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GEOMETRIC FORM DESIGN OF V-BELTS MADE OF NON-LINEAR ELASTIC MATERIALS

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A geometric form designing method for details made of rubber-like materials has been presented in the paper. With regard to real properties of materials of that kind a non-linearly elastic physical model was adopted as the most adequate one. Large finite deformations as well as a state of stress for composite load have been determined too. Eventually, it allows one to design for example detail cross-sections of desirable shape.

1. Introduction

Many applied constructional materials have non-linearly elastic properties. Polymers and elastomers among other materials belong to that group. These materials are able to carry, under a load, quite big finite deformations. Internal structure of these materials differs entirely from crystal and symmetric atomic structure of metals. Their internal structure has a molecular construction and molecules are connected each other with main and transverse bonds. Concluding, physical properties of rubber-like materials demand for application of more complex theory to their description, namely non-linear elasticity. These methods not only allow us to take into account large elastic deformations, but also to design machine elements made of these materials in more optimal way. It can consists, among others, in an appropriate selection of the element cross-section geometric form in states before and after load, respectively. Direction of further research is taking into account internal friction and its outcomes in the process.

The foundations of non-linear elasticity theory presented, among others, by Green and Zerna (1954) and Wesołowski and Woźniak (1970) were utilized in the paper. While formulating physical relations for elastomers it was assumed that these materials are homogeneous, isotropic, hyperelastic and incompressible.

The goal of the paper is a description of construction transversal deformation for a V-belt complex form as an example.

2. Geometry of deformation

An initial state B^0 of dimensionaly undefined rubber V-belt undergoes deformation as a result of belt gird around a pulley. For a deformation state B the belt assumes a cylindrical form of inner radius r_1, outer radius r_2, pulley bend angle 2α and trapezoidal cross-section (Fig.1), respectively. It is convenient to use for an actual state B the cylindrical coordinates $x_1(r, \alpha, z)$, while the Cartesian coordinates X_i for an initial state B^0.

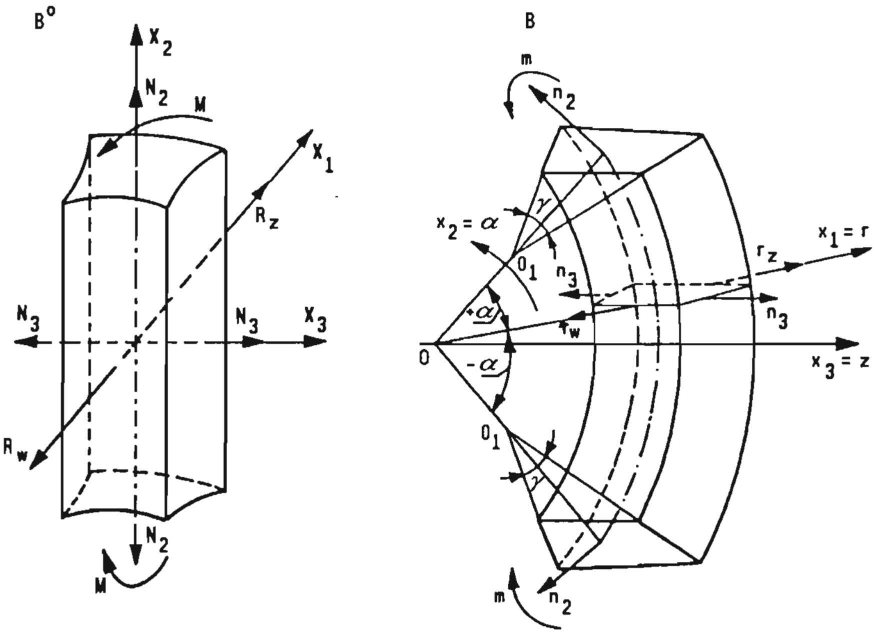

Fig. 1. Belt deformation geometry: state B^0 – initial configuration, state B – final configuration

Belt deformation can be described as follows

$$
\begin{aligned}
x_1 &= f(r) - az \\
x_2 &= b\alpha \\
x_3 &= cz
\end{aligned}
\tag{2.1}
$$

where $f(r)$, a, b, c are the characteristic quantities for belt deformations and are to be determined.

Metric tensors defined as

$$g_{ij} = \frac{\partial X^m}{\partial x^i}\frac{\partial X^m}{\partial x^j}$$

for the state B^0 will take the following form

$$\mathbf{g} = \begin{bmatrix} \left(\dfrac{\partial f}{\partial r}\right)^2 & 0 & -a\dfrac{\partial f}{\partial r} \\ 0 & b^2 & 0 \\ -a\dfrac{\partial f}{\partial r} & 0 & a^2 + c^2 \end{bmatrix} \tag{2.2}$$

and for the state B

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.3}$$

After determination of invariant I_3 based on the metric tensors

$$I_3 = \frac{\det \mathbf{G}}{\det \mathbf{g}} = \frac{G}{g}$$

and its comparing to the unit and after taking into account a condition of rubber incompressibility

$$f(r) = \frac{1}{2bc}r^2 + C \tag{2.4}$$

The constant C will be determined from the boundary conditions, which result from the assumed design conditions.

An effective determination of belt cross-section form design principles for a natural state B^0 using the method of non-linear elasticity requires proper assumptions conformable to design criteria.

The final effect of deformation is integrally connected with assumed geometric parameters determining the belt cross-section for the state B^0.

Assumptions of

- constant height in the belt central cross-section
- variant wedge angle $\gamma^0 > \gamma$ for the state B^0 or of constant angle $(\gamma^0 = \gamma)$
- A and B border points of belt side surface fixed during deformation (Fig.2)

are justified from technical point of view (description is given in a further part of the paper).

An assumption of belt constant height in the center cross-section CD (Fig.2) leads to the following boundary conditions

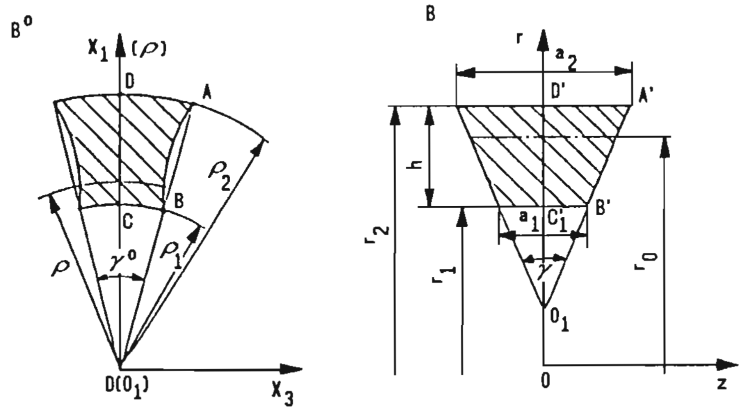

Fig. 2. Deformation of the belt cross-section: state B^0 – initial configuration, state B – final configuration

— for $z = 0$ and $r = r_1$

$$f(r_1) = \rho_1 = \frac{a_1}{2\tan\frac{\gamma}{2}}$$

thus

$$\frac{1}{2bc}r_1^2 + C = \frac{a_1}{2\tan\frac{\gamma}{2}}$$

hence

$$C = \frac{a_1}{2\tan\frac{\gamma}{2}} - \frac{1}{2bc}r_1^2 \tag{2.5a}$$

— for $z = 0$ and $r = r_2$

$$f(r_2) = \rho_2 = \frac{a_2}{2\tan\frac{\gamma}{2}}$$

thus

$$\frac{1}{2bc}r_2^2 + C = \frac{a_2}{2\tan\frac{\gamma}{2}}$$

hence

$$C = \frac{a_2}{2\tan\frac{\gamma}{2}} - \frac{1}{2bc}r_2^2 \tag{2.5b}$$

Comparing Eqs (2.5a) and (2.5b) we receive the function

$$r_1 + r_2 = 2bc \tag{2.6}$$

which links deformation parameters b and c.

The following boundary conditions result from the assumption that the location of belt side surface border points A and B does not change during the deformation ($A = A'$, $B = B'$)

— for $r = r_1$ and $z = a_1/2$ it occours

$$f(r_1) = r_1 = \frac{1}{2bc} r_1^2 + C - a\frac{a_1}{2}$$

so

$$C = a\frac{a_1}{2} + r_1 - \frac{1}{2bc} r_1^2 \tag{2.7a}$$

— for $r = r_2$ and $z = a_2/2$

$$f(r_2) = r_2 = \frac{1}{2bc} r_2^2 + C - a\frac{a_2}{2}$$

so

$$C = a\frac{a_2}{2} + r_2 - \frac{1}{2bc} r_2^2 \tag{2.7b}$$

Comparison of Eqs (2.7a) and (2.7b) leads to the function containing parameters a, b and c

$$r_1 + r_2 = 2bc\Big(1 + a\tan\frac{\gamma}{2}\Big) \tag{2.8}$$

The belt deformation described by functions (2.1) substituting for the constant C Eq (2.5a), can be written in detailed form

$$\begin{aligned} x_1 &= \frac{1}{2bc} r^2 - az + \frac{a_1}{2\tan\frac{\gamma}{2}} - \frac{1}{2bc} r_1^2 \\ x_2 &= b\alpha \\ x_3 &= cz \end{aligned} \tag{2.9}$$

and for the value of C from Eq (2.7a)

$$\begin{aligned} x_1 &= \frac{1}{2bc} r^2 - az + \frac{a_1}{2} a r_1 - \frac{1}{2bc} r_1^2 \\ x_2 &= b\alpha \\ x_3 &= cz \end{aligned} \tag{2.10}$$

Metric tensor $\mathbf{g}$ components, i.e. covariant and contravariant ones, are of the following form regardless of the earlier made assumptions

$$\Big[g_{ij}\Big] = \begin{bmatrix} \frac{r^2}{b^2c^2} & 0 & -\frac{a}{bc} r \\ 0 & b^2 & 0 \\ -\frac{a}{bc} r & 0 & a^2 + c^2 \end{bmatrix} \tag{2.11}$$

$$\Big[g^{ij}\Big] = \begin{bmatrix} g^{11} & 0 & g^{13} \\ 0 & \frac{1}{b^2} & 0 \\ g^{31} & 0 & g^{33} \end{bmatrix} \tag{2.12}$$

while

$$g^{11} = \frac{b^2c^2}{r^2} + \frac{a^2b^2c^2}{a^2b^2c^2 + b^2c^4 - a^2r^2}$$

$$g^{13} = g^{31} = \frac{ab^3c^3}{r(a^2b^2c^2 + b^2c^4 - a^2r^2)}$$

$$g^{33} = \frac{2a^2b^2c^2 + b^2c^4 - a^2r^2}{(a^2+c^2)(a^2b^2c^2 + b^2c^4 - a^2r^2)}$$

For rubber regarded as an incompressible material (cf Green and Zerna, 1954; Wesołowski and Woźniak, 1970; Dudziak and Mielniczuk, 1989) deformation invariants have the form

$$\begin{aligned} I_1 &= g^{rs}G_{rs} = g^{11} + \frac{r^2}{b^2} + g^{33} \\ I_2 &= g_{rs}G^{rs}g_{rs}G^{rs}I_3 = \frac{r^2}{b^2c^2} + \frac{b^2}{r^2} + a^2 + c^2 \\ I_3 &= \frac{\det G_{ij}}{\det g_{ij}} = 1 \end{aligned} \tag{2.13}$$

The state of deformation will be fully determined, when deformation parameters a, b, and c are known. Additional an still missing, two equations describing these quantities, besides of already got equations (2.6) or (2.8), can be obtained from the proper stress conditions. In order to achieve that, determination of a stress state in the deformed belt (2.1) seems to be indispensable.

3. State of stress for a finite deformation

A stress tensor in a non-linear elastic (hyperelastic) isotropic medium

$$t^{ij} = \phi g^{ij} + \psi B^{ij} + pG^{ij} \tag{3.1}$$

where

$$\phi = \frac{2}{\sqrt{I_3}}\frac{\partial W}{\partial I_3} \qquad \psi = \frac{2}{\sqrt{I_3}}\frac{\partial W}{\partial I_2} \qquad p = 2\sqrt{I_3}\frac{\partial W}{\partial I_3}$$

W is an elastic potential (cf Wesołowski and Woźniak, 1970; Dudziak and Mielniczuk, 1989) and B^{ij} is a deformation tensor

$$B^{ij} = I_1 g^{ij} - g^{ir}g^{js}G_{rs}$$

The stress tensor of rubber-like materials can be real described by the potential proposed by Mooney (cf Zahorski, 1962)

$$W = C_1(I_1 - 3) + C_2(I_2 - 3) \tag{3.2}$$

where $C_1 > 0, C_2 \leq 0$ are the material constants.

Stress tensor components can be determined with an accuracy of scalar quantity p

$$\begin{aligned}
t^{11} &= \phi\Big(\frac{b^2c^2}{r^2} + \frac{a^2b^2c^2}{a^2b^2c^2 + b^2c^4 - a^2r^2}\Big) + p \\
t^{22} &= \phi\frac{1}{b^2} + \frac{p}{r^2} \\
t^{33} &= \phi\frac{2a^2b^2c^2 + b^2c^4 - a^2r^2}{(a^2 + c^2)(a^2b^2c^2 + b^2c^4 - a^2r^2)} + p \\
t^{13} &= \phi\frac{ab^3c^3}{r(a^2b^2c^2 + b^2c^4 - a^2r^2)} \\
t^{12} &= t^{23} = 0
\end{aligned} \tag{3.3}$$

It was assumed that $\phi = 2C_1$ and $\psi = 0$ because $C_2 = 0$. Then a function of hydrostatic pressure p for an incompressible material will be determined from the balance equation and the boundary conditions, respectively. Omitting both mass and inertia forces the balance equation can be written in a general form

$$t^{ij}\Big|_i = t^{ji}_{,i} + \Gamma^i_{ir}t^{rj} + \Gamma^j_{ir}t^{ir} = 0 \tag{3.4}$$

while the Christoffel symbols for a change of coordinates from the Cartesian to cylindrical ones are as follows

$$\Gamma^1_{22} = -r \qquad\qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$$

other $\Gamma = 0$. After writting out Eq (3.4) we obtain

$$\begin{aligned}
&t^{11}_{,1} - rt^{22} + \frac{1}{r}t^{11} = 0 \\
&T^{22}_{,2} + T^{32}_{,3} = 0 \\
&T^{23}_{,2} + T^{33}_{,3} = 0
\end{aligned} \tag{3.5}$$

From Eqs $(3.5)_2$ and $(3.5)_3$ it results, that the parameter p is merely the function of r. Basing on Eq $(3.5)_1$ and relations (3.3) one obtains following function

$$\frac{1}{\phi}\frac{dp}{dr} = \frac{r}{b^2} + \frac{b^2c^2}{r^3} - \frac{a^2b^2c^2}{r(a^2b^2c^2 + b^2c^4 - a^2r^2)} - \\ - \frac{2a^4b^2c^2r}{(a^2b^2c^2 + b^2c^4 - a^2r^2)^2} \tag{3.6}$$

and after integration

$$\frac{p}{\phi} = \frac{r^2}{2b^2} - \frac{b^2c^2}{2r^2} - \frac{b^2c^2}{2A^2}\ln\frac{r^2}{A^2 - r^2} - \frac{b^2c^2}{A^2 - r^2} + D \tag{3.7}$$

where $A^2 = b^2c^2 + \frac{b^2c^2}{a^2}$ and D is an integration constant.

From the engineering point of view it is more convenient to introduce the stress state tensor components. They will have then one mutual denomination, e.g. MPa. Current stress tensor physical components for the curvilinear coordinates are defined in the following way

$$\sigma_{ij} = \sqrt{\frac{G_{ij}}{G^{ii}}}\, t^{ij} = \sqrt{G_{ii}G_{jj}}\, t^{ij} = \frac{1}{\sqrt{G^{ii}G^{jj}}}\, t^{ij}$$

For the Cartesian coordinates

$$\sigma_{ij} = t_{ij}$$

According to Eq (2.10) the stress state physical components are

$$\begin{aligned} \sigma_{11} &= t^{11} & \sigma_{22} &= r^2t^{22} \\ \sigma_{33} &= t^{33} & \sigma_{13} &= t^{13} \\ \sigma_{23} &= rt^{23} \end{aligned} \tag{3.8}$$

The constant D appearing in Eq (3.7) will be determined from the boundary conditions. For $r = r_1$ it happens that $\sigma_{11} = 0$ and in that case

$$D = \frac{b^2c^2}{2A^2}\ln\frac{r_1^2}{A^2 - r_1^2} - \frac{b^2c^2}{2r_1^2} - \frac{r_1^2}{2b^2} \tag{3.9a}$$

Similarly for $r = r_2$

$$D = \frac{b^2c^2}{2A^2}\ln\frac{r_2^2}{A^2 - r_2^2} - \frac{b^2c^2}{2r_2^2} - \frac{r_2^2}{2b^2} \tag{3.9b}$$

From functions (3.9a) and (3.9b) one can achieve the second equation describing deformation parameters of a belt cross-section

$$\frac{b^2c^2}{A^2}\ln\frac{r_2^2(A^2 - r_1^2)}{r_1^2(A^2 - r_2^2)} - b^2c^2\Big(\frac{1}{r_2^2} - \frac{1}{r_1^2}\Big) - \frac{1}{b^2}(r_2^2 - r_1^2) = 0 \tag{3.10}$$

An assumption of a pure bending $(n_2 = 0)$ leads to formulation of the last equation describing deformation parameters a, b and c. Therefore

$$n_2 = \int\limits_{r_1}^{r_2} F\sigma_{22}\,dr = \frac{a_2 - a_1}{h}\int\limits_{r_1}^{r_2} r\sigma_{22}\,dr + \Big(a_1 - \frac{a_2 - a_1}{h}r_1\Big)\int\limits_{r_1}^{r_2}\sigma_{22}\,dr = 0 \tag{3.11}$$

After integration of appearing in Eqs (3.5), (3.7), (3.8) and (3.9a) one can obtain

$$\begin{aligned}
&L\Big[\frac{3}{8b^2}(r_2^4 - r_1^4) - \frac{b^2c^2}{2}\ln\frac{r_2}{r_1} + \frac{3}{4}b^2c^2L\ln\frac{A^2 - r^2}{A^2 - r_1^2} + \\
&+\frac{1}{2}(r_2^2 - r_1^2)\Big(\frac{b^2c^2}{2A^2}\ln\frac{r_1^2}{A^2 - r_1^2} - \frac{b^2c^2}{2r_1^2} - \frac{r_1^2}{2b^2}\Big)\Big] + \\
&+K\Big[\frac{1}{2b^2}(r_2^3 - r_1^2) + \frac{b^2c^2}{2}\Big(\frac{1}{r_2} - \frac{1}{r_1}\Big) - \frac{b^2c^2}{A}\ln\frac{(A + r_2)(A - r_1)}{(A - r_2)(A + r_1)} + \\
&+\Big(\frac{2b^2c^2}{A^2} + D\Big)(r_2 - r_1)\Big] + \frac{b^2c^2}{2A^2}K\Big(r_2\ln\frac{A^2 - r_2^2}{r_2^2} - r_1\ln\frac{A^2 - r_1^2}{r_1^2}\Big) - \\
&-\frac{b^2c^2}{4A^2}L\Big(r_2^2\ln\frac{A^2 - r_2^2}{r_2^2} - r_1^2\ln\frac{A^2 - r_1^2}{r_1^2}\Big) = 0
\end{aligned} \tag{3.12}$$

where

$$L = \frac{a_2 - a_1}{h}$$

$$K = a_1\frac{a_2 - a_1}{h}r_1$$

D – according to (3.9a) or (3.9b)

Finally for the presented conditions the deformation parameters a, b, c of belt cross-section can be determined from Eqs (2.6), (3.10) and (3.12) or (2.8), (3.10) and (3.12), respectively.

4. Determination of the neutral axis location

When the stress distribution over the bend belt cross-section is known, one can find for this cross-section the location neutral axis, as it was done for a flat belt (Zahorski, 1962). Knowledge of the neutral axis proper location is necessary not only for the cross-section shape design process but first of all for a cord thread layer positioning inside the belt. The slender and inextensible cord layer should be positioned in a zone of neutral axis radius. It makes the belt susceptible for bending, what affects significantly durability and efficiency of the transmission.

From Eq $(3.8)_2$ for $\sigma_{22} = 0$ one can find a function describing the neutral axis radius r_0

$$\sigma_{22} = r^2 t^{22} = 0 \tag{4.1}$$

and finally

$$\frac{1}{b^4c^2}(3r_0^2 - r_1^2) - \Big(\frac{1}{r_0^2} + \frac{1}{r_1^2}\Big) - \frac{1}{A^2}\ln\frac{r_0^2(A^2 - r_1^2)}{r_1^2(A^2 - r_0^2)} - \frac{2}{A^2 - r_0^2} = 0 \tag{4.2}$$

The neutral axis radii r_0 were determined from Eq (4.2) for bend radii r_1 or r_2 of B17 × 11 V-belt. Radii values are presented in Table 1 and neutral axis radius r_0 values variations relative to bend r adii are shown in Fig.3.

Table 1. List of belt bend radii values

division diameter D_p [mm]	r_1 [mm]	r_2 [mm]
90	40	51
125	57.5	68.5
135	62.5	73.5
140	65	76
150	70	81
160	75	86
170	80	91
201	95	106
280	135	146

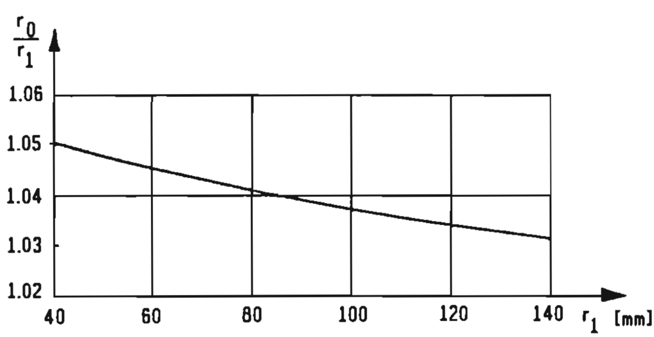

Fig. 3. Neutral axis radius location as a function of bend radius on pulley

In reality, a belt is loaded in a more complex way, because beside the bending moment axial forces like the force of initial pull and the effective force act as well. Dudziak (1990) proved that the axial forces affect the deformation of belt cross-section is negligible – about 10 times lower than the effect of pure bending moment. Due to that the achieved results are of significant theoretical and practical importance.

5. Examples of belt shape design

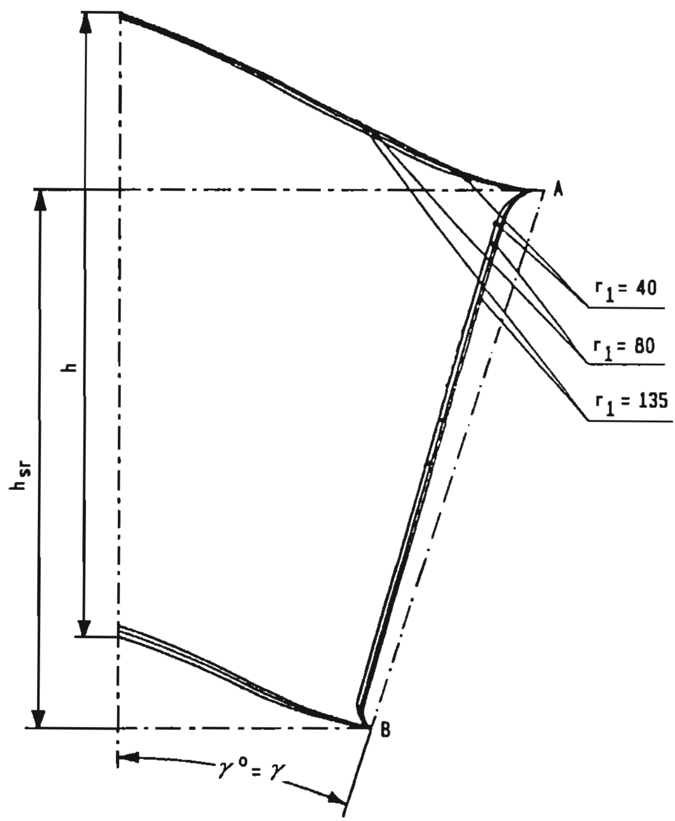

Fig. 4. Deformation of a belt cross-section for assumed fixed location of operating surface extreme points A, B: $\gamma^0 = \gamma$

A shape and dimensions of a standard V-belt B17 × 11 cross-section for the initial state, i.e. before vulcanization (Fig.2) will be established for two following basic assumptions:

a) a hight of the center cross-section varies during the deformation process,

b) the location of operational surface extreme points does not change during the deformation.

The deformations were determined for bend radii r_1 and r_2 of values presented in Table 1. The belt is made of OKB-1 type rubber, hardness of which is 55-56 Sh and other mechanical properties are available by Dudziak (1990).

Fig.4 and 5 show examples of a belt cross-section arbitrary form design dependent on the accepted assumptions.

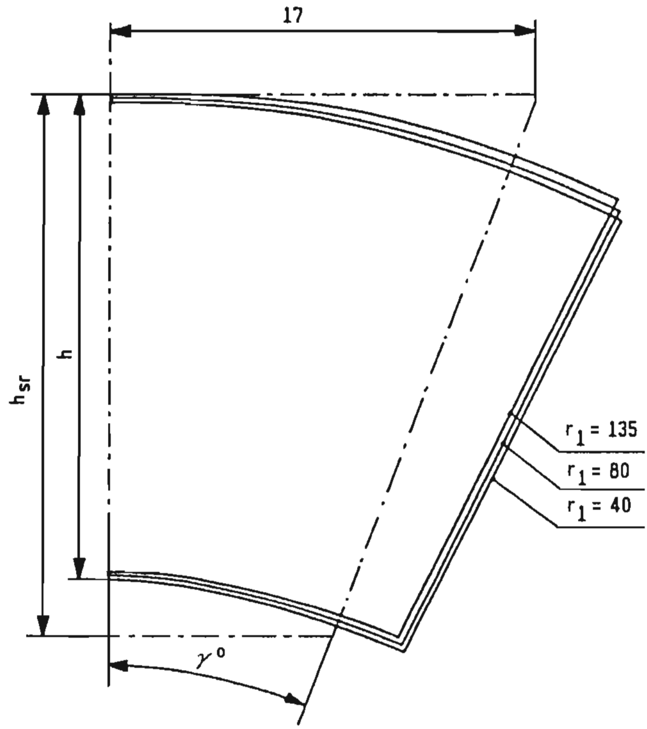

Fig. 5. Deformation of a belt cross-section for assumed variable angle $\gamma^0 = \gamma_0$ and $h \leq h_{sr}$

Summarizing, application of continuum non-linear mechanics methods using computer aided design leads to effective, closer to reality methods of calculations and designing of the complex form geometry constructions and details. The method allows one to form constructions of least internal friction effect.

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Projektowanie geomerycznej postaci pasów klinowych z materiałów nieliniowo sprężystych

Streszczenie

Przedstawiono metodę projektowania postaci geometrycznej elementów z materiałów gumopodobnych. Uwzględniając rzeczywiste właściwości tych materiałów przyjęto nieliniowo sprężysty model fizyczny jako najbardziej adekwatny. Określono duże skończone odkształcenia oraz stan naprężenia dla złożonego obciążenia. Pozwala to w efekcie projektować np. przekroje poprzeczne elementów o żądanym kształcie.

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