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ON THE WINDING PATHS OF THE THEORY OF PLATES

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The progress in the theory of plates formulation made in 1789-1988 has been carefully reviewed in the present paper. An attempt to answer the general question :"To study or to create" has been made. Facts concerning the history of this theory development given below, have proved that one is supposed to study the previous works before creating a new theory.

1. Introduction

This paper is based on my previous work (cf Jemielita, 1991) under the same title. Its purpose was to present the trends in the theory of elastic plates in years 1789-1988. I would like to refer here to the distinguished treatise by Toudhunter and completed after his death by Pearson (1886). I would like to present here the following Pearson's words, taken from his work, written on 23rd June 1886:

> "The use of a work of this kind is twofold. It forms on the one hand the history of a peculiar phase of intellectual development, worth studying for the many side lights it throws on general human progress. On the other hand it serves as a guide to the investigator in what has been done, and what ought *to* be done. In this latter respect the individualism of modern science has not infrequently led to a great waste of power; the same bit of work has been repeated in different countries at different times, owing to the absence of such histories as Dr Todhunter set himself to write. At it is, the would-be researcher either wastes much time in learning the history of his subject, or else works away regardless of earlier investigators. The latter course has been singularly prevalent with even some first-class British and French mathematicians."

One could think that the Pearson's opinion mentioned above became out of date in times of a stormy progress of communication. Unfortunately, that is not the case. But in my opinion this old text written 100 years ago is still up-to-date.

For the past two hundred years a great number of theories of plates have been developed. Some of them fell flat soon after their birth, others have been still applicable. For this period pioneering papers have appeared indicating new directions of researches, inspiring others to supplement and generalize the theories presented there. The old concepts are again recovered, the names of truly dedicated authors are forgotten and their hypotheses and theories are often associated with other personalities, sometimes with scientists of a great authority.

Many papers concerning the same problem were written almost simultaneously, but many more papers had the similar contents. Their origins were at the distance even of tens of years from one another.

There still have been published the papers whose authors claim that their theory as well as their approach is "new". These authors, roughly speaking didn't realize that earlier, in some cases even tens years ago, that sort of approach had been used and such theories had arisen already. One can find a really great number of papers covering the problems of the plate theory. Many review treatises are known. Usually those treatiesses concern the selected problems of the plate theory. In the previous work (cf Jemielita, 1991) I intendent to show the winding paths of the progress in the plate theory field on a possibly broad scale. I tried to do my best to reach out for the first papers where this subject was considered.

The materials for the aforementioned work have been gathered for many years. This work encompasses more than 3000 items, about 1500 of them being discussed.

This 217 pages survey contains eleven sections. Coming after a short introduction the second section shows the way in which the theory of thin plates since 1789 till 1850 has been formed. The following problems concerning the thin plates theory have been shown in the third section: nonlinear theory, stability, dynamics, anisotropic plates, plates of variable thickness, nonhomogeneous plates, plates of a periodic structure, micropolar plates, plates based on an elastic foundation, thermal problems, bond fields, the Green functions and finally, methods for the plate theory problems solving. Some of the latter ones being highlighted, namely: the methods of Navier, Lévy, Kączkowski (the method of superposition of wave surfaces of deflection) and Nowacki, respectively, together with the finite Fourier transform method. Hybrid methods (using both analytical and numerical techniques) together with the pure numerical methods have been discussed.

An exact 3D-solution is examined in the fourth section, while the ways of the plate theory formulation are shown in the fifth one. The methods for the asymptotic theories formulation have been discussed in the sixth section. The following methods are shown in details: the method of stress and displacement expansion into a power or trigonometric series, Legendre polynomials and Bernoulli polynomials, respectively, the half-inverse method, the method of infinite differential operators, the method of initial functions, the method of higher moments and the Birkhoff method as well. Some engineering theories of the moderate thickness plates have been reviewed in the seventh section. The details of the following methods are given: the theories of Lévy, Love, Stevenson, Alekseev, Reissner, Hencky, Bolle, Uflyand, Mindlin, Kromm, Vlasov, Ambartsumyan, Kączkowski and Jemelita. Other generalized theories are also mentioned. A variety of kinematic and static hypotheses being applied in plate theories has been presented. The boundary conditions are disscused in the eighth section.The problem of the shear coefficient choice met in theories basing upon the Hencky-Bolle assumptions has been shown in ninth section. Possible errors which can be involved due to the engineering theories application are discussed in the tenth section. Total and local estimations, respectively, have been given. Some Author's remarks concerning the material being gathered can be found in the last section of the aforementioned work.

In the present article due to the shortage of room the Author could present only some selected points of his work. In each case we shall emphasize that scientists should study carefully the papers written by their predecessors.

2. On forming the theory of thin plates. Years 1789-1850

The experiments of a great physicist Chladni (1756-1827) have triggered the interests concerning mathematical description of vibrations of thin plates. Chladni drew the bow across the glass pane edge covered with sand or powder. The vibrations of the pane made the layer of sand redistribute: along some strips the sand was thickened, while in some areas thinning occurred and he managed to determine the relevant vibration frequencies. In 1787 Chladni published the results of his experiments in a book "Discoveries in the theory of sound". This paper was found by Jacques Bernoulli to be so impressive that he decided to develop a mathematical description for these phenomena. In October 1788 Bernoulli submitted his paper on vibrations of an elastic rectangular plate to the Petersburg Academy. Bernoulli generalized the known equations of the theory of vibrating beams on the case of plates, the latter being thought of as a gridwork composed of two families of mutually orthogonal beams. This work of Bernoulli is the first one concerning the plate theory. In 1808 Chladni presented his results to the Paris Academy. Chladni, Laplace, Lacepede and Berthollete were invited to the Cesar Napoleon. For two hours Chladni had been presenting his experimental results. By virtue of the Cesar's initiative and thanks to his financial support (3000 of contemporary francs – what was equivalent to 967.5 grams of gold[1]) the Paris Academy threw it open to competition for development of a mathematical theory of vibrations

[1]This interesting information I have learned from Jan Kączkowski (Canada) and prof. Jerzy Borejsza (Polish Academy of Sciences Station in Paris) whom I would like to express my sincere appreciation for their help

of elastic plates capable of giving predictions matching the experimental results. Only one paper by Sophie Germain was submitted before the deadline (1st October 1811). That time Germain did not find a correct description and failed to win. The deadline for the competition was postponed two times (1813-10-1, 1815-10-1). The third version of the Germain's paper was eventually praised in 1816.

Navier obtained (1820) the following equation

$$D\nabla^4 v = q \tag{2.1}$$

q being an intensity of the loading.

Cauchy and Poisson described (from 1828 till 1829) 3D-state of stress and deformation in the plate with the help of some effective 2D-fields.

On the base of equation found by Cauchy one can get the proper equation (2.1) with a correctly determined stiffness D. Unfortunately, both Cauchy and Poisson failed to succeed in formulating boundary conditions on the free edge of the plate.

A proper equation (2.1) as well as an adequate number and form of boundary conditions were obtained by Kirchhoff (1850). Applying the variational calculus Kirchhoff reduced the number of boundary conditions on the free edge from three to two.

The range of applicability of the Kirchhoff's theory are thin elastic plates. Therefore this theory is referred to as a thin plate theory. It is based on two simple hypotheses: a kinematic and a stress-type (static).

The following three assumptions establish the kinematical hypothesis:

- the rectilinear fibers normal to the undeformed middle plane remain rectilinear and normal to the middle plane after deformation,
- the plate thickness does not undergo deformations,
- the fibers of the middle plane do not undergo extensions at small deflections of the plate.

The third assumption concerns plates transverse properties of which are symmetric with respect to the plate mid-plane.

The hypothesis outlined above is usually referred to as the Kirchhoff-Love hypothesis. In my opinion this hypothesis (for the case of plates) should be called Navier's or Navier-Kirchhoff hypothesis in honour of Navier who thirty years before Kirchhoff published his results and had submitted it to the Paris Academy a paper on the plate theory basing on the assumption that "the plate consists of particles located across the thickness and their displacements during bending are parallel to the middle plane and are proportional to the distance from a his plane".

It is worth noting that in his article Kirchhoff criticizes the approaches of Germain and Poisson, respectively, but does not mention the work of Navier.

The work of Kirchhoff closes the first stage of the progress of the theory of plates. It is worth noting that in this period the following two methods for deriving plate equations were applied:

- the variational approach (the method of final causes),
- the direct approach (the method of effective causes).

Both methods are known since the Euler's time. The former method was applied by Germain, Poisson and Kirchhoff, the latter being used by Navier, Poisson and Cauchy, respectively.

3. Theory of plates in the period of 1850-1988

Since 1850 the theory of plates has been developed parallel in two directions. Within the framework of the first trend the plate theory was developed on the basis of the Navier-Kirchhoff assumptions or using some of them. The aim of the second trend was to form new refined theories.

In these theories the original 3D-problems of deformable body mechanics are reduced to 1D or 2D-problems which are easier to handle. For instance, in the theory of plates all quantities should explicitly depend on the z coordinate characterizing a distance of a point to the basic plane. Since it can be achieved in different ways a variety of different refined theories has still been arising.

One can formulate the plate theory in two ways:

1. directly – when the plate is a priori assumed to be 2D-surface capable of bending bearing;
2. indirectly – when the 3D-problem is replaced by relevant 2D-problem.

The first case fits well with the Cosserats type model, while the theories arising according to the second way are based upon the linear or nonlinear elasticity theory (symmetric or asymmetric).

The theories formulated indirectly can be divided into two groups:

1. asymptotic theories;
2. engineering theories.

So called thick plate theories form the first class, while in the second one theories of moderate thickness plates can be found. One can make a lot of different divisions according to the hypothesis being applied, the way of differential equation and boundary conditions deriving or the kind of problems being solved (static or dynamic), respectively.

The asymptotic theory formulation consists in approximation of the primary 3D-problem to the series of relevant 2D-problems. The accuracy of approximation depends on the number of series elements being employed.

The heart of engineering theories approximating 3D-problems to relevant 2D-problems is a priori acceptance of some simplifying assumptions concerning, respectively, eitherthe displacement or the stress field which can be dealt with or both these fields together. These assumptions, roughly speaking, are obvious and the plain 2D-theories can be obtained. Unfortunately, they prevent one passing to more accurate 2D-theories. The choice of the theory being considered depends both on the problem which is supposed to be solved and the anticipated accuracy. Each theory can be applied to the specified field of problems. To arrive at the governing equations two approaches are at our disposal:

- a direct approach, called here the effective causes method,
- a variational approach (final causes method).

In the direct method the governing equations are obtained employing the local equations of the theory of elasticity. Some of these equations are satisfied pointwise and other are averaged. As usual we average the equlibrium equations by integrating over the thickness with the z^n weighting functions (n being a certain natural number). These weighting functions can be individually chosen for particular equations.

In variational approach one should employ the virtual work principle, the minimum potential energy principle, the minimal complementary energy principle, Hamilton's principle and other hybrid variational principles, respectively.

One of the basic differential equations creating the plate theory is the plate deflection equation $v(x^\alpha)$.

The midplane deflection can take the part of this function

$$\breve{w}(x^\alpha) \stackrel{\mathrm{df}}{=} u_3(x^\alpha, 0) \tag{3.1}$$

the mean value

$$\overset{*}{w}(x^\alpha) \stackrel{\mathrm{df}}{=} \frac{1}{h} \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} u_3(x^\alpha, z)dz \tag{3.2}$$

the mean value with a weight

$$\widetilde{w}(x^\alpha) \stackrel{\mathrm{df}}{=} \frac{1}{h} \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \Big(1 - 4\frac{z^2}{h^2}\Big) u_3(x^\alpha, z)dz \tag{3.3}$$

or in the case when only the bended state is considered (in plates of symmetrical transversal non-homogeneity) – the deflections of the plate faces

$$\hat{w}(x^\alpha) \stackrel{\text{df}}{=} u_3(x^\alpha, \pm\frac{h}{2}) \tag{3.4}$$

All the aforemetioned deflections reveal a simple mathematical interpretation. Deflections $\breve{w}$ and $\hat{w}$, respectively, are the first components of the u_3 displacement into power series expansion, while the averages $\overset{*}{w}$ and $\widetilde{w}$ represent the first components of the u_3 expansion into the Legendre polynomials

$$u_3 = \sum_{i=0,2,\dots}^{\infty} f_i(\zeta) v_i(x^\alpha) \qquad u_3 = \sum_{i=1,3,\dots}^{\infty} \frac{df_i(\zeta)}{d\zeta} w_i(x^\alpha) \tag{3.5}$$

where

$$\zeta = \frac{2z}{h} \qquad v_0 = \overset{*}{w} \qquad w_1 = \widetilde{w} \qquad f_n = \frac{1}{2^n n!} \frac{d^n((\zeta^2-1)^n)}{d\zeta^n}$$

The differential equation in each of these functions can be written in one of the following forms

$$\begin{aligned} D\mathcal{L}\nabla^4 v(x^\alpha) &= p_3(x^\alpha) \\ D\mathcal{M}\nabla^4 v(x^\alpha) &= \mathcal{R} p_3(x^\alpha) \\ D\nabla^4 v(x^\alpha) &= \mathcal{P} p_3(x^\alpha) \end{aligned} \tag{3.6}$$

where $v = \breve{w}$, $\hat{w}$, $\overset{*}{w}$, $\widetilde{w}$, and D stands for the plate stiffness.

Depending on the theory being employed the differential operators $\mathcal{L}$, $\mathcal{M}$, $\mathcal{R}$, $\mathcal{P}$ are infinite (in asymptotic theories) or the finite ones (in engineering theories). If one applies the finite operator, he is supposed to derive an approximate equation. The correct approximate equation is derived when the coefficients of the proper differentials agree with those coming in the correct infinite operators. Unfortunately, the incorrect approximate equations are given in many works.

The bibliography of the refined asymptotic and engineering theories, respectively, is rather broad. A lot of monographs and surveys have been presented. A majority of known monographs concerns the thin plates theory. The shortage in monographs on refined theories is observed.

For the past 141 years practically all fundamental problems concerning thin plates have been solved.

In the second half of the nineteenth century the equations of thin plates undergoing large deflections were derived by Kirchhoff and Clebsh. In their present form these equations were reported by Kármán in 1910, but without any explanation. Twelve years later Prescott arrived at the same equations, without referring to the Kármán's paper.

A very effective method for solving the problem of clamped plates is due to Lardy (1953). According to this method the sought for function of deflection was represented as a product of functions standing for vibration modes of a clamped beam. Four years later another author considered vibrations of a clamped rectangular plate and represented the deflection in a form of a product of eigenfunctions of a vibrating beam elastically clamped. In this paper the Lardy's work is not recalled.

Ponyatovskiĭ built an asymptotic theory of plates basing on the representation of stress fields with the help of Legendre polynomials (1962-1965). One of the world-wide authorities 21 years later proposed a theory which turned out to be a specific case of that of Ponyatovskiĭ.

In my previous papers of 1972 there had been proposed a generalized Dirac distribution describing the loading acting along an arbitrary curve. Seven years later two Americans obtained similar, but less general results.

The method of infinite differential operators (symbolic method) was developed by Lur'e in years 1936-1942 with the help of Dougall's results (1904). Lur'e arrived at all three groups of fundamental plate equations. Using the symbolic notation one of these equations can be written as follows

$$\cos(\alpha\psi) = 0 \tag{3.7}$$

where $\alpha = \frac{h}{2}\sqrt{\nabla^2}$ and ψ is a rotational part of the displacement vector in the midplane case.

It is noteworthy that this equation is given in a Lur'e's paper of 1942 but is not recalled in his world-wide known monograph of 1955. Being not aware of the Lur'e result (3.7) several authors have rederived this equation 21, 22 and 37 years later, respectively.

The method of higher moments using the Legendre polynomials had been applied in 1975 by Shlenev and Vashakmadze. Five years later the same approach was used by another author who named it "new approach".

In 1922 Birkhoff proposed an original asymptotic method. This method amounts to expanding the unknown functions into a series

$$u_i(x^\alpha, z) = \sum_{k=0} t^k u_{ik}(x^\alpha, z) \qquad i = 1,2,3 \tag{3.8}$$

where t is a natural parameter characterizing the plate thickness. The method of Birkhoff was applied by Garabedian (1923) and Goodier (1938). The paper mentioned above was forgotten for 40 years. This method was discovered once again in 1949.

The first refined theories of plates are due to Lévy's paper of 1877. One author proposes a certain generalization of Michell and Love solutions, respectively by assuming a kinematic hypothesis proposed by Lévy (obviously without recalling Lévy's papers) 77 years later.

The next refined theories were developed in almost the same time. The famous paper of Reissner was published in 1944. The paper of Hencky was written in 1944 in November, but published in 1947. Hencky assumed the linear distribution of displacements across the thickness

$$u_\alpha = -z\varphi_\alpha(x^\beta) \qquad u_3 = v(x^\beta) \tag{3.9}$$

A paper in which one modified the assumption $(3.9)_1$ to the form $u_\alpha = -z\varphi_{,\alpha}$ was published 38 years later. The authors of this article used, similarly as Hencky did, the variational calculus and arrived at less general results. Hencky's paper was not mentioned.

In 1947 there were published two papers of Bolle basing on the (3.9) assumptions. On the contrary to Hencky, Bolle introduced a shear correction factor. Bolle gave, in my opinion, the best examples revealing the boundary effect. He considered a circular plate of radius R subject to a boundary twisting moment $M_{r\varphi}(R) = -m\sin 2\varphi$ and to a transverse force $Q_r(R) = \frac{2m}{R}\cos 2\varphi$. It is readily seen that in the Kirchhoff plate subject to these forces no deformation occurred. Obviously, the predictions of a moderately thick theory are non-trivial: they describe the edge effect. In 1958 an author gave the same example and didn't refer to the paper of Bolle either. The edge effect was named the Reissner effect.

In 1981 Reissner generalized his theory. Three years later there was published a paper in which Reissner (1981) assumptions were adopted and the equations obtained were called "new theory". The Reissner's paper was not, obviously, referred to.

Simultaneously with the progress of the theory of Reissner the theory of Hencky-Bolle has been generalized. Generalization of it on the dynamic case is due to Uflyand (1948). Similarly as Bolle, Uflyand assumed the state of deformation in the form compatible with (3.9) assumptions. By using a direct approach and introducing a shear correction factor k Uflyand obtained the set of three differential equations describing the motion of the plate. Uflyand assumed that $k = 2/3$. In 1953 basing on the Hencky-Bolle assumptions and applying a direct method Mindlin derived the equations of motion of a plate coinciding with those being previously found by Uflyand. Mindlin introduced also a shear correction factor k similarly as it had previously been done by Bolle and Uflyand. The novelty of the Mindlin's paper consists in finding a criterion for the determination of the k coefficient.

The static theory of plates based on the Hencky-Bolle assumptions is usually referred to as a theory of Mindlin even then, if k is assumed to be equal to 5/6 (the value proposed by Bolle)! Bearing in mind the facts mentioned above I think, that according to the historical truth, instead of the notion "Mindlin's theory" one should use the term "theory of Hencky-Bolle" and in the case of dynamics "the theory of Uflyand with the shear correction factor k equal to...". This change will not cast any shadow on the great achievements of Mindlin in the field of mechanics.

In 1963 Aĭnola assumed the displacement distribution in the form

$$u_\alpha(x^\beta, z) = f_1(z)\varphi_\alpha(x^\beta) \qquad u_3(x^\alpha, z) = f_2(z)v(x^\alpha) \tag{3.10}$$

One author simplified Eq (3.10) putting $\varphi_\alpha = v_{,\alpha}$ 21 years later. To derive the governing equations he applied a variational method previously used by Ajnola (1963/1964) and called his method "a novel approach".

In 1957 B.F.Vlasov put forward a theory of plates basing on the kinematic assumptions of the form

$$u_\alpha = -zt_\alpha(x^\beta) - \frac{4z^3}{3h^2}(v_{,\alpha} - t_\alpha) \qquad u_3 = v(x^\alpha) \tag{3.11}$$

By using a direct method he found a set of differential equations of 6th order for three fields $v(x^\alpha)$, $t_\alpha(x^\beta)$. This set was identical to that previously found by Bolle. One author stipulated the same kinematics and found the same equations by a direct method 23 years later. He announced his results as a "new theory".

Collecting the materials to the previous paper (Jemielita, 1991) I have concluded that the content of various papers is often repeated and old laws are discovered again. Moreover, one can note that some brilliant ideas have fallen flat and only after many years new papers have come up in which the same results were recorded (cf other comments by Jemielita, 1991). For the sake of illustration I have recorded the up and downs of three papers.

The Lévy's paper (1877) brought about a loud and long discussion between Boussinesque and Lévy, its fruit being five polemical articles published in 1877/1878. Even a great authority, B.de Saint-Venant joined to this discussion by publishing his comments in the notes about the Clebsch's book. The paper of Lévy as well as this discussion were described in popular, in those times, the book of Todhunter and Pearson (1893) devoted to the history of elasticity theory. One could think that this paper would be remembered by the next generation of researchers and its results taken into account and referred to in further developments. Unfortunately, in dozen years this paper had fallen into oblivion. In 1900 Michell obtained the Lévy results by a different, more complicated manner. This work was made popular by Love by outlining it in the world-wide known monograph "A treatise on the mathematical theory of elasticity". In the paper of Michell as well as in the Love's monograph there is no reference to the paper of Lévy. Since the book of Love became popular the researchers of further generations repeated after this book that Michell (or Love) was the man who solved the problem of generalized state of stress. In this way, perhaps intentionally, the Lévy's paper sank into oblivion for at least half a century.

Similarly as the Lévy's paper the paper of Cauchy (1829) has been forgotten. In this paper, for the first time, a correct differential equation of vibrations of an anisotropic plate was derived. This paper was unknown to Boussinesque (1879), Voigt (1910) and to Huber (1929) who is considered to be a founder of the modern theory of anisotropic plates. Also such famous researches as Timoschenko, Lekhnickii, Ambartsumyan et al. did not refer to this work[2].

Huber's contribution to forming the theory of orthotropic plates, determining the effective stiffnenes of reinforced – concrete plates and other plates of engineering anisotropy are beyond discussion. The fundamental equation of the orthotropic plate in bending is referred to as a Huber's equations. One can find, however, papers in which not Huber's papers but the monograph of Lekhnitskii is referred to. This is due to publishing an English translation of this book in 1947. Other forgotten paper, not only in Poland, is a work of Dougall from 1904. The only copy of this article available in Poland remained virgin (the pages were uncut) until I found it. Hence for 82 years nobody in Poland have touched this copy.

One can ask the question: how many virgin copies of this work and other one can fin in libraries? How many papers have fallen into obscurity?

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Meandry teorii płyt

Streszczenie

W pracy przedstawiono krytyczne spojrzenie na rozwój teorii płyt w okresie 1789-1988. Próbowano znaleźć odpowiedź na pytanie: Studiować czy tworzyć? Podane przykłady z historii rozwoju teorii płyt wskazują na to, że tworząc należy studiować prace poprzedników.

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[2]I asked for the Poisson's paper of 1929. However, I received the paper of Cauchy from Paris. Upon a detail looking through an available literature I found only two papers (1961, 1964) in which this paper of Cauchy was adduced.