it is sufficient to keep only first upper-rows in the above block-matrix equations, and (4.16) reduces to

$$\mathbf{C}_{vv}\mathbf{t}_v - \mathbf{B}_{vv}\mathbf{u}_v = \mathbf{B}_{vp}\mathbf{u}_p - \mathbf{C}_{vp}\mathbf{t}_p \tag{4.17}$$
$$\mathbf{A}_{vv}\mathbf{t}_v = -\mathbf{A}_{vp}\mathbf{t}_p. \tag{4.18}$$

Let us consider now the elements of the left-hand side of the equation (4.18)

$$A^{Kq}_{Lst} t^{Lst}_{(\bullet)} = \sum_e g^{(K)q}_{(L)s} \int_{\mathcal{E}} P^K S_{Lr} g^r_{(L)t} dV t^{Lst}_{(\bullet)}. \tag{4.19}$$

The right-hand side of this expression can be rewritten in such a way that

$$\begin{aligned} A^{Kq}_{Lst} t^{Lst}_{(\bullet)} &= \sum_e g^{(K)q}_{(L)s} \int_{\mathcal{E}} (P^K S_L g^r_{(L)t})_{,r} dV t^{Lst}_{(\bullet)} - \\ &- \sum_e g^{(K)q}_{(L)s} \int_{\mathcal{E}} P^K_{,r} S_L g^r_{(L)t} dV t^{Lst}_{(\bullet)}. \end{aligned} \tag{4.20}$$

After the application of the divergence theorem to the first term of the right-hand side of (4.20), one obtains the relationship

$$\begin{aligned} A^{Kq}_{Lst} t^{Lst}_{(\bullet)} &= \sum_e g^{(K)q}_{(L)s} \int_{\partial\mathcal{E}} P^K S_L g^r_{(L)t} n_r dV t^{Lst}_{(\bullet)} - \\ &- \sum_e g^{(K)q}_{(L)s} \int_{\mathcal{E}} P^K_{,r} S_L g^r_{(L)t} dV t^{Lst}_{(\bullet)}. \end{aligned} \tag{4.21}$$

Furthermore, using the interpolation rule for the stresses (4.3b), it can be found from the boundary condtitions (2.4) that the kernel of the boundary integral in (4.21) can be transformed in a following way

$$S_L g^r_{(L)t} n_r t^{Lst}_{(\bullet)} = p^k_{(\bullet)} g^{(L)s}_k. \tag{4.22}$$

Note that, by definition, $\mathbf{A}_{vv}$ in (4.18) connects the unknown nodal displacements $\mathbf{u}_v$ and unknown nodal stresses $\mathbf{t}_v$. However, at the outer part of a boundary, where the displacements are unknown (variable), $\partial\mathcal{E} \subset \partial\mathcal{B}_t$ the boundary tractions are known (prescribed), and hence "unknown" boundary tractions $p^k_{(v)}$ do not exist. In addition, at the interelement boundaries, $\partial\mathcal{E} \subset \partial\mathcal{E}_i$, the boundary tractions are in equilibrium, in the accordance with the definition of a two-field model. Consequently, the first element of the right-hand side of (4.21) can be written as

$$\sum_e g^{(K)q}_{(L)s} \int_{\partial\mathcal{E}} P^K p^k_{(v)} g^{(L)s}_k dV \equiv 0 \tag{4.23}$$

and finally

$$A^{Kq}_{Lst} = -\sum_e g^{(K)q}_{(L)s} \int_{\mathcal{E}} P^K_r S_L g^r_{(L)t} dV. \tag{4.24}$$

From the comparison of this expression with (4.14) it follows that

$$\left[A^{Kq}_{Lst}\right]_{vv} \equiv -\left[B^{Kq}_{Lst}\right]^{\mathsf{T}}_{vv} \tag{4.25}$$

i.e.

$$\mathbf{A}_{vv} \equiv -\mathbf{B}^{\mathsf{T}}_{vv}. \tag{4.26}$$

Hence the system of equations (4.17), (4.18) is symmetric. It is convenient to rewrite this system in a block-matrix form

$$\begin{bmatrix} \mathbf{C}_{vv} & -\mathbf{B}_{vv} \\ -\mathbf{B}^{\mathsf{T}}_{vv} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{t}_v \\ \mathbf{u}_v \end{bmatrix} = \begin{bmatrix} -\mathbf{C}_{vp} & \mathbf{B}_{vp} \\ -\mathbf{A}_{vp} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{t}_p \\ \mathbf{u}_p \end{bmatrix}. \tag{4.27}$$

This result is very important from the practical point of view. The left-hand side of (4.27) has a symmetric form, typical for the mixed finite element formulations. Hence, it can be processed with the aid of any solving tool suitable for mixed systems, or reduced to the displacement-type set of equations [3].

It is useful to note that, in the special case, the entries of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ matrices respectively, take a relatively simple forms

$$A^{Kq}_{Lst} = \sum_e \int_{\mathcal{E}} P^K \delta^q_s S_{L,t} dV \tag{4.28}$$

$$B^{Kq}_{Lst} = \sum_e \int_{\mathcal{E}} S_L \delta^q_t P^K_{,s} dV \tag{4.29}$$

$$C_{LstMmn} = \sum_e \int_{\mathcal{E}} S_L C_{stmn} S_M dV \tag{4.30}$$

where δ^q_s is a Kronecker delta-symbol.

5. An illustrative example

The purpose of the present, very simple, hand–manageable but still meaningful finite element example, is to show how the proposed procedure works, in order to clarify the theoretical considerations, and to give some idea of the solution to the practical problems.

The exercise to be considered here is the cantilever beam (Fig.1.), under the shear loading triangularly distributed. Total loading of the beam is equal to the resultant force Q.

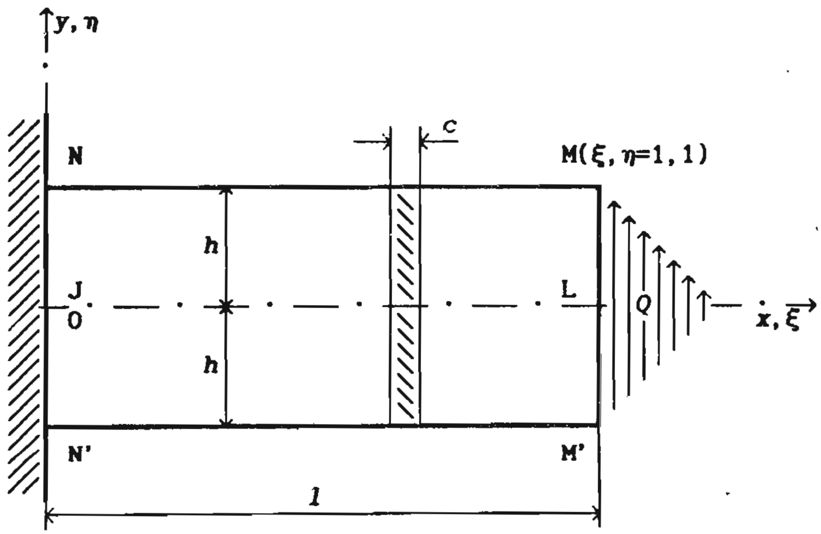

Fig. 1.

5.1. Coordinate systems and interpolation functions

Due to the symmetry of the system under consideration, only a half of it, say the element $JLMN$, can be considered. Its parametric coordinates ξ, η incidentally do overlap the common Cartesian coordinates $x = x_1$, $y = x_2$. However, when $x = l$, $y = h$ then the parametric nondimensional coordinates take the values $\xi = 1$, $\eta = 1$, respectively. In accordance with the theory exposed in this paper, the derivatives of the interpolation functions should be taken with respect to the common coordinates, i.e.

$$P_1^K = \frac{\partial P^K}{\partial \xi}\frac{\partial \xi}{\partial x} \qquad P_2^K = \frac{\partial P^K}{\partial \eta}\frac{\partial \eta}{\partial y}. \tag{5.1}$$

In this particular example, we will use the same, kind of bilinear, interpolation functions for both the stresses and the displacements. In addition to the apparent simplicity of this choice, it is, in accordance with [8] and [9], which do guarantee the uniqueness of the mixed finite element solutions. Let us consider now the values of the interpolation functions and their derivatives, corresponding to the nodes, J, L, M and N, respectively

$$\begin{aligned}
P^J &= (1-\xi)(1-\eta) & P_1^J &= -\frac{1-\eta}{l} & P_2^J &= -\frac{1-\xi}{h} \\
P^L &= \xi(1-\eta) & P_1^L &= \frac{1-\eta}{l} & P_2^L &= -\frac{\xi}{h} \\
P^M &= \xi\eta & P_1^M &= \frac{\eta}{l} & P_2^M &= \frac{\xi}{h} \\
P^N &= (1-\xi)\eta & P_1^N &= \frac{-\eta}{l} & P_2^N &= \frac{1-\xi}{h}
\end{aligned} \tag{5.2}$$

5.2. Boundary conditions

In addition to the interpolation functions and their derivatives, the next group of the necessary input data consists of the prescribed nodal displacements and stresses, determined from the boundary conditions and tabulated as follows

Table 1. Nodal displacements and stresses

node	displacement		stress		
	u_1	u_2	t^{11}	t^{12}	t^{22}
J	0	0	0	t^{J12}	0
L	0	u_{L2}	0	Q/hc	0
M	u_{M1}	$u_{M2} = u_{L2}$	0	0	0
N	0	0	t^{N11}	0	0

According to the Table 1 it is evident that the number of the nodal variables, due to the boundary conditions, symmetry of the body, and the skew symmetry of its loading, reduces to only four, i.e. to the dispalcements u_{L2} and u_{M1}, and the stresses t^{J12} and t^{N11}. Hence, the system to be solved, (4.27), is of the size 4×4. One can easily construct submatrices of the system, on the basis of the expressions (3.28)–(4.2). Note also that c in Table 1 denote the thickness of a beam.

5.3. On the symmetry of a system

However, it is also possible to use the system (4.17) and (4.18), appearing to be asymmetric. For the better understanding of the symmetry properties of the equations under consideration, let us perform the manual calculations of the entries of $\mathbf{A}_{vv}$ and $\mathbf{B}_{vv}$ in (4.17) and (4.18).

$$\mathbf{A}_{vv} = \begin{bmatrix} A^{L2}_{J21} & 0 \\ A^{M1}_{J12} & A^{M1}_{N11} \\ A^{M2}_{J21} & 0 \end{bmatrix} \qquad \mathbf{B}_{vv} = \begin{bmatrix} B^{L2}_{J21} & B^{M1}_{J12} & B^{M2}_{J21} \\ 0 & B^{M1}_{N11} & 0 \end{bmatrix}. \tag{5.3}$$

According to these expressions it is evident that overall topology of $\mathbf{A}_{vv}$ and the transpose of $\mathbf{B}_{vv}$ are identical, i.e. these matrices have the nonzero terms, connecting in both cases the same nodes, at the same positions. The next step is to replace now the symbolic notions for the terms in (5.3) by their values. Then

$$\mathbf{A}_{vv} = \frac{V}{l}\int_0^1\int_0^1 \begin{bmatrix} -(1-\eta)^2\xi & 0 \\ -\eta\xi(1-\xi)\frac{l}{h} & -\eta^2\xi \\ -\eta(1-\eta)\xi & 0 \end{bmatrix} d\xi d\eta \tag{5.4}$$

$$\mathbf{B}_{vv} = \frac{V}{l}\int_0^1\int_0^1 \begin{bmatrix} (1-\eta)^2(1-\xi) & (1-\eta)\xi(1-\xi)\frac{l}{h} & \eta(1-\eta)(1-\xi) \\ 0 & \eta^2(1-\xi) & 0 \end{bmatrix} d\xi d\eta \tag{5.5}$$

where $V = lhc$ is the volume of the element $JLMN$. Now, it should be noted that the corresponding terms in (5.4) and (5.5) are entirely different. However, after the integration is performed, these terms become identical (to a sign), and (4.26) is satisfied, as it should be expected on the basis of the theoretical considerations in Section 4.3. So

$$\mathbf{A}_{vv} = -\frac{V}{12l}\begin{bmatrix} 2 & 0 \\ \frac{l}{h} & 2 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{B}_{vv} = \frac{V}{12l}\begin{bmatrix} 2 & \frac{l}{h} & 1 \\ 0 & 2 & 0 \end{bmatrix}. \tag{5.6}$$

In a similar way one can calculate other matrices of the system (4.17), (4.18), (or (4.27)). In the calculation of the elasticity matrices, the plane stress is assumed. Finally, a system to be solved reads

$$\begin{bmatrix} \frac{V}{9E}\begin{bmatrix} 2(1+\nu) & 0 \\ 0 & 1 \end{bmatrix} & -\frac{V}{12l}\begin{bmatrix} 3 & l/h \\ 0 & 2 \end{bmatrix} \\ -\frac{V}{12l}\begin{bmatrix} 3 & 0 \\ l/h & 2 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} t^{J12} \\ t^{N11} \\ u^{L2} \\ u^{M1} \end{bmatrix} = -\frac{Q}{hc}\begin{bmatrix} \frac{V}{18E}\begin{bmatrix} 2(1+\nu) \\ 0 \end{bmatrix} \\ \frac{V}{12l}\begin{bmatrix} 3 \\ -2l/h \end{bmatrix} \end{bmatrix} \tag{5.7}$$

5.4. The results and discussion

Due to the simplicity of the problem model, it is possible to find closed form of solutions of (5.7), and to compare the results with the existing ones.

The shear at the root of the beam

From the third equation of a system, it directly follows that $t^{J12} = Q/hc$, i.e. the shear stress is homogeneous along the beam, as it should be expected.

The direct stress at the extreme fiber

From the fourth equation it follows that $t^{N11} = -3Ql/2h^2c$, exactly as in ETB (Engineering Bending Theory). No wonder, because the interpolation functions for stresses are linear, as in ETB.

The displacement at the tip of beam

Applying the second and first equations in (5.7), one finds that

$$u_{L2} = \frac{1}{3E}\left(\frac{l}{h}\right)^3 \frac{Q}{t}\left[1 + 4(1+\nu)\left(\frac{h^2}{l}\right)\right]. \qquad (5.8)$$

At this point, it is useful to disccuss (5.8) in comparison with some well-known expressions. Let us cite the ETB, Gurney [10] and Rankine - Grashoff [11] solutions, respectively

$$u_{L2} = \frac{1}{2E}\left(\frac{l}{h}\right)^3 \frac{Q}{t} \qquad (5.9)$$

$$u_{L2} = \frac{1}{2E}\left(\frac{l}{h}\right)^3 \frac{Q}{t}\left[1 + 2(1+\nu)\left(\frac{h^2}{l}\right)\right] \qquad (5.10)$$

$$u_{L2} = \frac{1}{2E}\left(\frac{l}{h}\right)^3 \frac{Q}{t}\left[1 + 3(1+\nu)\left(\frac{h^2}{l}\right)\right]. \qquad (5.11)$$

From the comparison between these eexpressions, it follows that the bending part in (5.8) is equal to $\frac{2}{3}$ of the corresponding one in (5.9) - (5.11). However, one should bear in mind that equations (5.8) - (5.11) do represent the displacement at tip of the beam, and that the finite element interpolation (approximation) of the beam displacements is linear for (5.8). Evidently, a linear approximation having a $\frac{2}{3}$ of the tip displacement of the cubic elastic, can be considered as a reasonable linear fit to the latter one.

It is also interesting to compare the shear terms (the second term in the brackets) in (5.8), (5.10) and (5.11). Note that the magnitude of the shear term in (5.8) lies between of these in (5.9) and (5.10), respectively, i.e. $1 < \frac{4}{3} < \frac{3}{2}$.

6. Conclusions

It has been demonstrated that practically desirable, and theoretically interesting, treatment of the boundary traction conditions in a two–field model as the essential ones, is a feasible goal, from the point of view both of the finite element theory and the computational practice. In order to verify this assertion, three main tasks were performed in the present paper.

First of all, it has been proved that, by the suitable choice of local common coordinates, it is possible to establish one–to–one correspondence between the boundary tractions and the stress coordinates. Hence, the latter can be determined if the former are prescibed.

Secondly the resulting system of equations is subdivided to the known (prescibed) and the uknown (variable) stresses and displacements. This decomposition problem, although failry straightforward, is not so simple as in the classical displacement formulation.

Thirdly, it has been shown that the resulting system of equations is symmetric. This highly desirable property, although obvious in the classical finite element analysis, is far from being self–evident in the present formulation. In the sequel, it can be concluded that, for the solution of the present problem, the existing solving tool for the mixed finite element equations can be used.

Finally, the example analysed confirmes the predicted behaviour of a system, and indicates that resonable results can be expected even in the case of a very crude mesh.

Acknowledgment

The authors is indebted to Prof. J.Jarić (Mathematical Faculty, Belgrade and The City College, New York), for the valuable discussions, concerning primarily the content of the Section 3 of this paper.

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Streszczenie

W artykule rozpatrywane są stosunkowo dobrze znane metody przybliżeń dwupolowymi (mieszanymi) elementami skończonymi w zastosowaniu do mechaniki ciała stałego. Zaproponowano jednak pewną znaczącą modyfikację metody elementów skończonych – siłowe więzy powierzchniowe traktowane są jako podstawowe warunki brzegowe. Aproksymacja wielopolowa w zakresie sprężystości nie ma żadnych ograniczeń, jednak w celu sprawdzenia koncepcji autor rozpatruje zakres liniowej sprężystości, dla którego dostępne są porównywalne rozwiązania analityczne.

Praca wpłynęła do Redakcji dnia 28 lipca 1989 roku