JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
50, 1, pp. 215-229, Warsaw 2012
50th anniversary of JTAM

MOTION OF A RIGID ROD ROCKING BACK AND FORTH AND CUBIC-QUINTIC DUFFING OSCILLATORS

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In this work, we implemented the first-order approximation of the Iteration Perturbation Method (IPM) for approximating the behavior of a rigid rod rocking back and forth on a circular surface without slipping as well as Cubic-Quintic Duffing Oscillators. Comparing the results with the exact solution, has led us to significant consequences. The results reveal that the IPM is very effective, simple and convenient to systems of nonlinear equations. It is predicted that IPM can be utilized as a widely applicable approach in engineering.

Key words: nonlinear oscillation, iteration perturbation method (IPM), rocking rigid rod, cubic-quintic Duffing oscillator

1. Introduction

With the rapid development of nonlinear science, it appears an ever-increasing interest of scientists and engineers in the analytical asymptotic techniques for nonlinear problems. Though it is easy for us now to find solutions to linear systems by means of numerical simulations, it is still very difficult to solve nonlinear problems analytically. Duffing oscillators comprise one of the canonical examples of Hamilton systems. However, simple generalizations of such oscillators, such as cubic-quintic Duffinng oscillators, have not been studied extensively (Hamdan and Shabaneh, 1997; Lin, 1999; Wu *et al.*, 2006). Belendez *et al.* (2011) presented a closed-form solution for the quintic Duffing equation using a cubication method. The restoring force is expanded in Chebyshev polynomials through their work and the governing nonlinear equation is approximated by a cubic Duffing equation in which the coefficients for the linear and cubic terms depend on the initial amplitude. The coupled Newton method with harmonic balancing was also utilized by Lai *et al.* (2009) for approximating higher-order solutions for strongly nonlinear Dufing oscillators with the cubic-quintic nonlinear restoring force. In addition, Ganji *et al.* (2009a) applied a new approximate method, so-called Energy Balance Method, to analyze these types of nonlinear oscillators with different engineering parameters of α, β and γ.

Principally, analytical methods to solve a nonlinear oscillator are limited to the perturbation approach (Nayfeh, 1981). However, as with other analytical techniques, certain limitations restrict the wide application of perturbation methods, the most important of which is the dependence of these methods on the existence of a small parameter in the equation. Disappointingly, the majority of nonlinear problems have no small parameter at all. Even in cases where a small parameter does exist, the determination of such a parameter does not seem to follow any strict rule, and is rather problem-specific. Furthermore, the approximate solutions solved by the perturbation methods are valid, in most cases, only for small values of the parameters. It is obvious that all these limitations come from the assumption of the small parameter. Therefore, new analytical techniques should be developed to overcome these analytical deficiencies (Barari *et al.*, 2008; Sfahani *et al.*, 2010).

Bayat *et al.* (2010) employed the Energy Balance Method to obtain analytical expressions for the non-linear fundamental frequency and deflection of Euler-Bernoulli beams. Their approximations were valid for a wide range of vibration amplitudes, unlike the solutions obtained by other analytical techniques, such as perturbation methods. The periodic solution for nonlinear free vibration of conservative, coupled mass-spring systems with linear and nonlinear stiffnesses as well as two mass-spring systems and buckling of a column were investigated within the works presented by Bayat *et al.* (2011) and Ganji *et al.* (2011). In the first work, the energy balance methodology was utilized for the approximations while, in the latter, after finding the maximal and minimal solution thresholds of the nonlinear problem, an approximate solution to the nonlinear equation was easily achieved using He Chengtian's interpolation. The other techniques recently proposed to eliminate the small parameter are listed as: homotopy perturbation (Barari *et al.*, 2008; Belendez *et al.*, 2007; He, 2005; Sfahani *et al.*, 2010; Yıldırım and Özis, 2007; Miansari *et al.*, 2010), differential transformation (Ganji *et al.*, 2010; Omidvar *et al.*, 2010), max-min (Ibsen *et al.*, 2010; Ganji *et al.*, 2011), parameterized perturbation (Barari *et al.*, 2011), frequency-amplitude formulation (Fereidon *et al.*, 2011; Ganji *et al.*, 2009b), harmonic balance (Gottlieb, 2006; Lim *et al.*, 2006), energy balance (Bayat *et al.*, 2010, 2011; Ganji *et al.*, 2009d; Momeni *et al.*, 2011; Sfahani *et al.*, 2011), variational iteration (Barari *et al.*, 2008; Fouladi *et al.*, 2010; Hosseinzadeh *et al.*, 2010) and variational approach (He, 2006; Ganji *et al.*, 2009c). In this letter, we present the periodic solution based on the iteration perturbation method (IPM) (He, 2001) for nonlinear oscillators. The method is applied to two cases, and the results are compared with those obtained by the exact solutions. In Sections 4 and 5, the cubic-quintic Duffing oscillator (Hamdan and Shabaneh, 1997) and motion of a rigid rod rocking back (Nayfeh and Mook, 1979; Wu *et al.*, 2003) are analyzed as well.

The mentioned problems can be written in the following forms

$$\begin{aligned} &x'' + f(x) = 0 && f(x) = \alpha x + \beta x^3 + \gamma x^5 \\ &x(0) = A && x'(0) = 0 \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &\Big(\frac{1}{12} + \frac{1}{16}u^2\Big)u''^2 + \frac{1}{16}uu'^2 + \frac{g}{4l}u\cos u = 0 \\ &u(0) = \beta \qquad \frac{du}{dt}(0) = 0 \end{aligned} \tag{1.2}$$

where $g > 0$ and $l > 0$ are known positive constants.

2. Basic idea of the iteration perturbation method

In this paper, we consider the following differential equation

$$u'' + f(u, u', u'', t) = 0 \tag{2.1}$$

We introduce the variable $y = du/dt$, and then Eq. (2.1) can be replaced by an equivalent system

$$u'(t) = y(t) \qquad y'(t) = -f(u, y, y', t) \tag{2.2}$$

Assume that its initial approximate guess can be expressed as

$$u(t) = A\cos(\omega t) \tag{2.3}$$

where ω is the angular frequency of oscillation. Then we have

$$u'(t) = -A\omega\sin(\omega t) = y(t) \qquad u''(t) = -A\omega^2\cos(\omega t) = y'(t) \tag{2.4}$$

Substituting Eqs. (2.3) and (2.4) into Eq. $(2.2)_2$, we obtain

$$y'(t) = -f(u, y, y', t) = -\sum_{n=0}^{\infty} \alpha_{2n+1}\cos[(2n+1)\omega t] \tag{2.5}$$

Substituting Eq. (2.5) into Eq. $(2.2)_2$, yields

$$y'(t) = -[\alpha_1\cos(\omega t) + \alpha_3\cos(3\omega t) + \ldots] \tag{2.6}$$

Integrating Eq. (2.6), gives

$$y(t) = -\frac{\alpha_1}{\omega}\sin(\omega t) - \frac{\alpha_3}{3\omega}\sin(3\omega t) - \ldots \tag{2.7}$$

Comparing Eqs. $(2.4)_1$ and (2.7), we obtain

$$(2.8) - A\omega = -\frac{\alpha_1}{\omega} \qquad \omega = \sqrt{\frac{\alpha_1}{A}} \qquad T = 2\pi\sqrt{\frac{A}{\alpha_1}} \tag{2.8}$$

3. Illustration of the problems

In this Section, IPM which was presented in Section 2 is applied to two smooth oscillators with odd nonlinearities in the displacement, and the results are compared with the exact solution.

Case 1. In this example, we consider the following nonlinear oscillator (Lim and Wu, 2003; Ramos, 2009)

$$u'' + \frac{u^3}{1+u^2} = 0 \qquad u(0) = A \qquad u'(0) = 0 \tag{3.1}$$

From Eq. (3.1), we have

$$u'' = -u''u^2 - u^3 \qquad u(0) = A \qquad u'(0) = 0 \tag{3.2}$$

Equation (3.2) is equivalent to the two-dimensional system

$$u' = y \qquad\qquad y' = -y'u^2 - u^3 \tag{3.3}$$

Substituting $u = A\cos(\omega t)$ into the right-hand side of Eqs. (3.3), gives

$$u' = -A\omega\sin(\omega t) = y \qquad\qquad y' = A^3\cos^3(\omega t)(\omega^2 - 1) \tag{3.4}$$

It is possible to perform the following Fourier series expansion

$$\begin{aligned} &A^3\cos^3(\omega t)(\omega^2-1) = \alpha_1\cos(\omega t) + \alpha_3\cos(3\omega t) + \ldots \\ &\alpha_1 = \frac{4}{\pi}\int\limits_0^{\pi/2} A^3\cos^4(\theta)(\omega^2-1)\ d\theta = \frac{3A^3(\omega^2-1)}{4} \\ &\alpha_3 = \frac{4}{\pi}\int\limits_0^{\pi/2} A^3\cos^3\theta\cos(3\theta)(\omega^2-1)\ d\theta = \frac{A^3(\omega^2-1)}{4} \end{aligned} \tag{3.5}$$

Substituting Eqs. (3.5) into Eq. $(3.4)_2$, yields

$$y' = \frac{A^3(\omega^2-1)}{4}[3\cos(\omega t) + \cos(3\omega t)] \tag{3.6}$$

By integrating Eq. (3.6), we obtain

$$\begin{aligned} y &= \frac{A^3(\omega^2-1)}{4}\int[3\cos(\omega t) + \cos(3\omega t)]\ dt \\ &= \frac{A^3(\omega^2-1)}{\omega}\Big[\frac{3}{4}\sin(\omega t) + \frac{1}{12}\sin(3\omega t)\Big] \end{aligned} \tag{3.7}$$

Comparing Eqs. $(3.4)_1$ and (3.7), gives

$$\omega = \frac{3A}{\sqrt{9A^2+12}} \qquad\qquad T = \frac{2\pi\sqrt{9A^2+12}}{3A} \tag{3.8}$$

The exact frequency ω_{ex} of Eqs. (3.15) is (Lim and Wu, 2003)

$$\omega_{ex} = \pi\Bigg[2\int\limits_0^{\pi/2}\frac{A^2\cos^2\theta}{\sqrt{A^2\cos^2\theta + \ln\Big(1 - \frac{A^2\cos^2\theta}{1+A^2}\Big)}}\ d\theta\Bigg]^{-1} \tag{3.9}$$

In case 1, we assume $A = 0.01$, 0.05, 0.1, 0.5, 1, 5, 10, 50, and 100. The obtained exact results are expressed in Eq. (3.8). The results for the

Table 1. Comparison between IPM and exact solution for Example 1

A	ω	ω_{ex}	$\lvert(\omega_{ex}-\omega)/\omega_{ex}\rvert$
0.01	0.00866	0.00847	2.242
0.05	0.04326	0.04232	2.22
0.1	0.08627	0.08439	2.22
0.5	0.39736	0.38737	2.58
1.0	0.65465	0.63678	2.81
5.0	0.97435	0.96698	0.763
10.0	0.99340	0.99092	0.250
50.0	0.99973	0.99961	0.012
100.0	0.99993	0.99990	0.003

approximate frequency ω with the exact frequency ω_{ex} are also compared and tabulated in Table 1. From the illustrated results, the maximum error 2.22% can be obtained. Hence, it is concluded that there is an excellent agreement with the exact solutions for the nonlinear systems.

Case 2. This example corresponds to

$$u'' + \frac{u}{1+\varepsilon u^2} = 0 \qquad u(0) = A \qquad u'(0) = 0 \tag{3.10}$$

From Eq. (3.10), we have

$$u'' = -u''\varepsilon u^2 - u \qquad u(0) = A \qquad u'(0) = 0 \tag{3.11}$$

Equation (3.11) is equivalent to the two-dimensional system

$$u' = y \qquad y' = -y'\varepsilon u^2 - u \tag{3.12}$$

Substituting $u = A\cos(\omega t)$ into the right-hand side of Eqs. (3.12), gives

$$u' = -A\omega\sin(\omega t) = y \qquad y' = A\cos(\omega t)[A^2\varepsilon\omega^2\cos^2(\omega t) - 1] \tag{3.13}$$

It is possible to carry out the following Fourier series expansion

$$\begin{aligned} &A\cos(\omega t)[A^2\varepsilon\omega^2\cos^2(\omega t) - 1] = \alpha_1\cos(\omega t) + \dots \\ &\alpha_1 = \frac{4}{\pi}\int\limits_0^{\pi/2} A\cos^2\theta[A^2\varepsilon\omega^2\cos^2(\theta) - 1]\,d\theta = \frac{A(3A^2\omega^2\varepsilon - 4)}{4} \end{aligned} \tag{3.14}$$

Substituting Eqs. (3.14) into Eq. $(3.13)_2$, yields

$$y' = \frac{A(3A^2\omega^2\varepsilon - 4)}{4}\cos(\omega t) + \dots \tag{3.15}$$

Integration of Eq. (3.15) leads to

$$y = \int\Big(\frac{A(3A^2\omega^2\varepsilon - 4)}{4}\cos(\omega t) + \ldots\Big)\,dt = \frac{A(3A^2\omega^2\varepsilon - 4)}{4\omega}\sin(\omega t) + \ldots \tag{3.16}$$

Comparing Eqs. $(3.13)_1$ and (3.16), gives

$$\omega = \frac{2}{\sqrt{3\varepsilon A^2 + 4}} \qquad\qquad T = \pi\sqrt{3\varepsilon A^2 + 4} \tag{3.17}$$

Equation $(3.17)_1$ gives the same frequency as the one resulting from the application of the harmonic balance method to Eq. (3.10). It is also exactly the same as that obtained by the artificial parameter Linstedt-Poincare method (Ramos, 2009).

4. Cubic-quintic Duffing equations

Now, we consider the nonlinear cubic-quintic Duffing equations. From Eq. (1.1), we have

$$x'' = -\alpha x - \beta x^3 - \gamma x^5 \tag{4.1}$$

Equation (4.1) is equivalent to the two-dimensional system

$$x' = y \qquad\qquad y' = -\alpha x - \beta x^3 - \gamma x^5 \tag{4.2}$$

Substituting $u = A\cos(\omega t)$ into the right-hand side of Eqs. (4.2), gives

$$\begin{aligned} x' &= -A\omega\sin(\omega t) = y \\ y' &= -A\cos(\omega t)[\alpha + \beta A^2\cos^2(\omega t) + \gamma A^4\cos^4(\omega t) \end{aligned} \tag{4.3}$$

Expanding the above in the Fourier series, we have

$$\begin{aligned} &-A\cos(\omega t)[\alpha + \beta A^2\cos^2(\omega t) + \gamma A^4\cos^4(\omega t)] = \alpha_1\cos(\omega t) + \ldots \\ &\alpha_1 = \frac{4}{\pi}\int\limits_0^{\pi/2} A\cos^2\theta[\alpha + \beta A^2\cos^2\theta + \gamma A^4\cos^4\theta]\,d\theta \\ &\quad = 4A\Big(\frac{\alpha}{4} + \frac{3\beta A^2}{16} + \frac{5\gamma A^4}{32}\Big) \end{aligned} \tag{4.4}$$

Substituting Eqs. (4.4) into Eq. $(4.3)_2$, yields

$$y' = 4A\Big(\frac{\alpha}{4} + \frac{3\beta A^2}{16} + \frac{5\gamma A^4}{32}\Big)\cos(\omega t) + \ldots \tag{4.5}$$

Integrating Eq. $(4.4)_2$, yields

$$\begin{aligned} y &= \int \Big[4A\Big(\frac{\alpha}{4} + \frac{3\beta A^2}{16} + \frac{5\gamma A^4}{32}\Big)\cos(\omega t) + \ldots\Big]\, dt \\ &= \frac{4A}{\omega}\Big(\frac{\alpha}{4} + \frac{3\beta A^2}{16} + \frac{5\gamma A^4}{32}\Big)\sin(\omega t) + \ldots \end{aligned} \tag{4.6}$$

Comparing Eqs. $(4.3)_1$ and (4.6), gives

$$\omega = \frac{\sqrt{16\alpha + 12A^2\beta + 10\gamma A^4}}{4} \qquad\qquad T = \frac{8\pi}{\sqrt{16\alpha + 12A^2\beta + 10\gamma A^4}} \tag{4.7}$$

The exact frequency ω_{ex} for the cubic-quintic Duffing oscillator is (Wu *et al.*, 2003)

$$\omega_e(A) = \pi k_1 \Big(2\int\limits_0^{\pi/2} \frac{1}{\sqrt{1 + k_2 \sin^2 t + k_3 \sin^4 t}}\, dt\Big)^{-1} \tag{4.8}$$

where

$$k_1 = \sqrt{\alpha + \frac{\beta A^2}{2} + \frac{\gamma A^4}{3}} \qquad\qquad k_2 = \frac{3\beta A^2 + 2\gamma A^4}{6\alpha + 3\beta A^2 + 2\gamma A^4}$$

$$k_3 = \frac{2\gamma A^4}{6\alpha + 3\beta A^2 + 2\gamma A^4}$$

The above result from Eq. $(4.7)_1$ is in good agreement with the result obtained by the exact solution as given in Eq. (4.8). Comparisons between the IPM and exact solutions for the cubic-quintic Duffing system are illustrated in Fig. 1 and Table 2.

5. Motion of a rocking rigid rod

In this Section, we present an example of motion of a rigid rod rocking back and forth on a circular surface without slipping as presented in Eq. (1.2)

$$u'' = -\frac{3}{4}u^2 u'' - \frac{3}{4}uu'^2 - \frac{3gu\cos u}{l} \tag{5.1}$$

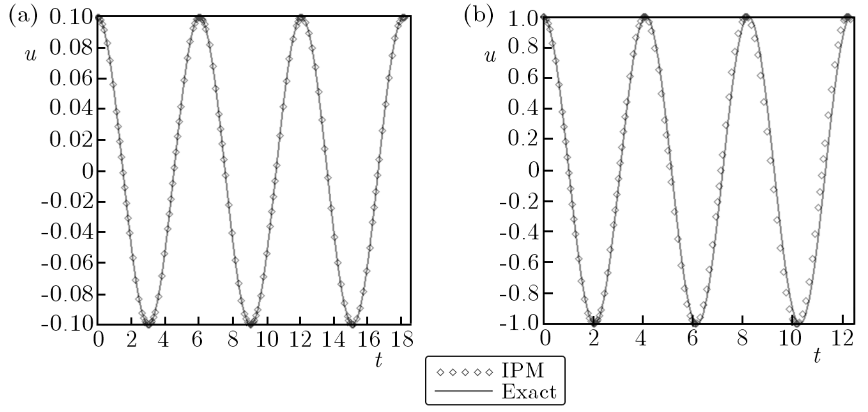

Fig. 1. Comparison between IPM and the exact solution for cubic-quintic Duffing oscillator (Eq. (1.1)); (a) $\alpha = 1$, $\beta = 10$, $\gamma = 100$, $A = 0.1$, (b) $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $A = 1.0$

Table 2. Comparison between IPM and exact solution for cubic-quintic Duffing oscillator

A	$\alpha=\beta=\gamma=1$			$\alpha=1,\ \beta=10,\ \gamma=100$		
	ω	ω_{ex}	$\mathcal{B}$	ω	ω_{ex}	$\mathcal{B}$
0.1	1.00377	1.00377	0.0	1.03983	1.03970	0.01250
0.5	1.10750	1.10654	0.06757	2.60408	2.52469	3.14468
1	1.54110	1.52359	1.14926	8.42615	8.01005	5.19472
5	20.2577	19.1815	5.61061	198.119	187.199	5.83318
10	79.5361	75.1774	5.79795	791.044	747.323	5.85038
50	1976.90	1867.57	5.85413	19764.71	18671.34	5.85587
100	7906.17	7468.83	5.85553	79057.42	74683.91	5.85602
500	197642.83	186709.04	5.85606	1976424.01	1867085.99	5.85608
1000	790569.89	746834.69	5.85608	7905694.62	7468342.49	5.85608

ω_{ex} – Ramos (2009); $\mathcal{B} = |(\omega_{ex} - \omega)/\omega_{ex}|$

Equation (5.1) is equivalent to the two-dimensional system

$$u' = y \qquad\qquad y' = -\frac{3}{4}u^2y' - \frac{3}{4}uy^2 - \frac{3gu\cos u}{l} \tag{5.2}$$

Substituting $u = A\cos(\omega t)$ into the right-hand side of Eqs. (5.2), gives

$$\begin{aligned} x' &= -A\omega\sin(\omega t) = y \\ y' &= -\frac{3}{4l}A\cos(\omega t)[-2A^2\omega^2 l\cos^2(\omega t) + A^2\omega^2 l + 4g\cos(A\cos(\omega t))] \end{aligned} \tag{5.3}$$

with the application of Fourier series expansion, we have

$$\begin{aligned}
&-\frac{3}{4l}A\cos(\omega t)[-2A^2\omega^2 l\cos^2(\omega t)+A^2\omega^2 l+4g\cos(A\cos(\omega t))]\\
&\quad=\alpha_1\cos(\omega t)+\dots\\
&\alpha_1=\frac{4}{\pi}\int\limits_0^{\pi/2}-\frac{3}{4l}A\cos^2\theta[-2A^2\omega^2 l\cos^2\theta+A^2\omega^2 l+4g\cos(A\cos(\theta))]\,d\theta\\
&\quad=\frac{3}{8l}[A^3\omega^2 l-16gAJ(0,A)+16gJ(1,A)]
\end{aligned}\tag{5.4}$$

where J – Bessel function.

Substituting Eqs. (5.4) into Eq. $(5.3)_2$, yields

$$y'=\frac{3}{8l}[A^3\omega^2 l-16gAJ(0,A)+16gJ(1,A)]\cos(\omega t)+\dots\tag{5.5}$$

Integrating Eq. (5.5), yields

$$\begin{aligned}
y&=\int\Big(\frac{3}{8l}[A^3\omega^2 l-16gAJ(0,A)+16gJ(1,A)]\cos(\omega t)+\dots\Big)\,dt\\
&=\frac{3}{8\omega l}[A^3\omega^2 l-16gAJ(0,A)+16glJ(1,A)]\sin(\omega t)+\dots
\end{aligned}\tag{5.6}$$

Comparing Eqs. $(5.2)_2$ and (5.5), gives

$$\begin{aligned}
\omega&=\frac{\sqrt{48[lA(3A^2+8)g(AJ(0,A)-J(1,A))]}}{lA(3A^2+8)}\\
T&=\frac{2\pi lA(3A^2+8)}{\sqrt{48[lA(3A^2+8)g(AJ(0,A)-J(1,A))]}}
\end{aligned}\tag{5.7}$$

The exact period T_{ex} for Eq. (1.2) is (Wu *et al.*, 2003)

$$T_{ex}=4\sqrt{\frac{l}{3g}}\int\limits_0^{\pi/2}\sqrt{\frac{(4+3\beta^2\sin^2\varphi)\beta^2\cos^2\varphi}{8[\beta\sin\beta+\cos\beta-\beta\sin\varphi\sin(\beta\sin\varphi)-\cos(\beta\sin\varphi)]}}\,d\varphi\tag{5.8}$$

For comparison, the approximate period computed by Eq. $(5.7)_2$, and the exact period T_{ex} obtained by Eq. (5.8) are given in Fig. 2 and Table 3.

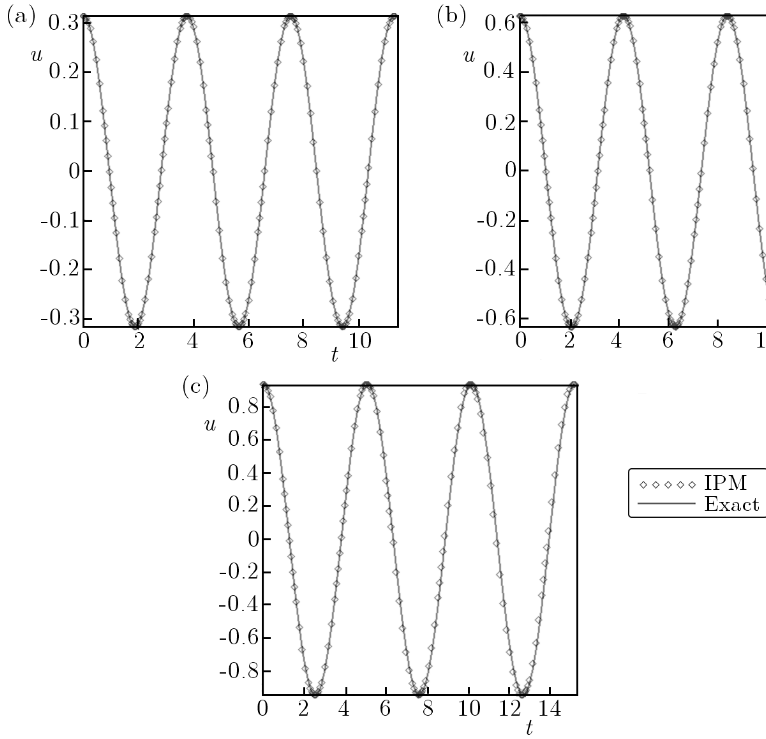

Fig. 2. Comparison between IPM and the exact solution for motion of the rocking rigid rod (Eq. (1.2)); (a) $g = l = 1$, $A = 0.10\pi$, (b) $g = l = 1$, $A = 0.20\pi$, (c) $g = l = 1$, $A = 0.30\pi$

Table 2. Comparison between IPM and exact solution for motion of the rocking rigid rod, when $g = l = 1$

β	T	T_{ex}	$\lvert (T_{ex} - T)/T_{ex} \rvert$
0.05π	3.66129	3.66109	0.0054
0.10π	3.76394	3.76397	0.0008
0.15π	3.94064	3.94086	0.0056
0.20π	4.20116	4.20292	0.04187
0.25π	4.56246	4.56948	0.15363
0.30π	5.05355	5.07728	0.46738
0.35π	5.72584	5.79770	1.23946
0.40π	6.67785	6.89564	3.1584

6. Conclusions

In this paper, the IPM has been implemented in order to analyze the equation of motion associated with a rocking rigid rod as well as cubic-quintic Duffing oscillators. We conclude from the obtained results that the IPM is an efficient method for finding periodic solutions for non-linear oscillatory systems. All the examples show that the presented results are in excellent agreement with those obtained by the exact solution. The general conclusion is that the IPM provides an easy and direct procedure for determining approximations of periodic solutions to Eqs. (1.1) and (1.2).

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Ruch pręta toczącego się wahadłowo po płaszczyźnie oraz oscylatora Duffinga piątego stopnia

Streszczenie

W pracy omówiono pierwszorzędową aproksymację zachowania się sztywnego pręta toczącego się bez poślizgu ruchem wahadłowym po kołowej powierzchni za pomocą iteracyjnej metody perturbacyjnej (IPM). Tę samą metodę zastosowano także do analizy dynamiki oscylatora Duffinga piątego stopnia. Porównanie otrzymanych wyników z rozwiązaniem dokładnym doprowadziło do istotnych wniosków. Wykazano przede wszystkim wysoką efektywność metody IPM przy jej jednoczesnej prostocie i wygodzie w stosowaniu do nieliniowych równań ruchu. Autorzy podkreślają duże walory aplikacyjne metody IPM w praktyce inżynierskiej.

Manuscript received February 28, 2011; accepted for print May 13, 2011