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THE METHOD OF THE BOUNDARY INTEGRAL EQUATION FOR THE POTENTIAL FLOW INSIDE THE PALISADE OF AIRFOILS IN THE SEMI-INFINITE PLANE REGION

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In the paper the boundary problem of the Neumann type for the Laplace equation in a certain periodic, 2-dimensional region is considered. The approach based on the boundary integral equation of the second kind has been suggested and the periodic version of such an equation has been obtained. The algorithm for solving this equation has been proposed and some sample results of numerical calculations have been presented.

1. Introduction

Solution to the fundamental problem of classical aerodynamics describe the motion of an ideal fluid flowing around the given 2- or 3-dimensional body, actually appears as a single task not so often. It seems that much more often problems of finding potential velocity fields arise as auxiliary steps incorporated in more complex algorithms designed for the flow models closer to reality. Typical examples are models of airfoils, wings or even the whole aircraft design obtained by iterative methods based on the external flow-boundary layer splitting.

Another example is the method of random vortices (blobs) for real liquid flow simulations. This method allows us to calculate, in a direct way, nonstationary, 2-dimensional flows of viscous liquid in domains of practically any geometry and at any Reynolds number. At each step of the "numerical evolution" of the flow one has to solve a certain boundary problem of the Neumann type to find an auxiliary potential velocity field, which is necessary to obtain the given normal velocity (usually to eliminate it) on the boundary contours. The Autor has been motivated mainly by a strong need for solving such a problem occurring when one wants to apply the random vortex method to calculate flows inside palisades of airfoils. Such a space-periodic potential problem can be also interesting in itself. The aim is to show that this task can be performed quite efficiently and directly in a physical periodic domain (i.e. without any additional trans-formations of the domain).

2. Formulation of the problem

We consider the flow of ideal fluid at one stage of the 2-dimensional, infinite palisade of airfoils. In such a case it is natural to postulate space-periodicity of the flow field. Thus we can divide the domain into the infinite number of semi-infinite rectangular regions. The width of each region is assumed to be equal 2π so the flow is 2π-periodic with respect to y-coordinate.

Farther on, the region of the range closed within the interval $<0, 2\pi>$ is considered and called Ω with the boundary $\partial\Omega$ oriented as it is shown in Fig.1.

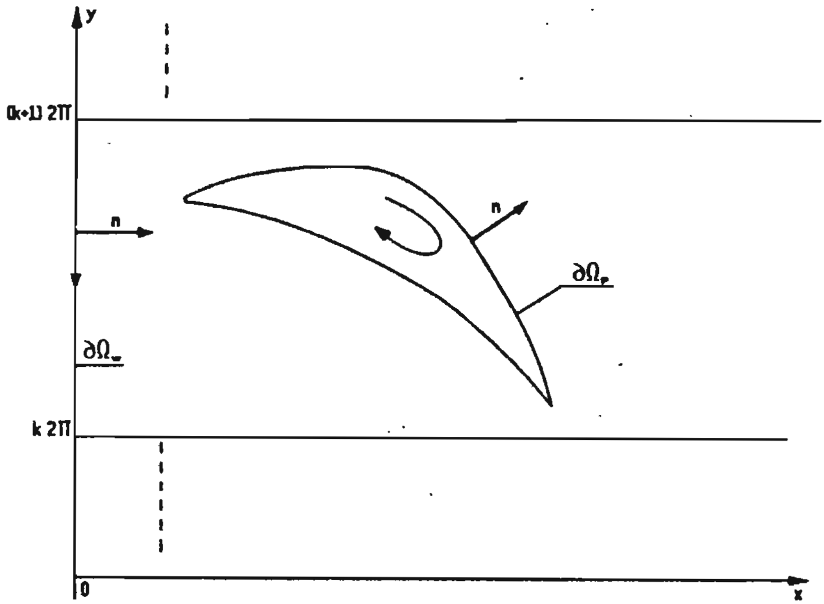


Fig. 1. The y-periodic computational domain

Now the formulation can be stated as follows

- find the velocity potential Φ which satisfies
  - the Laplace equation $\Delta\Phi = 0$ inside Ω
  - the periodicity condition $\Phi(x, y) = \Phi(x, y + 2k\pi)$, k – arbitrary integer number

– the Neumann boundary condition

$$\frac{D\Phi}{dn} = \begin{cases} V_p^n & \text{on } \partial\Omega_p \quad - \quad \text{contour of an airfoil} \\ u_w(y) & \text{on } \partial\Omega_w \quad - \quad \text{inlet line} \end{cases}$$

where $u_w(y) = u_w(y + 2k\pi)$, V_p^n is the prescribed, normal velocity on the airfoil.

Furthermore we assume that

C1 The circulation of the airfoil-connected vortex is chosen in the way which meets the Kutta-Zukovsky condition at the trailing edge (which, actually, is the point corresponding to the maximal curvature of the contour)

C2 The integral of the y-component of velocity along the inlet line is given and equals Γ_w.

3. Construction of the y-periodic velocity potential

We assume the following form of the potential sought for

$$\Phi(x,y) = \bar{u}_\infty x + \bar{v}_\infty y + \frac{\Gamma_p}{4\pi} y + \Gamma_p \Phi_\Gamma + \Phi_1 + \Gamma_p \Phi_2 \tag{3.1}$$

where

$\bar{u}_\infty$ – averaged x-component of the inlet velocity

$$\bar{u}_\infty = \frac{1}{2\pi} \int_0^{2\pi} u_w(y)dy$$

$\bar{v}_\infty$ – velocity equivalent to given Γ_w

$$\bar{v}_\infty = \frac{\Gamma_w}{2\pi}$$

Γ_p – circulation of the airfoil-connected vortex

Φ_Γ – velocity potential corresponding to the airfoil-connected vortex with unitary circulation

Φ_1, Φ_2 – additional harmonic potentials selected to satisfy boundary conditions for Φ, derivatives of which vanish at infinity.

It is important to point out, that the airfoil-connected vortex is actually the infinite set of vortices with the center of each one located inside the corresponding airfoil. Such a y-periodic vortex with unitary circulation induces the following velocity field

$$u_r - \mathrm{i}v_r = V_r(z) = \coth\frac{z-z_0}{2} \tag{3.2}$$

The complex notation is convenient here

$z = x + \mathrm{i}y$ – any point within the flow domain,

$z_0 = x_0 + \mathrm{i}y_0$ – the center of the palisade vortex which is located inside Ω.

The corresponding velocity potential is given by

$$\Phi_r(z) = \mathrm{Re}\Big[\frac{1}{2\pi\mathrm{i}}\ln\sinh\frac{z-z_0}{2}\Big] \tag{3.3}$$

Unlike in the case of a single vortex, the y-periodic vortex inducing the velocity field does not vanish at infinity

$$\lim_{\mathrm{Re}(z)\to\infty} V_r(z) = \frac{1}{4\pi\mathrm{i}} \qquad\qquad \lim_{\mathrm{Re}(z)\to\infty} V_r(z) = -\frac{1}{4\pi\mathrm{i}}$$

Thus the nonzero y-component of the velocity of opposite sign exists at infinity on each side of the vortex. This explains the presence of the third component in Eq (3.1) – it satisfies C2 condition.

Farther on we consider the particular case $V_p^n = 0$ on the whole contour $\partial\Omega_p$. Then the additional potentials Φ_1 and Φ_2 should fulfill the following conditions on the boundary $\partial\Omega$

$$\frac{d}{dn}\Big(\bar{u}_\infty x + \bar{v}_\infty y + \Phi_1\Big) = \begin{cases} 0 & \text{on} \quad \partial\Omega_p \\ u_w(y) & \text{on} \quad \partial\Omega_w \end{cases}$$

$$\frac{d}{dn}\Big(\frac{1}{4\pi}y + \Phi_r + \Phi_2\Big) = 0 \qquad \text{on} \quad \partial\Omega$$

Thus we have the following Neumann boundary conditions

$$\begin{aligned} \frac{d\Phi_1}{dn} &= \begin{cases} -\bar{u}_\infty\dfrac{dx}{dn} - \bar{v}_\infty\dfrac{dy}{dn} & \text{on} \quad \partial\Omega_p \\ \bar{u}_\infty(y) - \bar{u}_\infty & \text{on} \quad \partial\Omega_w \end{cases} \\ \frac{d\Phi_2}{dn} &= -\frac{d}{dn}\Big(\frac{1}{4\pi}y + \Phi_r\Big) \qquad \text{on} \quad \partial\Omega \end{aligned} \tag{3.4}$$

In Eq $(3.4)_1$ the simple observation has been employed that the normal unit vector to $\partial\Omega_w$ is simply the versor of x-direction. In the next paragraph the method of calculating Φ_1 and Φ_2. Now, assuming, that both potentials have been already determined, the Kutta-Zukovsky condition can be applied and unknown Γ_p can be found in the following way.

The derivation of Eq (3.1) with respect to the arc-length coordinate s yields the tangent velocity on the boundary

$$V_t(s) = \bar{u}_\infty t_x + \bar{v}_\infty t_y + \Gamma_p\Big(\frac{1}{4\pi}t_y + \frac{d\Phi_\Gamma}{ds} + \frac{d\Phi_2}{ds}\Big) + \frac{d\Phi_1}{ds} \tag{3.5}$$

where $t = [dx/ds, dy/ds]$ is a tangent unit vector. Then, assuming that the trailing edge corresponds to $s = 0$ one has

$$\Gamma_p = -\frac{\bar{u}_\infty t_x + \bar{v}_\infty t_y + \frac{d\Phi_1}{ds}}{\frac{1}{4\pi}t_y + \frac{d\Phi_\Gamma}{ds} + \frac{d\Phi_2}{ds}}\Bigg|_{s=0} \tag{3.6}$$

which is due to $V_t(0) = 0$.

4. The method of the boundary integral equation

We have to find unknown potentials Φ_1 and Φ_2, which are harmonic, y-periodic functions in Ω with normal derivatives on $\partial\Omega$ given by Eq (3.4). It is well known that the Neumann boundary problem for the Laplace equation can be solved by means of introducing the Fredholm second kind integral equation called the boundary integral equation. In the case when Ω is unbounded, multi- but finite-connected region in R^2 this equation has the form

$$\varphi(s_P) - \frac{1}{\pi}\oint_{\partial\Omega} \frac{d}{dn_Q}\ln r_{PQ}\,\varphi(s_Q)ds_Q = \frac{1}{\pi}\oint_{\partial\Omega} \ln r_{PQ}\frac{d}{dn_Q}\varphi(s_Q)ds_Q \tag{4.1}$$

where

- n_Q – internal normal unit vector to $\partial\Omega$
- s_P, s_Q – arc-length coordinates on $\partial\Omega$
- r_{PQ} – distance between points P and Q lying on $\partial\Omega$
- $\frac{d\varphi}{dn_Q}$ – given boundary distribution of the normal velocity.

Eq (4.1) originates from the two dimensional Green identity and boundary properties of double layer integrals $(X \in \Omega)$

$$\varphi(X) = -\frac{1}{2\pi}\oint_{\partial\Omega} \frac{d}{dn_Q}\ln r_{XQ}\,\varphi(s_Q)ds_Q + \frac{1}{2\pi}\oint_{\partial\Omega} \ln r_{XQ}\frac{d}{dn_Q}\varphi(s_Q)ds_Q \tag{4.2}$$

becoming its limit form when $X \to P \in \partial\Omega$. (See for example Vladimirov, 1981).

It should be emphasized that Eq (4.1) can be obtained when $\partial\Omega$ is smooth enough. When one assumes that n_Q exists on the where $\partial\Omega$ and the curvature $\kappa(s)$ in bounded, then the kernel of Eq (4.1) is also bounded and the equation is not singular in fact. When the condition of existence of a solution is fulfilled

$$\oint_{\partial\Omega} \frac{d}{dn_Q}\Phi(s)ds = 0$$

then the boundary value of the potential produces the velocity field with no circulation and vanishing at infinity.

In the case of the palisade the physical domain is infinitely multi-connected. The assumption of the y-periodicity allows us to consider the problem in the semi-infinite region Ω.

The next step one has to do is to obtain the y-periodic version of Eq (4.1) so that the integration would be carried along $\partial\Omega_w$ and $\partial\Omega_p$. Such equation exists (see Appendix B) in the form

$$\begin{aligned} \varphi(s_p) \quad &+ \quad \frac{1}{\pi}\oint_{\partial\Omega} \mathrm{Re}\Big[\frac{1}{2}\coth\frac{z_Q - z_P}{2}n_Q\Big]\varphi(s_Q)ds_Q = \\ &= \quad \frac{1}{\pi}\oint_{\partial\Omega} \mathrm{Re}\Big[\ln\sinh\frac{z_Q - z_P}{2}\Big]\frac{d}{dn_Q}\varphi(s_Q)ds_Q \end{aligned} \tag{4.3}$$

Moreover Eq (4.3) corresponds to the following, y-periodic analog of the Green formula Eq (4.2)

$$\begin{aligned} \varphi(X) \quad &= \quad \frac{1}{2\pi}\oint_{\partial\Omega} \mathrm{Re}\Big[\frac{1}{2}\coth\frac{z_Q - z_X}{2}n_Q\Big]\varphi(s_Q)ds_Q + \\ &+ \quad \frac{1}{2\pi}\oint_{\partial\Omega} \mathrm{Re}\Big[\ln\sinh\frac{z_Q - z_X}{2}\Big]\frac{d}{dn_Q}\varphi(s_Q)ds_Q \end{aligned} \tag{4.4}$$

In Eqs (4.3) and (4.4) $\partial\Omega = \partial\Omega_w \cup \partial\Omega_p$, complex numbers z_X, z_Q, z_P represent points $(x_X, y_X) \in \Omega$ and $(x_Q, y_Q), (x_P, y_P) \in \partial\Omega$.

It is easy to notice that both the kernel and the right hand side of Eq (4.3) are y-periodic, hence the solution is also y-periodic. The numerical method of solving this equation is described in the next paragraph. Finally, the solution to Eq (4.3) allows as to rearrange the velocity field inside Ω in a simple way. It seems that the simplest way should consist of two steps

1. numerical derivation of the potential on the boundary with respect to s-coordinate gives the tangent component of velocity on the boundary (see Eq

(3.5)); having $v_t(s)$ and $v_n(s)$ (given) one can calculate the complex velocity

$$V(s) = u(s) - \mathrm{i}v(s) = \Big(v_t(s) - \mathrm{i}v_n(s)\Big)\bar{t}(s) \tag{4.5}$$

where u, v are the cartesian components and $\bar{t}(s) = t_x(s) - \mathrm{i}t_y(s)$,

2. calculation of y-periodic analog of the Cauchy integral

$$V(z) = \frac{1}{4\pi \mathrm{i}} \oint_{\partial\Omega} V(s) \coth \frac{z_B - z}{2} dz_B(s) \tag{4.6}$$

where $z_B(s)$ – s-parameterized complex representation of $\partial\Omega$,
$z = x + \mathrm{i}y \in \Omega$.

Thus, after determining Φ_1 and Φ_2, the corresponding velocity fields V_1 and V_2 can be found. The total complex velocity field can be calculated then, using formula

$$V(z) = \widehat{V}_\infty + \Gamma_P(V_\Gamma + V_2) + V_1 \tag{4.7}$$

where $\widehat{V}_\infty = \bar{u}_\infty - \mathrm{i}\Big(\bar{v}_\infty + \frac{\Gamma_P}{4\pi}\Big)$, V_Γ and Γ_P are determined by, Eqs (3.2) and (3.6), respectively.

5. The numerical method

In order to find the solution one has to determine the unknown potentials Φ_1 and Φ_2, satisfying the integral equation (4.3), which can be written in a form

$$\varphi(s) + \oint_{\partial\Omega_p} K(s,t)\varphi(t)dt + \int_{\partial\Omega_w} K(s,t)\varphi(t)dt = r(s) \tag{5.1}$$

The solution to Eq (5.1) is assumed to be periodic with respect to the arc-length coordinate s, separately on $\partial\Omega_p$ and on $\partial\Omega_w$, respectively, which is equivalent to y-periodicity. We introduce the nodal points on $\partial\Omega_p$, which divide it into segments $[s_k, s_{k+1}]$, $k = 0, ..., NP$, $s_{NP+1} = s_0$; the inlet line $\partial\Omega_w$ is uniformly partitioned into segments $[s_k, s_{k+1}]$, $k = 0, ..., NW$, $s_{NW+1} = 2\pi$.

The following set of the linear B-splines can be defined

$$L_K(s) = \begin{cases} \dfrac{s - s_{k-1}}{s_k - s_{k-1}} & \text{for} \quad s \in [s_{k-1}, s_k] \\[2ex] \dfrac{s_{k+1} - s}{s_{k+1} - s_k} & \text{for} \quad s \in [s_{k-1}, s_k] \\[2ex] 0 & \text{otherwise} \end{cases}$$

separately for $\partial\Omega_p$ and $\partial\Omega_w$. The solution to Eq (5.1) can be approximated by a piece-wise linear function of s in the form

$$\varphi(s) = \sum_{i=0}^{NT} \varphi_i L_i(s) \qquad NT = NP + NW + 1 \tag{5.2}$$

Accordingly to the introduced partitions of $\partial\Omega_p$ and $\partial\Omega_w$ B-splines $L_0, ..., L_{NP}$ have their supports included in $\partial\Omega_p$, and B-splines $L_{NP+1}, ..., L_{NT}$ in $\partial\Omega_w$, respectively.

Next, the kernel $\mathcal{K}$ and the right-hand-side of Eq (5.1) can be represented in $\{L_i\}$-basis

$$\mathcal{K}(s,t) = \sum_{i=0}^{NT}\sum_{j=0}^{NT} \mathcal{K}_{ij} L_i(s) L_j(t) \qquad r(s) = \sum_{i=0}^{NT} r_i L_i(s) \tag{5.3}$$

The decomposition coefficients φ_i, $\mathcal{K}_{ij}$, r_i are given by the values of corresponding functions in nodal points

$$\varphi_i = \varphi(s_i) \qquad \mathcal{K}_{ij} = \mathcal{K}(s_i, t_j) \qquad r_i = r(s_i)$$

After substitution of Eqs (5.2) and (5.3) in to Eq (5.1) we obtain

$$\sum_{i=0}^{NT} \varphi_i L_i(s) + \oint_{\partial\Omega_p} \sum_{i=0}^{NT}\sum_{j=0}^{NP} \mathcal{K}_{ij} L_i(s) L_j(t) \sum_{k=0}^{NP} \varphi_k L_k(t) dt +$$

$$+ \int_{\partial\Omega_w} \sum_{i=0}^{NT} \sum_{j=NT+1}^{NT} \mathcal{K}_{ij} L_i(s) L_j(t) \sum_{k=NP+1}^{NT} \varphi_k L_k(t) dt = \sum_{i=0}^{NT} r_i L_i(s)$$

Then, after some calculations the following formula holds

$$\sum_{i=0}^{NT} \Big[\varphi_i + \sum_{j=0}^{NP} \mathcal{K}_{ij} \sum_{k=0}^{NP} \varphi_k \oint_{\partial\Omega_p} L_j(t) L_k(t) dt +$$

$$+ \sum_{j=NP+1}^{NT} \mathcal{K}_{ij} \sum_{k=NP+1}^{NT} \varphi_k \int_{\partial\Omega_w} L_j(t) L_k(t) dt - r_i \Big] L_i(s) = 0$$

The B-splines $\{L_0, ..., L_{NT}\}$ are linearly independent, thus

$$\varphi_i + \sum_{k=0}^{NP} \Big(\sum_{j=0}^{NP} \mathcal{K}_{ij} I_{jk} \Big) \varphi_k + \sum_{k=NP+1}^{NT} \Big(\sum_{j=NP+1}^{NT} \mathcal{K}_{ij} I_{jk} \Big) \varphi_k = r_i$$

$$i = 0, ..., NT \tag{5.4}$$

In Eq (5.4), the following, 3-diagonal matrix L of L_{jk} entries has been introduced

$$L_{jk} = \int_{\partial\Omega} L_j(t) L_k(t) dt \tag{5.5}$$

One can define the new matrix

$$\mathsf{M} = \mathcal{K}\mathsf{L} \tag{5.6}$$

to obtain the final form of the set of linear equations

$$\begin{gathered}(\mathsf{I} + \mathsf{M})\boldsymbol{\varphi} = \boldsymbol{r} \\ \boldsymbol{\varphi} = [\varphi_0, ..., \varphi_{NT}] \qquad\qquad \boldsymbol{r} = [r_0, ..., r_{NT}]\end{gathered} \tag{5.7}$$

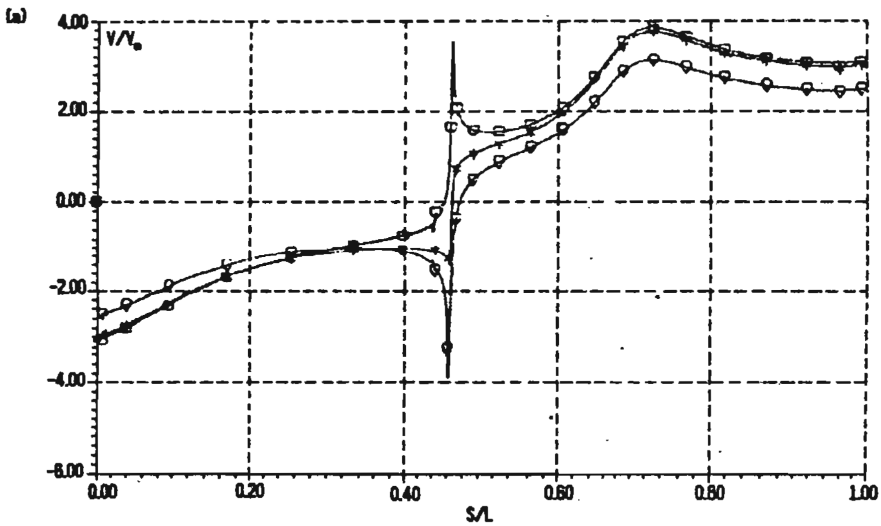
6. The results of the numerical analysis

The numerical computations have been performed for the palisade constructed of the blade-like airfoil obtained by the parametric, complex formula given in the Appendix. The airfoil partition into segments results from the transformation of the uniform partition of the circle, chosen to assure, that the point of the contour with the maximal curvature (trailing edge) belongs to the set of $NP + 1$ nodes.

The inlet line has been divided into $NW + 1$ segments.

The calculations were carried out for various values of NP and NW; hereinafter, the results for $NP = 599$ and $NW = 119$ are presented. Thus the set of 720 linear equations has been solved using the Crout's L-U decomposition algorithm (cf Press et al., 1989) and double-precision calculations. The velocity and the pressure distributions, respectively, acros the boundaries have been obtained for various normal-velocity conditions on the inlet line. The results are presented in Fig.2. The example of the flow (stream lines) within the periodicity region, computed in one of the cases under consideration, is shown in Fig.3. It should be emphasi-zed that the palisade airfoil has not to be, in general, totally included in the single periodicity region.

As it was shown by Szumbarski (1993), the 2-parameter family of symetric oval contours exists, for which a potential flow in the palisade can be determined in a close, analytic form. This fact allows as to compare the approximated solution (numerical) with the exact one and to acquire an idea how accurate the numerical method is. The only minor complication is due to the disappearent of the domains of the flow for the analytic and the numerical solutions, respectively. In the former, the domain is the whole plane and the flow is completely determined by the velocity at infinity in front of the palisade. In the latter the domain is obviously a half-plane, to the right of the inlet line, which is the part of the boundary with the



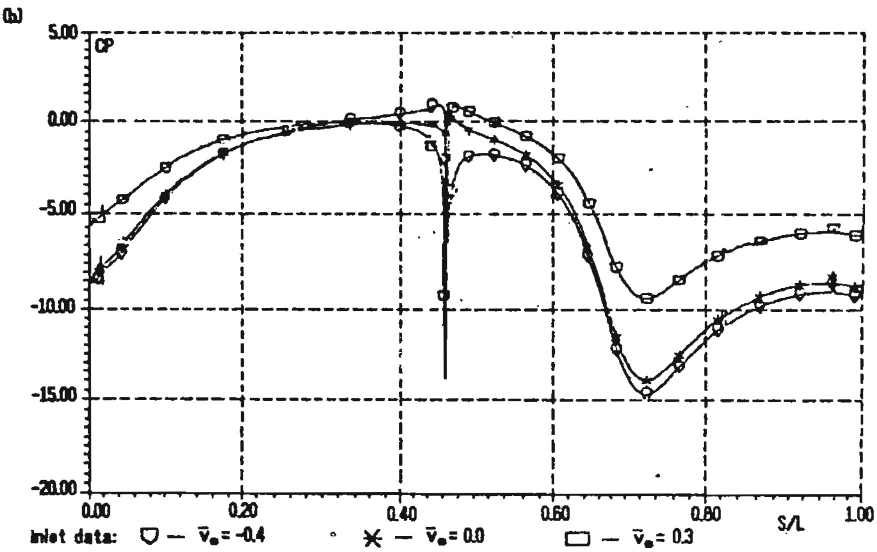


Fig. 2. The velocity (a) and pressure (b) distribution. In all cases the uniform distribution $u_w = 1.0$ has been applied

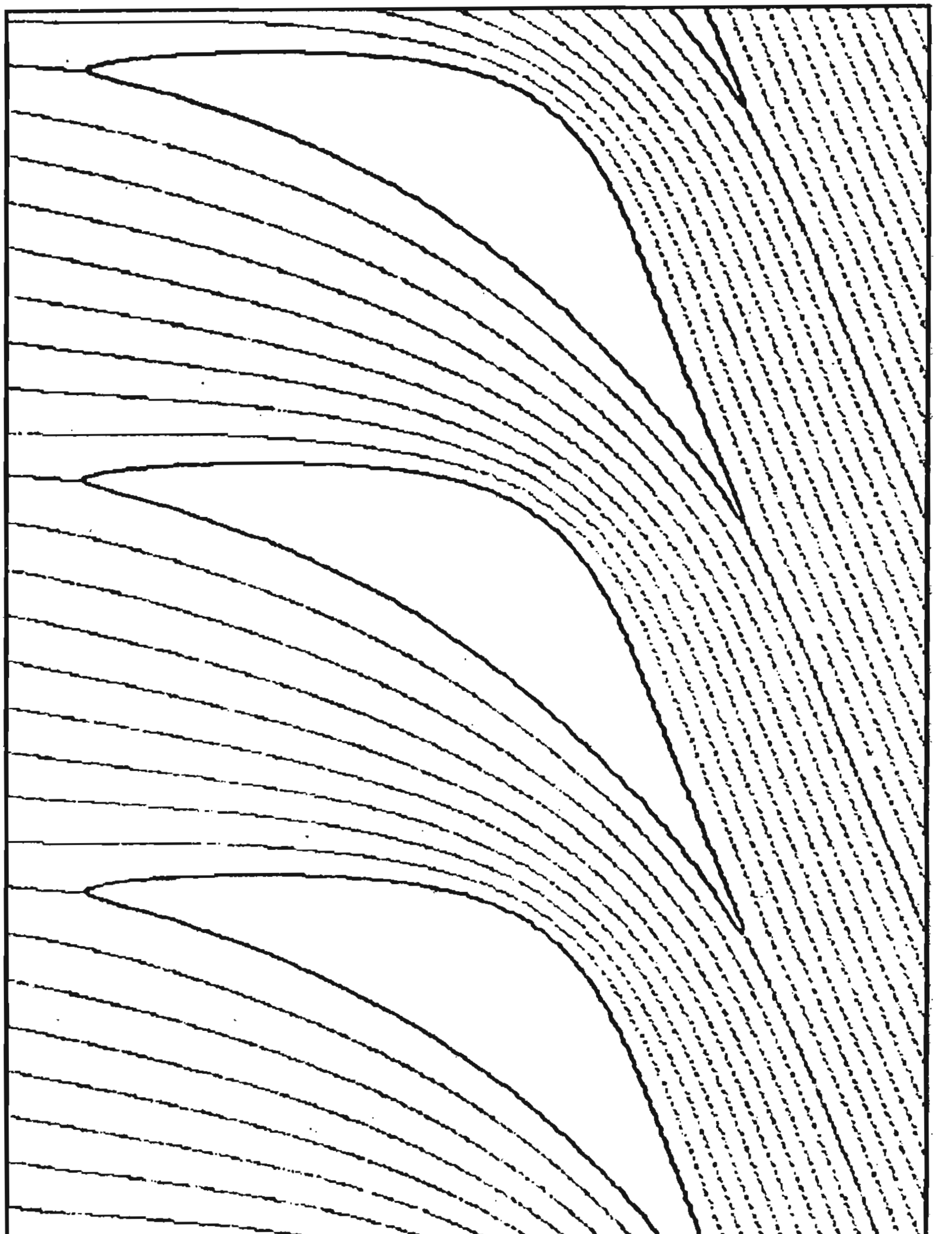


Fig. 3. The streamlines of the y-periodic flow computed for $u_w = 1.0$, $\bar{v}_\infty = -0.2$. The shape of the airfoil is determined by the parametrization given in the Appendix

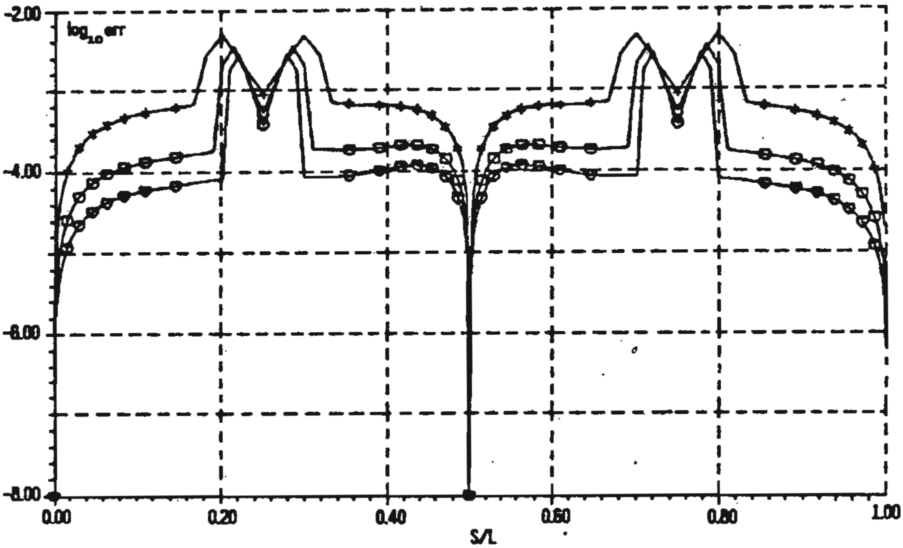


Fig. 4. The difference between the analytical and numerical solutions of the symmetric flow in the palisade of the oval contours ($r = 20$, $d = 204$) for uniform velocity distribution on the inlet $u_w = 1.0$

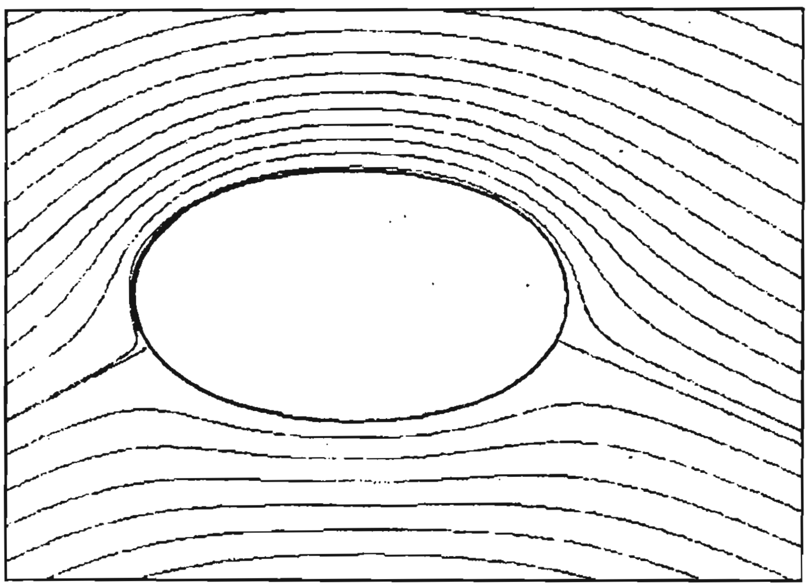


Fig. 5. The example of the nonsymmetric flow in the palisade of the oval contours (see text)

Neumann condition imposed on it. Thus, to obtain equivalent numerical solution, one has to evaluate, using analytical solution, the velocity distribution across the inlet line and then use it as a boundary condition for the numerical calculations.

The numerical results have been computed for the palisade of the ovals for parameter $r = 200$ and $d = 204$ (cf Szumbarski, 1993). The number of nodal points on the oval and on the inlet line has been varied, however the proportion of the number of nodes on the oval and on the inlet, respectively, has been fixed and equal to 3:1. The symmetric flow has been assumed and the absolute values of the differences between the exact and the approximated boundary velocities, respectively, in all nodal points have been calculated in each case. The results are shown in Fig.4. The example of the nonsy mmetric flow has been calculated for $NP = 599$ and $NW = 199$; the resulting stream lines are presented in Fig.5. The velocity distribution over the inlet line (nonuniform) corresponds to such an uniform flow at infinity, that the rear stagnation point is placed in the prescribed nodal point (its number is 50) on the oval. The velocity of the uniform flow has been obtained numerically with a high accuracy (approximately $v_\infty/u_\infty = 0.3$).

7. Conclusion

The method for solving a certain class of space-periodic potential flows has been proposed. The main feature of this method is the decomposition of the unknown ones using the boundary integral equation. The kernel and the right-hand-side function of this equation are y-periodic, like are the solutions: boundary values of unknown harmonic functions. To solvethe integral equation a simple, linear approximation has been applied, employing the idea of B-spline functions. Thisapproach has proved to be quite efficient, however the accuracy test (performed on the specific example, for which the analytic solution can be obtained for comparison) showed that relatively high errors could occur in some areas of the domain of the solution. One can expect that the application of higher-order numerical algorithms (square, cubic or more sophisticated approximations) would lower the level of errors and, probably, smooth their distribution.

Appendix

A The parametric definition of the shape of the airfoil used in the numerical computations

The palisade has been constructed from the airfoil given by the following complex parameterization

$$F(z) = c_0 + c_1 z + \sum_{k=-5}^{-1} c_k z^k \qquad z = \exp(\mathrm{i}Q) \tag{A.1}$$

The coefficients are as follows

$$\begin{aligned}
c_{-5} &= 0.0058548432 - 0.0058548432\mathrm{i} \\
c_{-4} &= 0.040983908 + 0.040983908\mathrm{i} \\
c_{-3} &= -0.078064588 + 0.078064588\mathrm{i} \\
c_{-2} &= -0.409839087 - 0.80016203\mathrm{i} \\
c_{-1} &= 1.366130302 - 1.366130302\mathrm{i} \\
c_0 &= 5.3 - 2.54\mathrm{i} \\
c_1 &= 2.14678 - 2.14678\mathrm{i}
\end{aligned}$$

Obviously, the coefficient c_0 causes only horizontal and vertical shift of the airfoil. Since the geometry of the flow domain is y-periodic the imaginary part can be given an arbitrary value.

B Derivation of the kernel functions of the integrals in Eq (4.3)

The kernel of the integral equation in the classical (nonperiodic) case can be expressed using complex notation as

$$\mathcal{K}(s_P, s_Q) = \frac{\cos(\vec{QP}, \vec{n_Q})}{r_{QP}} = -\frac{d}{dn_Q} \ln r_{QP} = -\mathrm{Re}\frac{n_Q}{z_Q - z_P} \tag{B.1}$$

where

$$\begin{aligned}
n_Q &= (\vec{n_Q})_X + \mathrm{i}(\vec{n_Q})_Y \\
z_Q &= x(s_Q) + \mathrm{i}y(s_Q) \\
z_P &= x(s_P) + \mathrm{i}y(s_P)
\end{aligned}$$

To obtain the periodic kernel the following infinite sum should be calculated

$$\sum_{k=-\infty}^{+\infty} \frac{d}{dn_Q} \ln \sqrt{(x_Q - x_P)^2 + [(y_Q + 2k\pi) - y_P]^2}$$

According to Eq (B.1) it is equivalent to calculate the sum

$$\sum_{k=-\infty}^{+\infty} \frac{1}{z_Q - z_P + 2k\pi\mathrm{i}}$$

and to take the real part of it.

The following identity plays the crucial role

$$\coth(\zeta) = \frac{1}{\zeta} + 2\sum_{k=1}^{+\infty} \frac{\zeta}{\zeta^2 + k^2\pi^2} \tag{B.2}$$

The calculation goes as follows

$$\sum_{k=-\infty}^{+\infty} \frac{1}{z_Q - z_P + 2k\pi i} = \sum_{k=1}^{+\infty}\Big[\frac{1}{z_Q - z_P + 2k\pi i} + \frac{1}{z_Q - z_P - 2k\pi i}\Big] + \frac{1}{z_Q - z_P} =$$

$$= \sum_{k=1} +\infty \frac{2(z_Q - z_P}{(z_Q - z_P)^2 + 4k^2\pi^2} + \frac{1}{z_Q - z_P} = \sum_{k=1}^{+\infty} \frac{\frac{z_Q - z_P}{2}}{\frac{(z_Q - z_P)^2}{4} + k^2\pi^2} + \frac{1}{2}\frac{1}{\frac{z_Q - z_P}{2}}$$

Comparing the foregoing equation with Eq (B.2), it is easy to notice that

$$\sum_{k=-\infty}^{+\infty} \frac{1}{z_Q - z_P + 2k\pi i} = \frac{1}{2}\coth\frac{z_Q - z_P}{2}$$

In order to obtain the kernel of the integral in the right-hand-side of Eq (4.3) one has to calculate the following sum

$$\sum_{k=-\infty}^{+\infty} \ln\sqrt{(x_Q - x_P)^2 + [(y_Q + 2k\pi) - y_P]^2}$$

Using complex notation it is equivalent to calculate the sum

$$\sum_{k=-\infty}^{+\infty} \ln(z_Q - z_P + 2k\pi i)$$

and taking the real part of it. However this sum does not converge unless a certain, properly chosen, set of constants is added. One is allowed to do it, because of the type of conditions imposed on the boundary data, namely

$$\int_{\partial\Omega} v_n(s)ds = 0$$

The set of constants is following

$$C_k = -\ln(2k\pi i) \qquad C_{-k} = -\ln(-2k\pi i) \qquad k = 1, 2, 3...$$

Additionally, to obtain the sum in a closed form, C_0 is defined: $C_0 = -\ln 2$.

Then the calculations go as follows

$$\sum_{k=-\infty}^{+\infty}\Big[\ln(z_Q - z_P + i2k\pi) + C_k\Big] = \sum_{k=1}^{+\infty}\Big[\ln(z_Q - z_P + i2k\pi) +$$

$$+\ln(z_Q - z_P + i2k\pi) - \ln(-i2k\pi) - \ln(i2k\pi)\Big] + \ln(z_Q - z_P) - \ln 2 =$$

$$= \sum_{k=1} +\infty\Big[\ln[(z_Q - z_P)^2 + 4k^2\pi^2] - \ln(4k^2\pi^2)\Big] + \ln\frac{z_Q - z_P}{2} =$$

$$= \sum_{k=1}^{+\infty}\ln\Big[1 + \frac{(z_Q - z_P)^2}{4k^2\pi^2}\Big]\ln\frac{z_Q - z_P}{2} = \ln\frac{z_Q - z_P}{2}\prod_{k=1}^{+\infty}\Big[1 + \Big(\frac{z_Q - z_P}{2k\pi}\Big)^2\Big]$$

Now, from the identity

$$\sinh(z) = z\prod_{k=1}^{+\infty}\Big[1 + \Big(\frac{z}{k\pi}\Big)^2\Big]$$

it follows, that

$$\sum_{k=1} +\infty\Big[\ln(z_Q - z_P + i2k\pi) + C_k\Big] = \ln\sinh\frac{z_Q - z_P}{2}$$

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Zastosowanie metody całkowego równania brzegowego do analizy przepływu potencjalnego wewnątrz palisady łopatkowej w obszarze płaskim

Streszczenie

W artykule omówiono metodę wyznaczania pola prędkości ruchu cieczy idealnej w stopniu okresowej palisady łopatkowej, opartą na wykorzystaniu całkowego równania brzegowego z okresowym jądrem. Zaproponowano elementarną metodę numeryczną

rozwiązania tego równania wykorzystującą aproksymację jądra, funkcji prawej strony i funkcji niewiadomej (brzegowego rozkładu potencjału prędkości) w bazie B-splajnów liniowych. Zaprezentowano wyniki przykładowych obliczeń oraz oszacowano dokładnoś' c metody stosując ją do przypadku ze znanym rozwiązaniem analitycznym.

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