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SOME PROBLEMS OF STRESS DISTRIBUTION IN A PERIODIC STRATIFIED ELASTIC STRATUM

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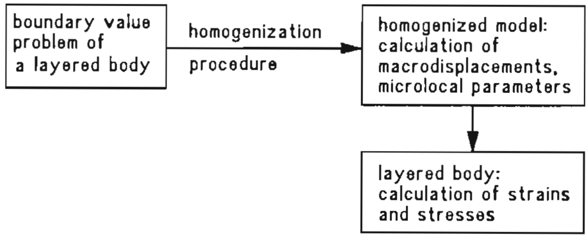
The paper deals with the problems of stress distribution caused by gravitation forces in a periodic stratified elastic layer resting on a rigid foundation. The surface of substrate is assumed to be a plane with an infinitely long slender overlay or a narrow excavation having rectangular cross-sections. The problem is solved within the framework of the homogenized model with microlocal parameters.

1. Introduction

The present paper is concerned with the analysis of stresses in a periodic stratified elastic layer resting on the rigid foundation. The surface of substrate is assumed to be a plane with infinitely long slender overlay or narrow excavation with rectangular cross-sections. The stresses in the stratified layer are caused by gravitation forces.

The problem is solved within the framework of the homogenized model with microlocal parameters given by Woźniak (1986) and (1987), Matysiak and Woźniak (1987). The proposed model permits to evaluate mean and local values of strains and stresses in every material components of the stratified body. The equations of the homogenized model are formulated in terms of the unknown macrodisplacement vector and certain extra unknowns being referred to as microlocal parameters. The algorithm of the microlocal modelling can be presented in the form



By using the theory of elasticity with microlocal parameters, certain problems connected with stress concentrations caused by cracks, rigid stamps as well as wave propagation in laminated composites have been solved (see for references Matysiak and Woźniak, 1988; Kaczyński and Matysiak, 1989; Kaczyński, 1993). This model can be also applied to some problems of rock mechanics (to description of sandstone-slate, sandstone-shale, shale, thin-layered limestone) and soil mechanics (warved clays, Miocene clays and flotation wastes, see R.Kaczyński and Matysiak, 1993).

The considered problems of a narrow excavation or a slender overlay in the periodic stratified body is particularly important in mining engineering. Similar investigations within the classical theory of elasticity have received considerable attention, see for references Dymek (1967), Gill (1991).

2. Basic equations

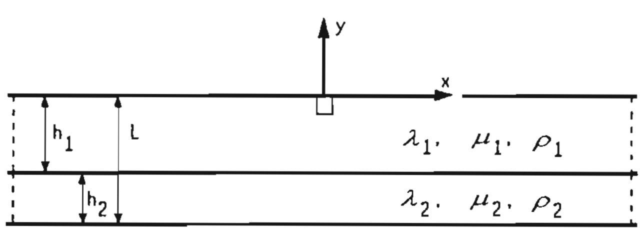


Fig. 1. The middle-cross section of repeated fundamental layer

We consider a periodic stratified body in which each repeated fundamental layer (unit) is composed of two different homogeneous isotropic elastic layers of thickness L characterized by the Lame constants λ_1, μ_1 and λ_2, μ_2 as well as the mass densities ρ_1, ρ_2, respectively. The scheme of the middle cross-section of the basic unit is given in Fig.1. Let (x, y, z) denote the Cartesian coordinate system such that the axis y is normal to the layering. Let h_1, h_2 be the thicknesses of the subsequent layers, so $L = h_1 + h_2$. The perfect bonding between the layers is assumed.

To determine the displacement and stress distributions in the periodic stratified layer we take into consideration the homogenized model with microlocal parameters presented by Woźniak (1987), Matysiak and Woźniak (1987), Kaczyński and Matysiak (1989). In the plane static case of strain, the displacement vector $(U, V, 0)$ is postulated in the form

$$\begin{aligned} U(x,y) &= u(x,y) + \underline{l(y)p(x,y)} \approx u(x,y) \\ V(x,y) &= v(x,y) + \underline{l(y)q(x,y)} \approx v(x,y) \end{aligned} \tag{2.1}$$

where $l : \mathcal{R} \to \mathcal{R}$ is a given δ-periodic sectionally linear function

$$l(y) = \begin{cases} y - \frac{1}{2}h_1 & \text{for} \quad 0 \le y \le h_1 \\ \frac{h_1 - \eta y}{1-\eta} - \frac{1}{2}h_1 & \text{for} \quad h_1 \le y \le \delta \end{cases} \tag{2.2}$$

where

$$\eta = \frac{h_1}{L} \tag{2.3}$$

and roughly speaking, the values of $l(y)$ are small and the underlined terms in Eqs (2.1) can be neglected for small L, but the values of the derivative $l'(y)$ are not small for very thin layers.

The functions $u(\cdot)$, $v(\cdot)$ and $p(\cdot)$, $q(\cdot)$ are unknown functions representing as the macrodisplacements and microlocal parameters, respectively.

According to the results given by Woźniak (1987), Matysiak and Woźniak (1987) the equations of the homogenized model with microlocal parametrs take the form

$$\begin{aligned} &(\widetilde{\lambda} + \widetilde{\mu})(u,_{xy} + v,_{yy}) + \widetilde{\mu}(v,_{xx} + v,_{yy}) + [\mu]p,_x + ([\lambda] + 2[\mu])q,_y = -\widetilde{\rho} b_y \\ &(\widetilde{\lambda} + \widetilde{\mu})(u,_{xx} + v,_{xy}) + \widetilde{\mu}(u,_{xx} + u,_{yy}) + [\mu]p,_y + [\lambda]q,_x = -\widetilde{\rho} b_x \\ &(\widehat{\lambda} + \widehat{\mu})q + [\lambda](u,_x + v,_y) + 2[\mu]v,_y = 0 \\ &\widehat{\mu}p + [\mu](u,_y + v,_x) = 0 \end{aligned} \tag{2.4}$$

where $(b_x, b_y, 0)$ is the body force vector

$$\begin{aligned} (\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\rho}) &= (\eta\lambda_1 + (1-\eta)\lambda_2, \eta\mu_1 + (1-\eta)\mu_2, \eta\rho_1 + (1-\eta)\rho_2) \\ ([\lambda], [\mu]) &= (\eta(\lambda_1 - \lambda_2), \eta(\mu_1 - \mu_2)) \\ (\widehat{\lambda}, \widehat{\mu}) &= \Big(\eta\lambda_1 + \frac{\eta^2\lambda_2}{1-\eta}, \eta\mu_1 + \frac{\eta^2\mu_2}{1-\eta}\Big) \end{aligned} \tag{2.5}$$

and the comma denotes partial differentiation.

Eliminating microlocal parametrs p, q from Eqs $(2.4)_{1,2}$ (by using Eqs $(2.4)_{3,4}$) the formulae for macrodisplacements u, v are obtained in the form (Kaczyński and Matysiak, 1989)

$$
\begin{aligned}
A_2 u,_{xx} + (B+C) v,_{xy} + C u,_{yy} &= -\tilde{\rho} b_x \\
A_1 v,_{yy} + (B+C) u,_{xy} + C v,_{xx} &= -\tilde{\rho} b_y
\end{aligned}
\tag{2.6}
$$

where

$$
\begin{aligned}
A_1 &= \frac{(\lambda_1+2\mu_1)(\lambda_2+2\mu_2)}{(1-\eta)(\lambda_1+2\mu_1)+\eta(\lambda_2+2\mu_2)} > 0 \\
A_2 &= \frac{4\eta(1-\eta)(\mu_1-\mu_2)(\lambda_1-\lambda_2+\mu_1-\mu_2)}{(1-\eta)(\lambda_1+2\mu_1)+\eta(\lambda_2+2\mu_2)} + A_1 > 0 \\
B &= \frac{(1-\eta)\lambda_2(\lambda_1+2\mu_1)+\eta\lambda_1(\lambda_2+2\mu_2)}{(1-\eta)(\lambda_1+2\mu_1)+\eta(\lambda_2+2\mu_2)} > 0 \\
C &= \frac{\mu_1\mu_2}{(1-\eta)\mu_1+\eta\mu_2} > 0
\end{aligned}
\tag{2.7}
$$

To formulate the stress boundary value-problems for Eqs (2.6) we have to take into account the following relations (Kaczyński and Matysiak, 1989)

$$
\begin{aligned}
\sigma_{yy}^{(j)} &= B u,_x + A_1 v,_y \\
\sigma_{xy}^{(j)} &= C(u,_y + v,_x) \\
\sigma_{xx}^{(j)} &= D_j v,_y + E_j u,_x \\
\sigma_{zz}^{(j)} &= (\sigma_{xx}^{(j)} + \sigma_{yy}^{(j)})\frac{\lambda_j}{2(\lambda_j+\mu_j)}
\end{aligned}
\tag{2.8}
$$

where

$$
\begin{aligned}
D_j &= \lambda_j \frac{A_1}{\lambda_j + 2\mu_j} \\
E_j &= 4\mu_j \frac{\lambda_j+\mu_j}{\lambda_j+2\mu_j} + \lambda_j \frac{B}{\lambda_j+2\mu_j}
\end{aligned}
\tag{2.9}
$$

and the index j, $j = 1,2$, is related to the layers of the first kind (with material constants λ_1, μ_1, ρ_1; then $j = 1$) or the second kind (with material constants λ_2, μ_2, ρ_2; then $j = 2$), respectively.

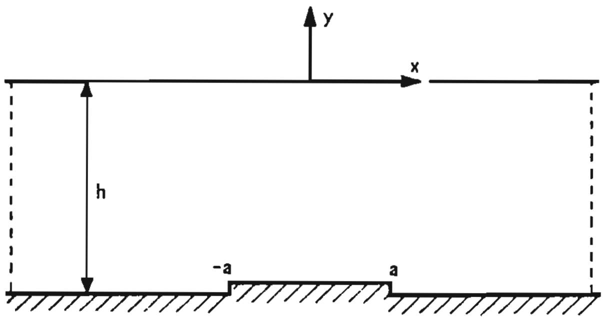


Fig. 2. The middle-cross section of the stratum resting on the foundation with an overlay

3. Problem of a slender overlay

Consider now the plane problem of a periodic stratified layer resting on the rigid substratum. The surface of the foundation is a plane with a slender overlay, see Fig.2. To determine the displacements and stresses in the periodic stratified layer, the homogenized model with microlocal parameters described by Eqs (2.6) and (2.7) is applied. Let the body forces be given by

$$b_x = 0 \qquad b_y = g \tag{3.1}$$

where $g =$ constant, is the acceleration of gravity. The problem under consideration is described by the following boundary conditions

$$\begin{aligned} &\sigma_{yy}^{(1)}(x, y = 0) = 0 \\ &\sigma_{xy}^{(1)}(x, y = 0) = 0 \qquad x \in \mathcal{R} \end{aligned} \tag{3.2}$$

and

$$\begin{array}{lll} v(x, y = -h) = v_0 & \text{for} & x \in (-a, a) \\ v(x, y = -h) = 0 & \text{for} & x \in \mathcal{R} \backslash < -a, a > \\ \sigma_{xy}^{(2)}(x, y = -h) = 0 & \text{for} & x \in \mathcal{R} \end{array} \tag{3.3}$$

where

h – thickness of the layer, ($h = nL$, n is a sufficiently large natural number)

v_0 – thickness of overlay, $v_0 \ll h$, see Fig.2.

Making use of the superposition principle the problem stated above is separated into two parts. The first part satisfies Eqs (2.6) with the RHS given by Eq (3.1) and boundary conditions (3.2). The solution of the problem takes the form

$$
\begin{aligned}
&u(x,y) = 0 \\
&v(x,y) = \frac{\tilde{\rho} g}{2A_1}(h^2 - y^2) \\
&\sigma_{xy}^{(j)} = 0 \\
&\sigma_{yy}^{(j)}(x,y) = -\tilde{\rho} g y \\
&\sigma_{xx}^{(j)}(x,y) = -D_j \tilde{\rho} g \frac{y}{A_1} \qquad x \in \mathcal{R} \qquad y \in < -h, 0 >
\end{aligned}
\tag{3.4}
$$

The second part satisfies Eqs (2.4) for $b_x = 0$, $b_y = 0$ and boundary conditions (3.2) and (3.3). To obtain the solution of this problem,the method of integral Fourier transforms (see, for instance Sneddon, 1951) is applied. Two different cases given below can be marked out.

3.1. Case 1. $\mu_1 \neq \mu_2$

For the case $\mu_1 \neq \mu_2$ one obtains

$$
\begin{aligned}
u(x,y) &= \sqrt{\frac{2}{\pi}} \mathcal{F}_s\{\xi^{-1}\varphi(\xi)\Phi_0(\xi,y); \xi \to x\} \\
v(x,y) &= \sqrt{\frac{2}{\pi}} \mathcal{F}_c\{\xi^{-1}\varphi(\xi)\Phi_1(\xi,y); \xi \to x\} \\
\sigma_{yy}^{(j)}(x,y) &= \sqrt{\frac{2}{\pi}} \mathcal{F}_c\{\varphi(\xi)\Phi_2(\xi,y); \xi \to x\} \\
\sigma_{xy}^{(j)}(x,y) &= \sqrt{\frac{2}{\pi}} C\mathcal{F}_s\{\varphi(\xi)\Phi_3(\xi,y); \xi \to x\} \\
\sigma_{xx}^{(j)}(x,y) &= \sqrt{\frac{2}{\pi}} \mathcal{F}_c\{\varphi(\xi)\Phi_4^{(j)}(\xi,y); \xi \to x\}
\end{aligned}
\tag{3.5}
$$

where

$$
\begin{aligned}
\Phi_0(\xi,y) &= e^{k_1\xi y} + \beta(\xi)e^{-k_1\xi y} + \beta_1\Big(e^{k_2\xi y} + \beta(\xi)e^{-k_2\xi y}\Big) + \\
&\quad + \beta_2\Big(e^{-k_2\xi y} + \beta(\xi)e^{k_2\xi y}\Big) \\
\Phi_1(\xi,y) &= G_1\Big(e^{k_1\xi y} - \beta(\xi)e^{-k_1\xi y}\Big) + G_2\Big[\beta_1\Big(e^{k_2\xi y} - \beta(\xi)e^{-k_2\xi y}\Big) -
\end{aligned}
$$

$$-\beta_2\Big(e^{-k_2\xi y} - \beta(\xi)e^{k_2\xi y}\Big)\Big]$$

$$\Phi_2(\xi,y) = (B + A_2G_1k_1)\Big(e^{k_1\xi y} + \beta(\xi)e^{-k_1\xi y}\Big) + (B + A_2G_2k_2)\cdot$$
$$\cdot\Big[\beta_1\Big(e^{k_2\xi y} + \beta(\xi)e^{-k_2\xi y}\Big) + \beta_2\Big(e^{-k_2\xi y} + \beta(\xi)e^{k_2\xi y}\Big)\Big]$$

$$\Phi_3(\xi,y) = k_1e^{k_1\xi y} - \beta(\xi)k_1e^{-k_1\xi y} + \beta_1k_2\Big(e^{k_2\xi y} - \beta(\xi)e^{-k_2\xi y}\Big) +$$
$$+\beta_2k_2\Big(\beta(\xi)e^{k_2\xi y} - e^{-k_2\xi y}\Big) - \Phi_1(\xi,y)$$

$$\Phi_4^{(j)}(\xi,y) = D_jG_1k_1\Big(e^{k_1\xi y} + \beta(\xi)e^{-k_1\xi y}\Big) + D_jG_2\beta_1k_2\Big(e^{k_2\xi y} + \beta(\xi)e^{-k_2\xi y}\Big) +$$
$$+D_jG_2\beta_2k_2\Big(\beta(\xi)e^{k_2\xi y} + e^{-k_2\xi y}\Big) + E_j\Phi_0(\xi,y) \tag{3.6}$$

$$\beta_1 = -\frac{1}{2}\Big(\frac{B + A_1G_1k_1}{B + A_1G_2k_2} + \frac{k_1 - G_1}{k_2 - G_2}\Big)$$
$$\beta_2 = -\frac{1}{2}\Big(\frac{B + A_1G_1k_1}{B + A_1G_2k_2} - \frac{k_1 - G_1}{k_2 - G_2}\Big)$$
$$k_1 = \sqrt{\frac{C^2 + A_1A_2 - (B + C)^2 + \sqrt{\Delta}}{2A_1C}} \in R$$
$$k_2 = \sqrt{\frac{C^2 + A_1A_2 - (B + C)^2 - \sqrt{\Delta}}{2A_1C}} \in R$$
$$G_j = \frac{Ck_j^2 - A_2}{B + C}k_j \qquad j = 1,2$$
$$\beta(\xi) = \Big[(k_1 - G_1)e^{-2k_1\xi y} + (k_2 - G_2)\Big(\beta_1e^{-\xi(k_1+k_2)h} - \beta_2e^{-\xi(k_1-k_2)h}\Big]\cdot$$
$$\cdot\Big[k_1 - G_1 + (k_2 - G_2)\Big(\beta_1e^{-\xi(k_1-k_2)h} - \beta_2e^{-\xi(k_1+k_2)h}\Big]$$
$$\Delta = [C^2 + A_1A_2 - (B + C)^2]^2 - 4A_1A_2C^2 > 0$$

and

$$\mathcal{F}_c\{f(\xi,y);\xi\to x\} = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(\xi,y)\cos(\xi x)\,d\xi$$
$$\mathcal{F}_s\{f(\xi,y);\xi\to x\} = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(\xi,y)\sin(\xi x)\,d\xi \tag{3.7}$$

The unknown function $\varphi(\cdot)$ will be determined satisfying boundary conditions

$(3.3)_{1,2}$. From Eqs $(3.5)_2$ and $(3.3)_{1,2}$ it follows that

$$\varphi(\xi) = \sqrt{\frac{2}{\pi}} v_0 \frac{\sin(a\xi)}{\Phi_1(\xi, y=-h)} \tag{3.8}$$

The second part of solution is given in form of the Fourier integrals by Eqs (3.5) and (3.11). The final solution describing displacements and stresses in the layered stratum can be written by summing up the results given in Eqs (3.4), (3.5) and function $\varphi(\cdot)$ defined in Eq (3.11). Thus, for $y = -h$ after some calculations one obtains

$$\begin{aligned} \sigma_{yy}^{(2)}(x, y=-h) &= -\widetilde{\rho}gh + 2v_0 X_0 \frac{a}{\pi(a^2 - x^2)} + \\ &+ 2v_0 \frac{1}{\pi} \int\limits_0^\infty \Big[X(\xi, y=-h) - X_0\Big] \sin(a\xi)\cos(\xi x)\, d\xi \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} X(\xi, y) &= \frac{\Phi_2(\xi, y)}{\Phi_1(\xi, y)} \\ X_0 &= \frac{(B + A_2G_1k_1)(k_2 - G_2) - (B + A_2G_2k_2)(k_1 - G_1)}{k_1G_2 - k_2G_1} \end{aligned} \tag{3.10}$$

From Eqs (3.10) and (3.6) it follows that

$$\lim_{x\to\infty} X(\xi, y) = \begin{cases} 0 & \text{for} \quad -h < y \leq 0 \\ X_0 & \text{for} \quad y = -h \end{cases} \tag{3.11}$$

so the integral in the RHS of Eq (3.9) is convergent. It yields

$$\sigma_{yy}^{(2)}(x, y=-h) = \frac{2v_0X_0a}{\pi(a^2 - x^2)} + O(1) \tag{3.12}$$

3.2. Case 2. $\mu_1 = \mu_2$

In the case $\mu_1 = \mu_2$ the solution of Eqs (2.6) together with $b_x = 0$, $b_y = 0$ and boundary conditions (3.2), $(3.3)_3$ can be written in the form

$$u(x, y) = \sqrt{\frac{2}{\pi}} \mathcal{F}_s\{\xi^{-1}\widehat{\varphi}(\xi)\widehat{\Phi}_0(\xi, y); \xi \to x\}$$

$$v(x, y) = \sqrt{\frac{2}{\pi}} \mathcal{F}_c\{\xi^{-1}\widehat{\varphi}(\xi)\widehat{\Phi}_1(\xi, y); \xi \to x\}$$

$$
\begin{aligned}
\sigma_{yy}^{(j)}(x,y) &= \sqrt{\frac{2}{\pi}}\mathcal{F}_c\{\widehat{\varphi}(\xi)\widehat{\Phi}_2(\xi,y);\xi\to x\} \\
\sigma_{xy}^{(j)}(x,y) &= \sqrt{\frac{2}{\pi}}C\mathcal{F}_s\{\widehat{\varphi}(\xi)\widehat{\Phi}_3(\xi,y);\xi\to x\} \\
\sigma_{xx}^{(j)}(x,y) &= \sqrt{\frac{2}{\pi}}\mathcal{F}_c\{\widehat{\varphi}(\xi)\widehat{\Phi}_4^{(j)}(\xi,y);\xi\to x\}
\end{aligned}
\tag{3.13}
$$

where

$$
\begin{aligned}
\widehat{\Phi}_0(\xi,y) &= \Big[1+\widehat{D}\xi y\Big(1-\widehat{\beta}(\xi)\Big)\Big]e^{-\xi y}+ \\
&\quad +\Big[\widehat{\beta}(\xi)+\widehat{D}\Big(1-\widehat{\beta}(\xi)\Big)\xi y\Big]e^{\xi y} \\
\widehat{\Phi}_1(\xi,y) &= \Big[\widehat{G}_1+\widehat{D}\Big(1-\widehat{\beta}(\xi)\Big)\Big(\widehat{G}_2+\widehat{G}_1\xi y\Big)\Big]e^{-\xi y}+ \\
&\quad +\Big[-\widehat{G}_1\widehat{\beta}(\xi)+\widehat{D}\Big(1-\widehat{\beta}(\xi)\Big)\Big(\widehat{G}_2-\widehat{G}_1\xi y\Big)\Big]e^{\xi y} \\
\widehat{\Phi}_2(\xi,y) &= -(2B+C)\Big(1-\widehat{\beta}(\xi)\Big)\Big(e^{-\xi y}-e^{\xi y}\Big) \\
\widehat{\Phi}_3(\xi,y) &= -\Big[1+\widehat{D}\Big(1-\widehat{\beta}(\xi)\Big)\xi(-1+y)\Big]e^{-\xi y}+ \\
&\quad +\Big[\widehat{\beta}(\xi)+\widehat{D}\Big(1-\widehat{\beta}(\xi)\Big)\xi(1+y)\Big]e^{\xi y}-\Phi_1(\xi,y) \\
\widehat{\Phi}_4^{(j)}(\xi,y) &= -D_j\Big[\widehat{G}_1+\widehat{D}\Big(1-\widehat{\beta}(\xi)\Big)\Big(\widehat{G}_2+\widehat{G}_1\xi y-\widehat{G}_1\xi\Big)\Big]e^{-\xi y}+ \\
&\quad +D_j\Big[-\widehat{G}_1\widehat{\beta}(\xi)+\widehat{D}\Big(1-\widehat{\beta}(\xi)\Big)\Big(\widehat{G}_2-\widehat{G}_1\xi y-\widehat{G}_1\xi\Big)\Big]e^{\xi y}+E_j\Phi_0(\xi,y) \\
\widehat{D} &= -\frac{2B+C}{2C} \qquad \widehat{G}_1=\frac{B}{B+C} \qquad \widehat{G}_2=\frac{B+2C}{B+C}
\end{aligned}
\tag{3.14}
$$

$$
\widehat{\beta}(\xi)=\frac{-1-\widehat{G}_1+\widehat{D}(1-\widehat{G}_2)+\widehat{D}(1+\widehat{G}_1)\xi h+\widehat{D}\Big[1-\widehat{G}_2-(1+\widehat{G}_2)\xi h\Big]e^{-2\xi h}}{\widehat{D}\Big[1-\widehat{G}_2+(1+\widehat{G}_1)\xi h\Big]+\Big[-1-\widehat{G}_1+\widehat{D}(1-\widehat{G}_2)-\widehat{D}(1+\widehat{G}_1)\xi h\Big]e^{-2\xi h}}
$$

The unknown function $\widehat{\varphi}(\cdot)$ will be determined satisfying the boundary conditions $(3.3)_{1,2}$. From (3.13) and $(3.3)_{1,2}$ it follows that

$$
\widehat{\varphi}(\xi)=\sqrt{\frac{2}{\pi}}\frac{v_0\sin(a\xi)}{\Phi_1(\xi,y=-h)}
\tag{3.15}
$$

Similarly as in Case 1 the final solution describing displacements and stresses in the stratified layer can be written by summing up the results given in (3.4), (3.13), (3.14) and function $\widehat{\varphi}(\cdot)$ determined in Eq (3.15). For $y = -h$ the normal stress $\sigma_{yy}^{(2)}$ is expressed in the form

$$\begin{aligned}\sigma_{yy}^{(2)}(x, y = -h) &= -\widetilde{\rho} g h + 2v_0 \widehat{X}_0 \frac{a}{\pi(a^2 - x^2)} + \\ &+ 2v_0 \frac{1}{\pi} \int\limits_0^\infty \Big[\widehat{X}(\xi, y = -h) - \widehat{X}_0\Big] \sin(a\xi)\cos(\xi x)\, d\xi\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}\widehat{X}(\xi, y) &= \frac{\widehat{\Phi}_2(\xi, y)}{\widehat{\Phi}_1(\xi, y)} \\ \widehat{X}_0 &= -\frac{(2B + C)(1 + \widehat{G}_1)}{\widehat{D}(\widehat{G}_1 + \widehat{G}_2)}\end{aligned} \tag{3.17}$$

From Eq (3.16) it follows that

$$\sigma_{yy}^{(2)}(x, y = -h) = 2v_0 \widehat{X}_0 \frac{a}{\pi(a^2 - x^2)} + O(1) \tag{3.18}$$

3.3. Remark

The case of homogeneous layer resting on the rigid foundation with an overlay is given on assumption that

$$\lambda_1 = \lambda_2 \equiv \lambda \qquad \mu_1 = \mu_2 \equiv \mu \qquad \rho_1 = \rho_2 \equiv \rho \tag{3.19}$$

Then, from Eqs (2.5), (2.6), (2.9), and (3.6) we obtain

$$\begin{aligned}&A_1 = A_2 = \lambda + 2\mu \qquad &&B = \lambda \qquad &&C = \mu \\ &D_j = \lambda &&E_j = \lambda + 2\mu\end{aligned} \tag{3.20}$$

Eqs (2.6) and (2.8) together with Eq (3.20) lead to the case of homogeneous body described by the equations of the classical theory of elasticity.

Substituting Eqs (3.19), (3.20) into Eqs (3.13), (3.14) and (3.4) we obtain the solution of the problem for homogeneous layer resting on the rigid foundation with an overlay.

3.4. Numerical example

As an example, we consider the laminated stratum such that

$$\lambda_1 = \mu_1 \qquad \lambda_2 = \mu_2 \tag{3.21}$$

so that $\nu_1 = \nu_2 = 0.25$, where ν_1, ν_2 are the Poisson ratios of the subsequent layers. Then, from Eqs (2.7) it follows that

$$\begin{aligned} A_1 &= 3C & A_2 &= \frac{8\widetilde{\mu} + C}{3} \\ B &= C & C &= \widetilde{\mu} - \frac{[\mu]^2}{\widehat{\mu}} \end{aligned} \tag{3.22}$$

On introducing the notations

$$\alpha \equiv \frac{\mu_2}{\mu_1} \tag{3.23}$$

and

$$w \equiv \frac{A_2}{C} = \frac{1}{3} + \frac{8}{3}\frac{(1-\eta)\eta + (2\eta^2 - 2\eta + 1)\alpha + (1-\eta)\eta\alpha^2}{(1-\eta+\eta\alpha)[\eta + (1-\eta)\alpha] - (1-\eta)(1-\alpha)^2\eta} \tag{3.24}$$

and using Eqs (3.22), (3.6) and $(3.10)_2$ we obtain

$$\begin{aligned} k_1 &= \sqrt{\frac{1}{2}(w-1) + \frac{1}{6}\sqrt{9(w-1)^2 - 12w}} \\ k_2 &= \sqrt{\frac{1}{2}(w-1) - \frac{1}{6}\sqrt{9(w-1)^2 - 12w}} \\ G_j &= \frac{1}{2}\frac{k_j^2 - w}{k_j} \\ X_0 &= C\frac{(1 + wG_1k_1)(k_2 - G_2) - (1 + wG_2k_2)(k_1 - G_1)}{k_1G_2 - k_2G_1} \end{aligned} \tag{3.25}$$

The effects of the periodic laminated structure of the stratum on the nondimensional coefficient of normal stress concentration (see Eq (3.12)) X_0^* for two cases of $\delta \equiv v_0/a$, where

$$X_0^* \equiv 2v_0X_0\frac{a}{\pi(2B + C)} \tag{3.26}$$

is presented in Fig.3. Fig.4 shows the nondimensional stresses $\sigma_{yy}^* \equiv \sigma_{yy}^{(2)}(x, y = -h)/(2B+C)$ (see Eq (3.9)) for $\delta = 0.05$, $\delta = 0.03$ and $\delta = 0.01$.

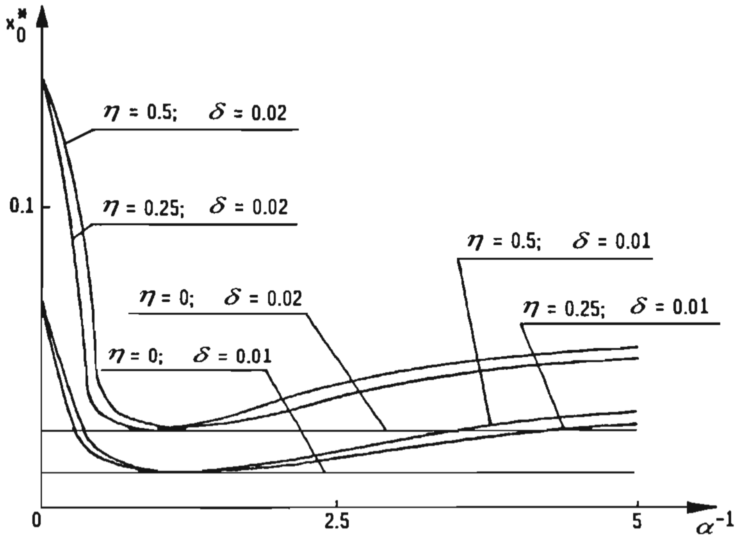


Fig. 3. The nondimensional coefficient of normal stress concentration versus $\alpha^{-1} = \mu_1/\mu_2$

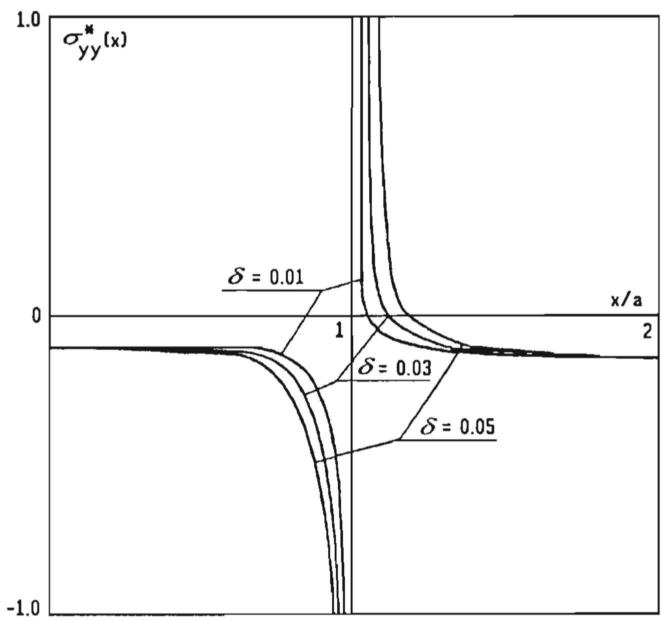


Fig. 4. The nondimenional normal stresses versus x/a

4. Problem of a narrow excavation

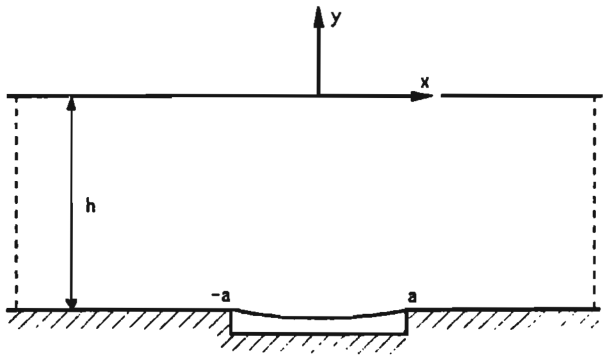


Fig. 5. The middle-cross section of the stratum resting on the foundation with excavation

Consider now the periodically laminated ponderable layer resting on the rigid substratum, where the surface of foundation is assumed to be a plane with a narrow deep excavation with the rectangular cross-section, see Fig.5. The problem under consideration is described by the boundary conditions (3.2) for $y = 0$ and

$$\begin{aligned} &\sigma_{yy}^{(2)}(x, y = -h) = 0 && \text{for} \quad |x| < a \\ &v(x, y = -h) = 0 && \text{for} \quad |x| > a \\ &\sigma_{xy}^{(2)}(x, y = -h) = 0 && \text{for} \quad x \in \mathcal{R} \end{aligned} \tag{4.1}$$

Making use of the superposition principle the problem stated above is separated into two parts. The solution of the first one is given by Eqs (3.4). the second part satisfies Eqs (2.4) with $b_x = b_y = 0$, the boundary conditions (3.2) and

$$\begin{aligned} &\sigma_{yy}^{(2)}(x, y = -h) = -\tilde{\rho} g h && \text{for} \quad |x| < a \\ &v(x, y = -h) = 0 && \text{for} \quad |x| > a \\ &\sigma_{xy}^{(2)}(x, y = -h) = 0 && \text{for} \quad x \in \mathcal{R} \end{aligned} \tag{4.2}$$

Below we consider two different cases.

4.1. Case 1. $\mu_1 \neq \mu_2$

For the case $\mu_1 \neq \mu_2$ the solution of the second part is given by Eqs (3.5) together with Eq (3.6), where function $\varphi(\cdot)$ has to satisfy the following dual integral equations

$$\begin{aligned} &\mathcal{F}_c\{\varphi(\xi)\Phi_2(\xi, y = -h); \xi \to x\} = -\sqrt{\tfrac{2}{\pi}}\tilde{\rho} g h && \text{for} \quad 0 < x < a \\ &\mathcal{F}_c\{\xi^{-1}\varphi(\xi)\Phi_1(\xi, y = -h); \xi \to x\} = 0 && \text{for} \quad x > a \end{aligned} \tag{4.3}$$

Denoting by

$$\psi(\xi) \equiv \xi^{-1}\varphi(\xi)\Phi(\xi, y = -h) \tag{4.4}$$

and using Eqs (4.3), (3.10), (3.11) we obtain

$$\begin{array}{lll} \mathcal{F}_c\{\xi\psi(\xi)[X_0 + H(\xi)]; \xi \to x\} = -\sqrt{\frac{2}{\pi}}\tilde{\rho}gh & \text{for} & 0 < x < a \\ \mathcal{F}_c\{\psi(\xi); \xi \to x\} = 0 & \text{for} & x > a \end{array} \tag{4.5}$$

where

$$H(\xi) = X(\xi, y = -h) - X_0 \tag{4.6}$$

Let the solution of dual integral equations (4.5) have the form (Sneddon, 1966)

$$\psi(\xi) = \int\limits_0^a ts(t)J_0(\xi t)\, dt \tag{4.7}$$

where $s(t)$ is an unknown function and $J_0(\cdot)$ denotes the Bessel function of the first kind. With $\psi(\xi)$ given in Eq (4.7), equation $(4.5)_2$ is satisfied and Eq $(4.5)_1$ reduces to the following integral Fredholm equation of the second kind for determination of the unknown function $s(t)$

$$s(t) + \int\limits_0^a u\mathcal{K}(u,t)s(u)\, du = -\frac{\pi}{2}\tilde{\rho}gh \qquad \text{for} \quad 0 \le t \le a \tag{4.8}$$

where the kernel $\mathcal{K}(u,t)$ is given by

$$\mathcal{K}(u,t) = \int\limits_0^\infty \xi H(\xi)J_0(\xi t)J_0(\xi u)\, d\xi \tag{4.9}$$

Using (4.4) and (4.7) we obtain

$$\varphi(\xi) = \frac{\xi}{\Phi_1(\xi, y = -h)} \int\limits_0^a ts(t)J_0(\xi t)\, dt \tag{4.10}$$

Substituting for the function $\varphi(\cdot)$ given by Eqs (4.10) and (4.8) into Eq (3.5), the displacements and stresses in the laminated stratum are given in terms of the Fourier integrals and the function $s(\cdot)$. The normal stress component can be written in the form

$$\begin{aligned} \sigma_{yy}^{(2)}(x, y = -h) \quad &= \quad -\tilde{\rho}gh - \sqrt{\frac{2}{\pi}}X_0 x \int\limits_0^a s(\tau)\frac{d\tau}{\sqrt{(x^2 - \tau^2)^3}} + \\ &+ \sqrt{\frac{2}{\pi}} \int\limits_0^a s(\tau)S(\tau, x)\, d\tau \qquad \text{for} \quad x > a \end{aligned} \tag{4.11}$$

where

$$S(\tau, x) = \int_0^\infty [X(\xi, y = -h) - X_0] J_0(\xi\tau) \sin(\xi x) \, d\xi \tag{4.12}$$

The normal stress component has the singularity at the points $(\pm a, -h)$ as follows

$$\sigma_{yy}^{(2)}(x, y = -h) = -\sqrt{\frac{2}{\pi}} X_0 \frac{x s(a)}{\sqrt{x^2 - a^2}} + O(1) \qquad \text{for} \quad |x| > a \tag{4.13}$$

4.2. Case 2. $\mu_1 = \mu_2$

Similarly as in Case 1 the solution of the problem has to be composed of the displacements and stresses given by Eqs (3.4), (3.13) and (3.14), where function $\widehat{\varphi}(\xi)$ is determined by

$$\widehat{\varphi}(\xi) = \frac{\xi}{\Phi_1(\xi, y = -h)} \int_0^a t\widehat{s}(t) J_0(\xi t) \, dt \tag{4.14}$$

and $\widehat{s}(t)$ is the solution of the integral equation

$$\widehat{s}(t) + \int_0^a u\widehat{\mathcal{K}}(u, t)\widehat{s}(u) \, du = -\frac{\pi}{2}\widetilde{\rho} g h \qquad 0 \le t \le a \tag{4.15}$$

with the kernel

$$\widehat{\mathcal{K}}(u, t) = \int_0^\infty \xi H(\xi) J_0(\xi t) J_0(\xi u) \, d\xi \tag{4.16}$$

The normal stress component $\sigma_{yy}^{(2)}(x, y = -h)$ is given by (4.11) replacing X_0 by $widehat X_0$ and $s(a)$ by $\widehat{s}(a)$.

4.3. Numerical example

Consider the case given by Eq (3.21). The solution of the integral Fredholm equation (4.8) at point $t = a$ is presented in Fig.6. Fig.7 shows nondimensional stresses σ_{yy}^* (see Eq (4.11)) for $B_0 = 0.1$, $B_0 = 0.3$, $B_0 = 0.5$ where

$$B_0 \equiv \frac{\widetilde{\rho} g a}{2B + C} \tag{4.17}$$

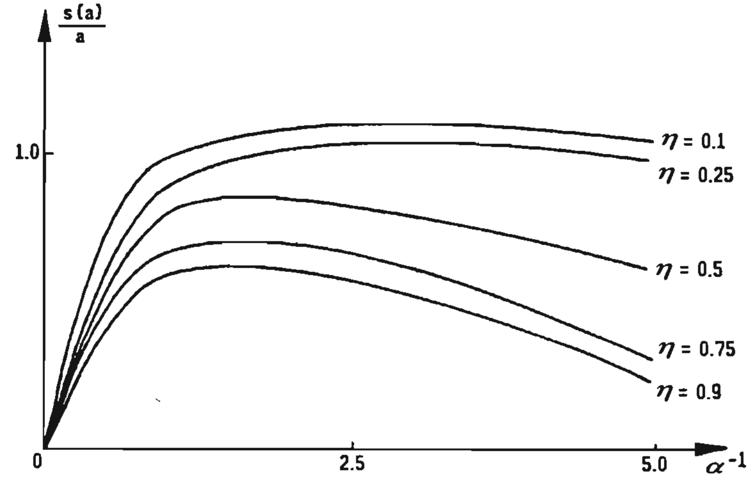


Fig. 6. The coefficient $s(a)/a$ versus $\alpha^{-1} = \mu_1/\mu_2$

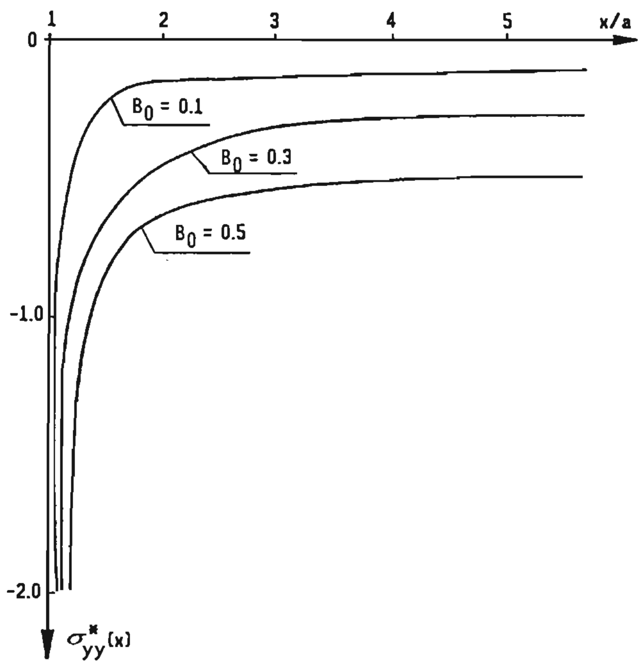


Fig. 7. The nondimensional normal stresses versus x/a

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Pewne zagadnienia rozkładu naprężeń w periodycznie uwarstwionym sprężystym pokładzie

Streszczenie

W pracy rozpatrzono zagadnienie rozkladu naprężeń wywołanych silami masowymi w periodycznie uwarstwionym pokladzie spoczywającym na sztywnym podlożu. Powierzchnia podloża jest przyjęta jako plaszczyzna z nieskończenie dlugą, cienką nadkladką lub wąskim wycięciem o prostokątnych przekrojach. Zagadnienie zostalo rozwiązane w ramach modelu homogenizowanego z parametrami mikrolokalnymi.

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