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THERMOELASTIC VIBRATIONS OF THE THIN PLATE INDUCED BY MOVING HEAT SOURCE

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The case of thermoelastic vibrations of the thin plate simply supported on the entire edge, induced by the moving heat source is considered. The problem is solved by using of the integral transform method. As the illustration of the problem, a numerical example is given.

1. Introduction

In this paper the case of transversal vibrations of thin plate produced by nonstationary temperature field, induced by moving heat source is considered. The problem is formulated on the basis of the linear theory of thermoelasticity (Nowacki, 1970) and the classical theory of thin plates (Čukić, 1979). Analytical solutions are calculated by using the finite integral transform technique (Čukić, 1979), and moving heat source is described by the Heaviside and similar functions (Rożnowski, 1989).

The following assumptions are taken

- temperature changes linearly across the thickness of the plate,
- the longitudinal vibrations of the plate are independent of the transverse vibrations (the disc stresses in the plane are small as compared with the bending stresses),
- the temperature field and the strain field are uncoupled.

2. Basic equations

On the basis of the assumption that the temperature θ changes linearly along the thickness of the plate we have

$$\theta(x_1,x_2,x_3,t) = \tau_0(x_1,x_2,t) + x_3\tau_1(x_1,x_2,t) \tag{2.1}$$

where

$$\tau_0(x_1,x_2,t) = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \theta dx_3 = \frac{\theta_1+\theta_2}{2} \tag{2.2}$$

$$\tau_1(x_1,x_2,t) = \frac{12}{h^3}\int_{-\frac{h}{2}}^{\frac{h}{2}} \theta x_3 dx_3 = \frac{\theta_1-\theta_2}{h} \tag{2.3}$$

and θ_1, θ_2 are the upper and lower face temperatures.

The system of equations defining the problem takes the form

$$\left(\nabla_1^2 - \frac{1}{\kappa}\partial_t\right)\tau_0 = -\frac{q+s}{h\lambda_0} - \frac{1}{\kappa h}\int_{-\frac{h}{2}}^{\frac{h}{2}} Q dx_3 \tag{2.4}$$

$$\left(\nabla_1^2 - \beta_1 - \frac{1}{\kappa}\partial_t\right)\tau_1 = \frac{\beta_1}{2\lambda_0}(s-q) - \frac{\beta_1}{\kappa h}\int_{-\frac{h}{2}}^{\frac{h}{2}} Q x_3 dx_3 \tag{2.5}$$

$$(\nabla_1^4 + K^2\partial_t^2)w + (1+\nu)\alpha_t\nabla_1^2\tau_1 = 0 \tag{2.6}$$

together with the boundary conditions

$$\lambda_0\frac{\partial\theta}{\partial x_3}\Big|_{\frac{h}{2}} = q(x_1,x_2,t) \qquad \lambda_0\frac{\partial\theta}{\partial x_3}\Big|_{-\frac{h}{2}} = -s(x_1,x_2,t) \tag{2.7}$$

where the following notations are applied

∇_1^2 – Laplace operator
∂_t – time derivative
w – deflection of the plate in x_3-direction
h – thickness of the plate
E – modulus of elasticity
κ – coefficient of thermal intensity
λ_0 – heat conduction coefficient

C_ε – specific heat at a constant strain
α_t – coefficient of thermal expansion
ρ – plate density per unit area of the middle surface
ν – Poisson ratio
W – quantity of heat generated in a unit volume and unit time
T_0 – temperature of the plate in its natural state
θ – temperature, $\theta = T - T_0$
N – flexural rigidity of the plate, $N = \frac{Eh^3}{12(1-\nu^2)}$

and

$$Q = Q(x_1, x_2, x_3, t) \qquad Q_0 = \frac{W}{C_\varepsilon} \qquad K^2 = \frac{\rho h}{N} \qquad \beta_1 = \frac{12}{h^2}$$

The bending stresses in the plate are given by

$$\sigma_{ij} = -\frac{Ex_3}{1-\nu^2}\Big\{(1-\nu)w_{,ij} + [\nu w_{,kk} + (1+\nu)\alpha_t\tau_1]\delta_{ij}\Big\} \tag{2.8}$$

2.1. Nonstationary temperature field in a thin rectangular plate produced by a moving heat source

Let the plate be under the influence of the temperature field caused by the heat source moving at a constant velocity v along the symetric axis of the upper surface of a plate (Fig.1).

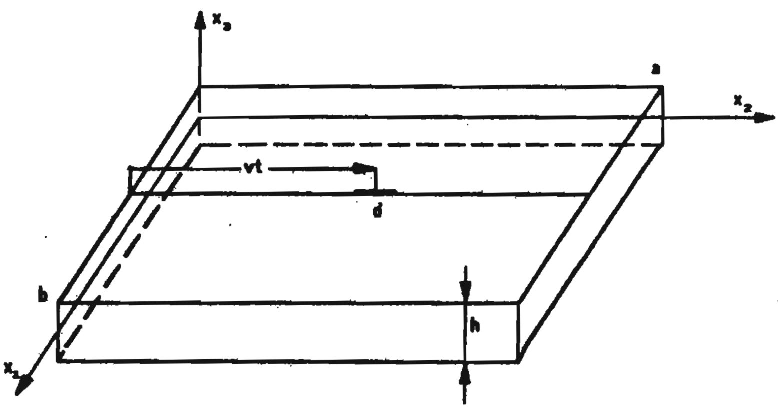

Fig. 1.

Let the thermal initial and boundary conditions be assumed in the form

$$
\begin{aligned}
&\theta\Big|_{t=0} = 0 \\
&\theta\Big|_{x_2=0} = 0 \qquad\qquad \theta|_{x_1=0} = 0 \\
&\theta\Big|_{x_2=a} = 0 \qquad\qquad \theta|_{x_1=b} = 0 \\
&\frac{\partial\theta}{\partial x_3}\Big|_{x_3=\pm\frac{h}{2}} = 0
\end{aligned}
\tag{2.9}
$$

The moving heat source (concentrated or linear) can be taken as

$$
Q = Q_0\delta(x_1 - \frac{b}{2})\delta(x_3 - \frac{h}{2})\Big[H(t) - H(t - \frac{a}{v})\Big]\begin{cases}\delta(x_2 - vt)\\ \frac{1}{d}\prod\left(\frac{x_2 - vt}{d}\right)\end{cases}
\tag{2.10}
$$

where d is the width of linear source, $\delta(\cdot)$ denotes the Dirac delta-function, $H(\cdot)$ denotes the Heaviside and $\prod(\cdot)$ is a pulsation function

$$
\prod\Big(\frac{x_2 - vt}{d}\Big) = H(x_2 - vt + \frac{d}{2}) - H(x_2 - vt - \frac{d}{2})
$$

The heat conduction equations (2.4) and (2.5) may be written in the form

$$
\Big(\nabla_1^2 - \frac{1}{\kappa}\partial_t\Big)\tau_0 = -\frac{Q_0}{\kappa h}\delta(x_1 - \frac{b}{2})\Big[H(t) - H(t - \frac{a}{v})\Big]\begin{cases}\delta(x_2 - vt)\\ \frac{1}{d}\prod\left(\frac{x_2 - vt}{d}\right)\end{cases}
\tag{2.11}
$$

$$
\Big(\nabla_1^2 - \beta_1 - \frac{1}{\kappa}\partial_t\Big)\tau_1 = -\frac{6Q_0}{\kappa h^2}\delta(x_1 - \frac{b}{2})\Big[H(t) - H(t - \frac{a}{v})\Big]\begin{cases}\delta(x_2 - vt)\\ \frac{1}{d}\prod\left(\frac{x_2 - vt}{d}\right)\end{cases}
\tag{2.12}
$$

To solve Eqs (2.11) and(2.12) subjected to the initial and boundary conditions (2.9), we shall use the double finite sine Fourier transform and the Laplace transform. Using the notations

$$
C_k = \begin{cases}\frac{Q_0}{h} & k = 0\\ \frac{6Q_0}{h^2} & k = 1\end{cases}
\qquad\qquad
\beta_k = \begin{cases}\frac{12}{h^2} & k = 1\\ 0 & k = 0\end{cases}
\tag{2.13}
$$

we arrive at the solution for the temperature field in the form

$$\tau_k = \frac{4}{ab} C_k \sum_{m=1}^{\infty} \sum_{n=1,3,\dots}^{\infty} \sin\frac{n\pi}{2}\Big(\frac{\sin\frac{\alpha_m d}{2}}{\frac{\alpha_m d}{2}}\Big) \cdot$$

$$\cdot\Big[f_k(m,n,t)H(t) - (-1)^m f_k(m,n,t-\frac{a}{v})H(t-\frac{a}{v})\Big] \sin\alpha_n x_1 \sin\alpha_m x_2 \qquad (2.14)$$

$$(k = 0,1)$$

where functions $f_k(m,n,t)$ are described by

$$f_k(m,n,t)H(t) = L^{-1}\Big[\frac{\alpha_m v}{[p^2 + (\alpha_m v)^2][p + \kappa(\alpha_n^2 + \alpha_m^2 + \beta_k)]}\Big] =$$

$$= \frac{\kappa(\alpha_n^2 + \alpha_m^2 + \beta_k)\sin\alpha_m vt - \alpha_m v\cos\alpha_m vt + \alpha_m v e^{-t\kappa(\alpha_n^2+\alpha_m^2+\beta_k)}}{(\alpha_m v)^2 + \kappa^2(\alpha_n^2 + \alpha_m^2 + \beta_k)^2} H(t) \qquad (2.15)$$

$$\alpha_n = \frac{n\pi}{b} \qquad \alpha_m = \frac{m\pi}{a}$$

and L^{-1} denotes the inverse Laplace transform.

2.2. Transversal vibrations

Let the plate be simply supported on the entire edge. So, the boundary conditions are given by

$$\begin{aligned}
&w\Big|_{x_1=0} = 0 \qquad\qquad w\Big|_{x_1=b} = 0\\
&M_{11}\Big|_{x_1=0} = \Big[\frac{\partial^2 w}{\partial x_1^2} + \nu\frac{\partial^2 w}{\partial x_2^2} + (1+\nu)\alpha_t\tau_1\Big]N\Big|_{x_1=0} = 0\\
&M_{11}\Big|_{x_1=b} = 0\\
&w\Big|_{x_2=0} = 0 \qquad\qquad w\Big|_{x_2=a} = 0\\
&M_{22}\Big|_{x_2=0} = \Big[\nu\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} + (1+\nu)\alpha_t\tau_1\Big]N\Big|_{x_2=0} = 0\\
&M_{22}\Big|_{x_2=a} = 0
\end{aligned} \qquad (2.16)$$

The initial conditions can be assumed in the form

$$w\Big|_{t=0} = 0 \qquad\qquad \frac{\partial w}{\partial t}\Big|_{t=0} = 0 \qquad (2.17)$$

Consider now the differential equation (2.6) governing the transverse vibrations of the plate. Using the double sine Fourier transform and Laplace transform we obtain the solution

$$
\begin{aligned}
w &= \frac{24Q_0\alpha_t(1+\nu)}{abh^2}\sum_{m=1}^{\infty}\sum_{n=1,3,\dots}^{\infty}\sin\frac{n\pi}{2}\Big(\frac{\sin\frac{\alpha_m d}{2}}{\frac{\alpha_m d}{2}}\Big)\cdot \\
&\cdot\Big[I(m,n,t)H(t)-(-1)^m I(m,n,t-\frac{a}{v})H(t-\frac{a}{v})\Big]\sin\alpha_n x_1\sin\alpha_m x_2
\end{aligned}
\tag{2.18}
$$

where

$$
\begin{aligned}
I(m,n,t) &= \frac{1}{\kappa[(\alpha_m v)^2+\kappa^2(\Delta+\beta_1)^2]}\Big\{\frac{\kappa(\Delta+\beta_1)}{(\alpha_m v)^2+\left(\frac{\Delta}{K}\right)^2}\cdot \\
&\cdot\Big(\alpha_m v\sin\frac{\Delta}{K}t-\frac{\Delta}{K}\sin\alpha_m vt\Big)+\frac{\alpha_m v\frac{\Delta}{K}}{(\alpha_m v)^2-\left(\frac{\Delta}{K}\right)^2}\Big(\cos\frac{\Delta}{K}t-\cos\alpha_m vt\Big)+ \\
&+\frac{\alpha_m v}{\left(\frac{\Delta}{K}\right)^2+\kappa^2(\Delta+\beta_1)^2}\Big[\kappa(\Delta+\beta_1)\sin\frac{\Delta}{K}t-\frac{\Delta}{K}\cos\frac{\Delta}{K}t+\frac{\Delta}{K}e^{-\kappa t(\Delta+\beta_1)}\Big]\Big\} \\
\Delta &= \alpha_m^2+\alpha_n^2
\end{aligned}
\tag{2.19}
$$

2.3. Thermal stresses

Let us use the following notations

$$
\begin{aligned}
\widetilde{w}_{mn}(t) &= C_w I(m,n,t)\sin\alpha_n x_1\sin\alpha_m x_2 H(t) \\
\widetilde{\tau}_{1mn}(t) &= C_\tau f_1(m,n,t)\sin\alpha_n x_1\sin\alpha_m x_2 H(t) \\
C_w &= \frac{24Q_0\alpha_t(1+\nu)}{abh^2}\sin\frac{n\pi}{2}\Big(\frac{\sin\frac{\alpha_m d}{2}}{\frac{\alpha_m d}{2}}\Big) \\
C_\tau &= \frac{C_w}{\alpha_t(1+\nu)}
\end{aligned}
$$

and

$$
\begin{aligned}
w &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big[\widetilde{w}_{mn}(t)-(-1)^m\widetilde{w}_{mn}(t-\frac{a}{v})\Big] \\
\tau_1 &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big[\widetilde{\tau}_{1mn}(t)-(-1)^m\widetilde{\tau}_{1mn}(t-\frac{a}{v})\Big]
\end{aligned}
$$

Substituting for w from Eq (2.18) into Eq (2.8), we obtain the final form of the stress functions

$$\begin{aligned}
\widetilde{\sigma}_{11\,\mathrm{max}(mn)}(t) &= C_\sigma \sin\frac{n\pi}{2}\Big(\frac{\sin\frac{\alpha_m d}{2}}{\frac{\alpha_m d}{2}}\Big)\cdot \\
&\cdot\big[(\alpha_n^2+\nu\alpha_m^2)I(m,n,t)-f_1(m,n,t)\big]\sin\alpha_n x_1 \sin\alpha_m x_2 H(t) \\
\widetilde{\sigma}_{22\,\mathrm{max}(mn)}(t) &= C_\sigma \sin\frac{n\pi}{2}\Big(\frac{\sin\frac{\alpha_m d}{2}}{\frac{\alpha_m d}{2}}\Big)\cdot \\
&\cdot\big[(\alpha_m^2+\nu\alpha_n^2)I(m,n,t)-f_1(m,n,t)\big]\sin\alpha_n x_1 \sin\alpha_m x_2 H(t) \\
\widetilde{\sigma}_{12\,\mathrm{max}(mn)}(t) &= (1-\nu)C_\sigma \sin\frac{n\pi}{2}\Big(\frac{\sin\frac{\alpha_m d}{2}}{\frac{\alpha_m d}{2}}\Big)\cdot \\
&\cdot\alpha_n\alpha_m I(m,n,t)\cos\alpha_n x_1 \cos\alpha_m x_2 H(t) \\
\sigma_{ij\,\mathrm{max}} &= \pm\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big[\widetilde{\sigma}_{ij\,\mathrm{max}(mn)}(t)-(-1)^m\widetilde{\sigma}_{ij\,\mathrm{max}(mn)}\big(t-\frac{a}{v}\big)\Big] \qquad (i,j=1,2) \\
C_\sigma &= \frac{12Q_0\alpha_t E}{(1-\nu)abh}
\end{aligned} \tag{2.20}$$

3. Numerical example

In order to analyse the obtained results we shall consider a numerical example, assuming that the plate is made of carbon steel with the following properties

Symbol	Unit	Value
E	kN/cm^2	$2.1\cdot10^4$
ρ	kg/cm^3	$7.85\cdot10^{-3}$
λ	W/(cm K)	0.502
C_ε	kJ/(kg K)	0.460
κ	cm^2/s	0.140
α	K^{-1}	$1.2\cdot10^{-5}$
ν	–	0.3

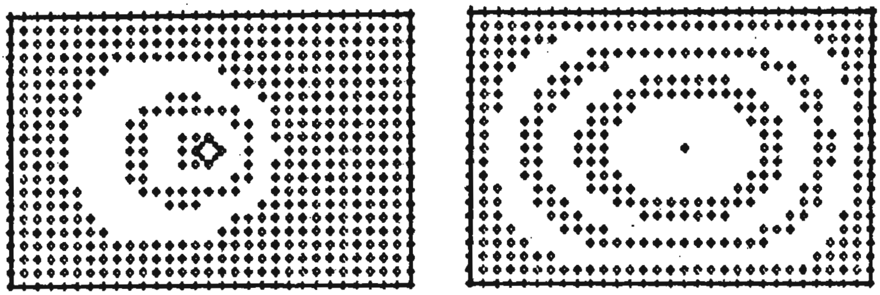

Fig. 2. Temperature field at the instant $t = 500$ s and $t = 1500$ s

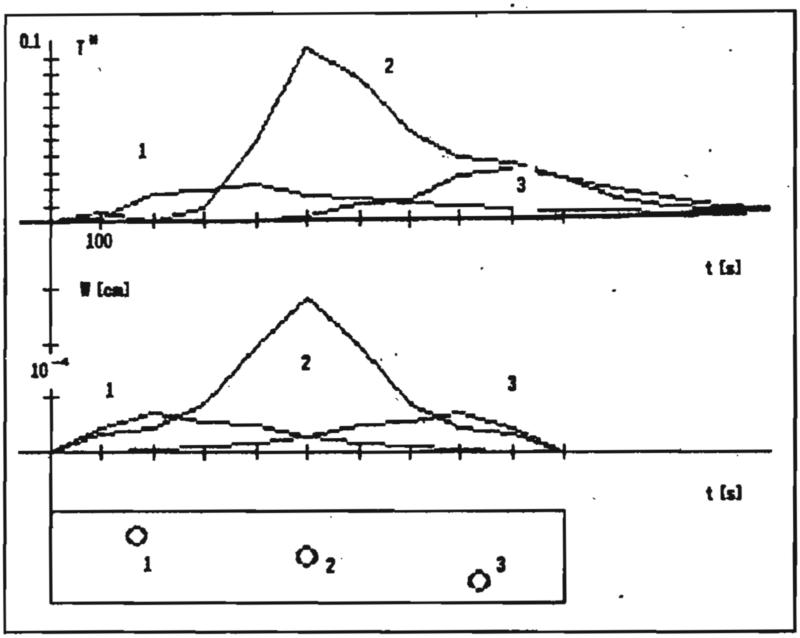

Fig. 3. Temperature $T^* = \frac{T-T_0}{T_0}$ and deflection w plotted as a functions of time t

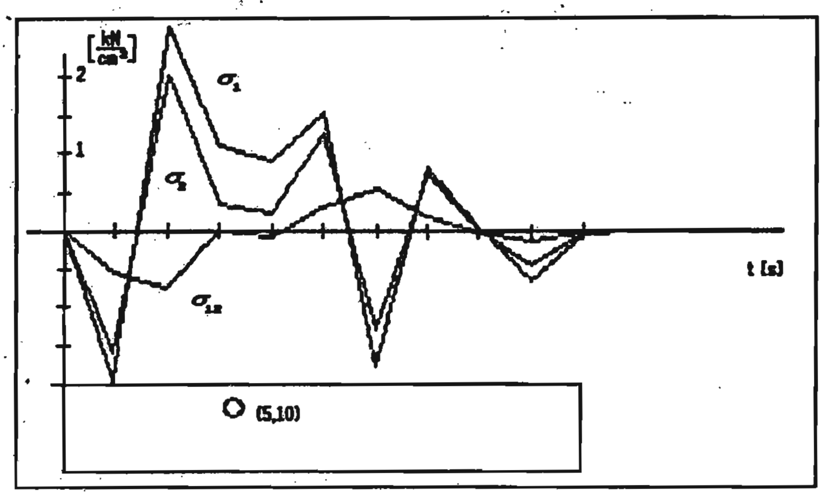

Fig. 4. Stresses $(\sigma_1, \sigma_2, \sigma_{12})$ plotted as a functions of time t

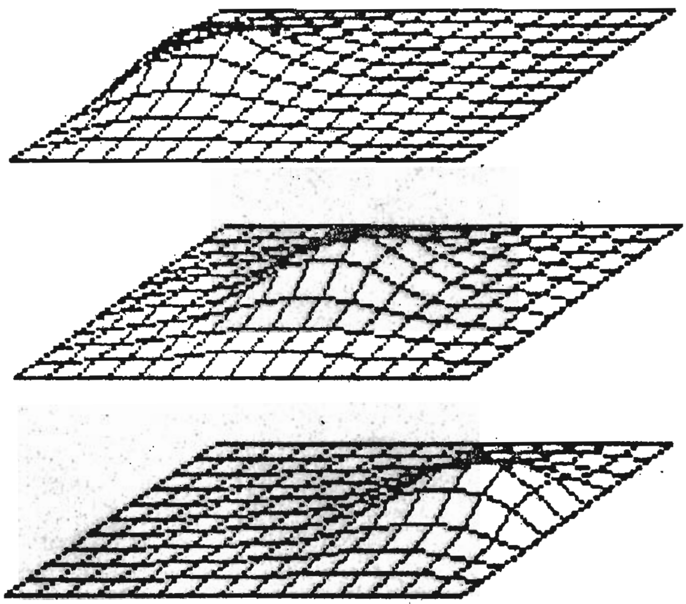

Fig. 5. Deflection of the middle surface in $t = 200$ s, $t = 500$ s, $t = 800$ s

The plate dimensions are taken as: $a = 30$ cm, $b = 20$ cm, $h = 1$ cm. The characteristics of the moving heat source are:

velocity	–	$v = 0.03$ cm/s
heat suorce intensity	–	$Q = 100$ W
initial temperature of plate	–	$T_0 = 20°$ C.

References

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Termosprężyste drgania cienkich płyt wywołane ruchomym źródłem ciepła

Streszczenie

W pracy rozpatrzono zagadnienie termosprężystych drgań cienkich płyt, podpartych na brzegach, wywołanych ruchomym źródłem ciepła. Problem rozwiązano stosując metodę transformacji całkowych. Rozwiązanie zilustrowano przykładem liczbowym.

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