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MAGNETOELASTIC PLATE THEORY

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The paper deals with homogeneous isotropic linearly elastic and non-perfectly conducting plates of constant thickness. A starting point consists in displacement thickness distribution and equations of motion of the Kirchhoff theory incorporating Lorentz force as combined with 3-D equations of electrodynamics. Both the bending and the stretching are taken into account. Mutually coupled 2-D governing equations are derived under the somewhat new E-M (electromagnetic) assumptions. The final equations are simplified so as to differ, but little, from the corresponding equations due to the modified hypothesis of magnetoelasticity of thin bodies.

1. Introduction

The aim of the present paper is to derive basic differential equations of the linear theory of thin magnetoelastic plates using a new E-M (electromagnetic) hypothesis. The purely elastic terms are adopted in the same form as in the classical (Kirchhoff) plate theory, attention being focused on the approximate two-dimensional description of the magnetoelastic coupling.

The general concept of including of magnetoelastic effects may be traced back to Alpher and Rubin (1954), however, in some previous works (cf Cagniard, 1952) the magnetomechanical interactions were considered in the framework of geophysics. By contrast to ferromagnetics as well as many other media endowed with various electromagnetomechanical properties (cf Rymarz, 1986; Maugin, 1988) the magnetoelastic body may be treated on retaining the dynamic assumptions of classical continuum (cf Rogula, 1982, p.20). According to the case of quasi-stationary E-M field (cf Ingarden and Jamiołkowski, 1980, p.196) the Lorentz force equals to a divergence of the symmetric Maxwell stress tensor, the latter being defined by the magnetic induction. As far as the perfectly conducting media are concerned the Maxwell stress tensor is expressible within the linear theory by a strong bias magnetic induction and the small displacement. Thereby, in the case of the perfectly conducting plates it is feasible to formulate the magnetoelastic plate theory as an approximation of the theory of magnetoelasticity entirely with the use of the purely mechanical hypotheses (cf Kaliski, 1962; Baghdasarian, 1983).

As regards the plates of finite conductivity the hypothesis of magnetoelasticity of thin bodies was deduced by Ambartsumian et al. (1971) for the plates under bending, and by Ambartsumian et al. (1973) for the plates under both bending and stretching. Initially the above-mentioned hypothesis has contained of the suitable E-M assumptions as well as the classical Kirchhoff assumptions. Since the E-M hypothesis may be formally joined to the more sophisticated mechanical assumptions as well (cf Ambartsumian, 1979), the hypothesis of magnetoelasticity of thin bodies is now referred exclusively to the corresponding E-M thickness distributions (cf Ambartsumian et al., 1984). Unfortunately, this hypothesis seems to be inadequate in an accurate representing of the frequency response of the plate under stretching (sheet) in the longitudinal magnetic field (cf Radovinskii, 1987). In an effort to improve the former hypothesis in the paper by Rudnicki (1984) the less restrictive E-M assumptions were employed.

Author [16] presented a quite different E-M hypothesis physical meaning of which consists in the vanishing of the resulting normal component of the current density at the top and bottom surfaces of the plate, respectively. Developing this last approach in the present paper we propose a somewhat more general E-M hypothesis than it was assumed by Rudnicki [16]. In Section 2 the "electromagnetically-exact" equations governing the magnetoelasticity problem of the plate thin enough allowing the Kirchhoff assumptions are presented. The two-dimensional reduction under the introduced hypothesis is carried out in Section 3. The approximate distribution of the secondary E-M field in the thickness direction and the resulting E-M terms in the reduced equations of motion are obtained. In Section 4 some further simplifications due to the thinness of the plate are proposed as well as the corresponding boundary and initial conditions of the plate theory are formulated. In Section 5 the special form of the equations due to the simplest version of the introduced hypothesis being employed by Rudnicki [16] is specified.

The MKSA unit system is used. The Latin indices i, k and l have the range (1,2,3) while the Greek indices α, β and γ the range (1,2).

2. Formulation of the problem

We consider an elastic non-perfectly conducting plate that at the undeformed reference configuration occupies a region of space $V = \Omega \times (-h/2, h/2)$ where Ω is the middle surface of the plate and h is the plate thickness. Let x_i refer to the orthogonal Cartesian co-ordinates related to the reference configuration so as the plane $x_3 = 0$ coincides with the surface Ω. We assume that the plate is surrounded by a perfectly rigid (motion-less) and perfectly conducting medium placed in the outer domain $|x_3| < h/2$. The domains $|x_3| > h/2$ remain free from any physical substance. We deal with homogeneous isotropic plates of diamagnetic or paramagnetic properties. The magnetic permeability μ is set equal to the magnetic permeability of vacuum.

At the initial configuration the plate is still at rest, however, deformed (statically) and subjected to a given magnetostatic field. We confine ourselves to the bias magnetic fields uniform in the thickness direction within the plate region, i.e.

$$B_i(x_i) = P_i(x_\alpha) \tag{2.1}$$

where B_i are the components of the bias magnetic induction. It follows from the Maxwell equations that under the condition (2.1) the normal component of the bias magnetic induction does not vary throughout the plate region, i.e.

$$P_3 = \text{const} \tag{2.2}$$

while the tangent components satisfy 2-D equations

$$P_{\alpha,\alpha} = 0 \qquad \varepsilon_{\alpha\beta} P_{\beta,\alpha} = 0 \tag{2.3}$$

where $\varepsilon_{\alpha\beta}$ is the permutation symbol, and comma represents the partial differentiation with respect to a position coordinate.

Surface tractions caused by the bias magnetic field, if any, are due to the jump of the magnetic permeability at the bounding surface which surrounds the plate region. Keeping in mind that the magnetostatic field does not cause any body force, the mechanical effects of the bias magnetic field are negligible so far as the initial configuration is concerned.

At the present configuration the induced secondary E-M field is coupled with the time-dependent elastic deformation determined by the vector displacement components $w_i = w_i(x_i, t)$ (t = time) as measured from the initial configuration. The magnetoelastic coupling is brought into play through the current dependent body force and the velocity dependent electric current. The secondary E-M field has a quasi-stationary character. We assume that the time-dependent mechanical and E-M quantities are small enough allowing the linearization of the governing equations. Moreover, we omit the influence of the initial static deformation and stresses on the time-dependent state (cf Eringen, 1989).

According to the classical plate theory the displacement field may be approximated as follows

$$\begin{aligned} w_\alpha(x_i, t) &= u_\alpha(x_\alpha, t) - x_3 u_3(x_\alpha, t)_{,\alpha} \\ w_3(x_i, t) &= u_3(x_\alpha, t) \end{aligned} \tag{2.4}$$

Thus, u_i may be identified as the displacement components at the middle surface of the plate.

Taking into account the Lorentz force we present the "displacement" equations of motion for the plate in the following form

$$\begin{aligned} &\frac{1-\nu}{2}u_{\alpha,\beta\beta}+\frac{1+\nu}{2}u_{\beta,\alpha\beta}-\frac{1}{c_0^2}\ddot{u}_\alpha+F_\alpha+F_\alpha^e=0 \\ &\frac{h^2}{12}u_{3,\alpha\alpha\beta\beta}+\frac{1}{c_0^2}\Big(\ddot{u}_3-\frac{h^2}{12}\ddot{u}_{3,\alpha\alpha}\Big)=F_3+F_3^e \end{aligned} \tag{2.5}$$

where superposed dot indicates the partial differentiation with respect to time, ν is Poisson ratio, c_0 is the speed of sound, i.e.

$$c_0^2=\frac{E}{\rho(1-\nu^2)} \tag{2.6}$$

with E and ρ being the Young modulus and the mass density, respectively, F_i and F_i^e are defined by formulas

$$\begin{aligned} F_\alpha&=\frac{1-\nu^2}{Eh}p_\alpha & F_3&=\frac{1-\nu^2}{Eh}(p_3+m_{\alpha,\alpha}) \\ F_\alpha^e&=\frac{1-\nu^2}{Eh}p_\alpha^e & F_3^e&=\frac{1-\nu^2}{Eh}(p_3^e+m_{\alpha,\alpha}^e) \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} p_i&=\int\limits_{-\frac{h}{2}}^{\frac{h}{2}}q_i\,dx_3+S_i^++S_i^- \\ m_\alpha&=\int\limits_{-\frac{h}{2}}^{\frac{h}{2}}q_\alpha x_3\,dx_3+\frac{h}{2}(S_\alpha^+-S_\alpha^-) \end{aligned} \tag{2.8}$$

with q_i and $S_i^\pm$ being the mechanical body force and the mechanical tractions at the surfaces $x_3=\pm h/2$, respectively, and finally

$$p_i^e=\varepsilon_{ikl}\int\limits_{-\frac{h}{2}}^{\frac{h}{2}}j_kB_l\,dx_3 \qquad m_\alpha^e=\varepsilon_{\alpha kl}\int\limits_{-\frac{h}{2}}^{\frac{h}{2}}j_kB_lx_3\,dx_3 \tag{2.9}$$

with $j_i=j_i(x_i,t)$ being the components of the current density.

The three-dimensional equations of the secondary E-M field in the quasi-stationary approximation within the plate region read

$$
\begin{aligned}
& b_{i,i} = 0 && \varepsilon_{ikl} b_{l,k} = \mu j_i \\
& \varepsilon_{ikl} e_{l,k} + \dot{b}_i = 0 && j_i = \lambda(e_i + \varepsilon_{ikl} B_l \dot{w}_k)
\end{aligned}
\tag{2.10}
$$

where b_i and e_i mean the components of the secondary magnetic induction and the electric field intensity, respectively, λ stands for the electric conductivity.

We denote the scalar potentials describing the secondary quasi-stationary magnetic field throughout the vacuum domains $|x_3| > h/2$ by Φ^α, i.e,, $\Phi^2 = \Phi^2(x_i, t)$ where $x_3 > h/2$ and $\Phi^1 = \Phi^1(x_i, t)$ where $x_3 < -h/2$. Each of these potentials has to meet the corresponding Laplace equation

$$
\Phi^\alpha{}_{,ii} = 0 \qquad (-1)^\alpha x_3 > \frac{h}{2}
\tag{2.11}
$$

under the boundary conditions

$$
\begin{aligned}
\Phi^2{}_{,3}(x_i, t)\Big|_{x_3 \to \frac{h}{2}} &= \begin{cases} 0 & \text{if} \quad x_\alpha \notin \Omega \\ b_3^+ & \text{if} \quad x_\alpha \in \Omega \end{cases} \\
\Phi^1{}_{,3}(x_i, t)\Big|_{x_3 \to -\frac{h}{2}} &= \begin{cases} 0 & \text{if} \quad x_\alpha \notin \Omega \\ b_3^- & \text{if} \quad x_\alpha \in \Omega \end{cases}
\end{aligned}
\tag{2.12}
$$

Here and afterwards the quantities assigned with "$\pm$" refer to the surfaces $x_3 = \pm h/2$, respectively. Thus

$$
b_i^\pm = b_i\Big|_{x_3 = \pm\frac{h}{2}}
\tag{2.13}
$$

At the top and bottom surfaces of the plate we should satisfy the following continuity conditions

$$
b_\alpha^+ = \Phi^{2+}{}_{,\alpha} \qquad b_\alpha^- = \Phi^1{}_{,\alpha}
\tag{2.14}
$$

In view of the preliminary assumptions the boundary conditions at the lateral surface $x_\alpha \in \partial\Omega$, $|x_3| < h/2$ read

$$
\begin{aligned}
& u_t = 0 \quad && u_n = 0 \quad && u_3 = 0 \quad && u_{3,n} = 0 \quad && \text{and} \quad && x_3 = 0 \\
& b_n = 0 && j_t = 0 && j_3 = 0 && && \text{and} && |x_3| < \tfrac{h}{2}
\end{aligned}
\tag{2.15}
$$

where the subscripts n and t, respectively, refer to the normal and tangent directions to the curve $\partial\Omega$ which bounds the surface Ω.

3. Reduction under a new hypothesis

First, we proceed to establish the through-the-thickness distributions of the secondary magnetic field and the current density, which in turn enable us to determine the E-M part of the reduced equations of motion. The starting point of our consideration consists of the following thickness assumptions (cf Ambartsumian, 1987, p.31)

$$b_\alpha(x_i,t) = \frac{1}{2}b^s_\alpha(x_\alpha,t) + \frac{x_3}{h}b^r_\alpha(x_\alpha,t) + \varphi(x_3)l_\alpha(x_\alpha,t) \tag{3.1}$$

where

$$b^r_\alpha = b^+_\alpha - b^-_\alpha \qquad\qquad b^s_\alpha = b^+_\alpha + b^-_\alpha \tag{3.2}$$

and φ is required to be the even function of x_3 undergoing the conditions

$$\varphi^+ = \varphi^- = 0 \tag{3.3}$$

Let us note that Eq (3.1) when evaluated for $x_3 = \pm h/2$ is identically satisfied.

Introducing Eq (3.1) into Eq $(2.10)_1$ and next carrying out the integration

$$L[...] = \frac{1}{h}\Big(\int\limits_0^{x_3}(...)dx_3 + \frac{1}{2}\int\limits_{-\frac{h}{2}}^{0}(...)dx_3 - \frac{1}{2}\int\limits_0^{\frac{h}{2}}(...)dx_3\Big) \tag{3.4}$$

we obtain

$$b_3(x_i,t) = f(x_\alpha,t) - L[\varphi](x_3)g(x_\alpha,t) - \frac{x_3}{2h}a^s(x_\alpha,t) + \frac{\varphi^1(x_3)}{8}a^r(x_\alpha,t) \tag{3.5}$$

where

$$\varphi^1(x_3) = 1 - 4\Big(\frac{x_3}{h}\Big)^2 \tag{3.6}$$

and

$$g = hl_{\alpha,\alpha} \qquad\qquad a^s = hb^s_{\alpha,\alpha} \qquad\qquad a^r = hb^r_{\alpha,\alpha} \tag{3.7}$$

Substituting Eqs (3.1) and (3.5) into the first two equations of the set $(2.10)_2$, i.e., for $i = 1,2$, we find

$$\begin{aligned} j_\alpha(x_i,t) = \frac{1}{\mu}\varepsilon_{\alpha\beta}\Big[f(x_\alpha,t)_{,\beta} - \frac{1}{h}b^r_\beta(x_\alpha,t) - \varphi(x_3)_{,3}l_\beta(x_\alpha,t) - \\ -L[\varphi](x_3)g(x_\alpha,t)_{,\beta} - \frac{x_3}{2h}a^s(x_\alpha,t)_{,\beta} + \frac{\varphi^1(x_3)}{8}a^r(x_\alpha,t)_{,\beta}\Big] \end{aligned} \tag{3.8}$$

Similarly, inserting Eq (3.1) into the third one of Eq $(2.10)_2$, i.e., for $i = 3$, with the aid of Eqs (3.2) and (2.14) yields

$$j_3(x_i,t) = \frac{\varphi(x_3)}{h}\frac{s(x_\alpha,t)}{\mu} \tag{3.9}$$

where

$$s = h\varepsilon_{\alpha\beta}l_{\beta,\alpha} \tag{3.10}$$

By taking the partial derivatives of Eqs $(3.7)_1$ and (3.10) with respect to x_α, and combining the results, we have

$$l_{\alpha,\beta\beta} - \frac{1}{h}(g_{,\alpha} - \varepsilon_{\alpha\beta}s_{,\beta}) = 0 \tag{3.11}$$

Making use of the appropriate thickness distributions, i.e., Eqs (3.8), (3.9), and (2.1), (2.9) gives

$$\begin{aligned} p^e_\alpha &= \varepsilon_{\beta\alpha}\frac{P_\beta}{\mu}\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{\varphi}{h}dx_3 s - \frac{P_3}{\mu}\Big[h\Big(f+\frac{a^r}{12}\Big)_{,\alpha} - b^r_\alpha\Big] \\ p^e_3 &= \frac{P_\alpha}{\mu}\Big[h\Big(f+\frac{a^r}{12}\Big)_{,\alpha} - b^r_\alpha\Big] \\ m^e_\alpha &= \frac{P_3}{\mu}\Big(\int_{-\frac{h}{2}}^{\frac{h}{2}} L[\varphi]x_3dx_3 g_{,\alpha} - \int_{-\frac{h}{2}}^{\frac{h}{2}}\varphi dx_3 l_\alpha + \frac{h^2}{24}a^s{}_{,\alpha}\Big) \end{aligned} \tag{3.12}$$

Introduction of Eqs (3.12) to Eqs $(2.7)_3$ and $(2.7)_4$, with the use of Eq $(3.7)_1$, results in

$$\begin{aligned} F^e_\alpha &= \varepsilon_{\beta\alpha}T_\beta\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{\varphi}{h}x_3 s - T_3\Big[h\Big(f+\frac{a^r}{12}\Big)_{,\alpha} - b^r_\alpha\Big] \\ F^e_3 &= T_\alpha\Big[h\Big(f+\frac{a^r}{12}\Big)_{,\alpha} - b^r_\alpha\Big] + \\ &\quad + T_3\Big(\int_{-\frac{h}{2}}^{\frac{h}{2}} L[\varphi]x_3dx_3 g_{,\alpha\alpha} - \int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{\varphi}{h}dx_3 g + \frac{h^2}{24}a^s{}_{,\alpha\alpha}\Big) \end{aligned} \tag{3.13}$$

where

$$T_i = \frac{1-\nu^2}{\mu Eh}P_i \tag{3.14}$$

In view of Eqs (3.13) the fully reduced equations of motion (2.5) involve the following two-dimensional unknowns: u_i, f, g, s, b^r_α, a^r and a^s. Now, we proceed to derive the appropriate E-M reduced equations involving at most the above-mentioned unknowns. For this purpose we shall directly use Ohm's law $(2.10)_4$ as well as the Maxwell equations $(2.10)_3$ which have not been employed so far.

The inverted relations $(2.10)_4$ for $i = 1,2$ read

$$e_\alpha = \frac{j_\alpha}{\lambda} + \varepsilon_{\alpha\beta}(B_\beta \dot{w}_3 - B_3 \dot{w}_\beta) \tag{3.15}$$

Using Eqs (2.1), (2.4) and (3.8) we replace Eq (3.15) by

$$\begin{aligned} e_\alpha(x_i,t) &= \varepsilon_{\alpha\beta}\Big\{\frac{1}{\mu\lambda}\Big[f(x_\alpha,t)_{,\beta} - \frac{1}{h}b^r_\beta(x_\alpha,t) - \varphi(x_3)_{,3} l_\beta(x_\alpha,t) - \\ &- L[\varphi](x_3)g(x_\alpha,t)_{,\beta} - \frac{x_3}{2h}a^s(x_\alpha,t)_{,\beta} + \frac{\varphi^1(x_3)}{8}a^r(x_\alpha,t)_{,\beta}\Big] + \\ &+ P_\beta(x_\alpha)\dot{u}_3(x_\alpha,t) - P_3[\dot{u}_\beta(x_\alpha,t) - x_3\dot{u}_3(x_\alpha,t)_{,\beta}]\Big\} \end{aligned} \tag{3.16}$$

Substituting Eqs (3.16) and (3.5) into the third equation of set $(2.10)_3$, i.e., for $i = 3$, with the aid of Eqs $(3.7)_1$, (2.2) and $(2.3)_1$ we find

$$\begin{aligned} & h^2\Big\{f_{,\alpha\alpha} - \mu\lambda\dot{f} - \frac{1}{h}\varphi_{,3}\, g - L[\varphi](g_{,\alpha\alpha} - \mu\lambda\dot{g}) + \\ & + \mu\lambda[P_\alpha \dot{u}_{3,\alpha} - P_3(\dot{u}_{\alpha,\alpha} - x_3\dot{u}_{3,\alpha\alpha})]\Big\} - \\ & - \frac{h}{2}x_3(a^s{}_{,\alpha\alpha} - \mu\lambda\dot{a}^s) - a^r + \frac{h^2}{8}\varphi^1(a^r{}_{,\alpha\alpha} - \mu\lambda\dot{a}^r) = 0 \end{aligned} \tag{3.17}$$

Integration of Eq (3.17) with the weight zero and one, with respect to x_3 between the limits $-h/2$ and $h/2$, yields

$$\begin{aligned} & h^2\Big[f_{,\alpha\alpha} - \mu\lambda\dot{f} + \mu\lambda(P_\alpha\dot{u}_{3,\alpha} - P_3\dot{u}_{\alpha,\alpha})\Big] - a^r + \frac{h^2}{12}(a^r{}_{,\alpha\alpha} - \mu\lambda\dot{a}^r) = 0 \\ & \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} L[\varphi]x_3 dx_3(g_{,\alpha\alpha} - \mu\lambda\dot{g}) - \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\varphi}{h}dx_3 g - \mu\lambda P_3\frac{h^3}{12}\dot{u}_{3,\alpha\alpha} + \frac{h^2}{24}(a^s{}_{,\alpha\alpha} - \mu\lambda\dot{a}^s) = 0 \end{aligned} \tag{3.18}$$

Similarly, with the aid of Eqs (2.1), (2.4) and (3.9), the inverted relation $(2.10)_4$ for $i = 3$, i.e.

$$e_3 = \frac{j_3}{\lambda} + B_1\dot{w}_2 - B_2\dot{w}_1 \tag{3.19}$$

becomes

$$
\begin{aligned}
e_3(x_i,t) &= \frac{\varphi(x_3)}{h}\frac{s(x_\alpha,t)}{\mu\lambda} + P_1(x_\alpha)[\dot{u}_2(x_\alpha,t) - x_3\dot{u}_3(x_\alpha,t)_{,2}] - \\
&- P_2(x_\alpha)[\dot{u}_1(x_\alpha,t) - x_3\dot{u}_3(x_\alpha,t)_{,1}])
\end{aligned} \tag{3.20}
$$

Differentiating Eq $(2.10)_3$ for $i = 1$ with respect to x_1, and for $i = 2$ with respect to x_2, then subtracting both results we arrive at the equation

$$
e_{3,\alpha\alpha} - e_{\alpha,\alpha 3} - \varepsilon_{\alpha\beta}\dot{b}_{\beta,\alpha} = 0 \tag{3.21}
$$

Substitution of Eqs (3.20), (3.16) and (3.1) into Eq (3.21) by accounting for Eq (3.10) leads to

$$
\varphi_{,33}\, s + \varphi(s_{,\alpha\alpha} - \mu\lambda\dot{s}) + \varepsilon_{\alpha\beta}\mu\lambda h[x_3(P_\beta\dot{u}_{3,\alpha})_{,\gamma\gamma} - (P_\beta\dot{u}_\alpha)_{,\gamma\gamma}] = 0 \tag{3.22}
$$

Integrating Eq (3.22) with respect to x_3 between the limits $-h/2$ and $h/2$ we obtain

$$
\begin{aligned}
&\Big(\varphi_{,3}\Big|_{x_3\to\frac{h}{2}} - \varphi_{,3}\Big|_{x_3\to-\frac{h}{2}}\Big)s + \int_{-\frac{h}{2}}^{\frac{h}{2}} \varphi dx_3 (s_{,\alpha\alpha} - \mu\lambda\dot{s}) - \\
&-\varepsilon_{\alpha\beta}\mu\lambda h^2(P_\beta\dot{u}_\alpha)_{,\gamma\gamma} = 0
\end{aligned} \tag{3.23}
$$

Eqs (3.18) and (3.23) constitute the desired set of the E-M reduced differential partial equations with unknowns: u_i, f, g, s, b^r_α, a^r and a^{s}.

Using Eqs (3.5), (3.6), $(3.7)_2$, $(3.7)_3$ and (2.14) the continuity conditions (2.12) as $x_\alpha \in \Omega$ read

$$
\begin{aligned}
\Phi^2{}_{,3}(x_i,t)\Big|_{x_3\to\frac{h}{2}} + \frac{h}{4}\Big(\Phi^{2+}_{,\alpha\alpha} + \Phi^{1-}_{,\alpha\alpha}\Big) &= f - L[\varphi]^+ g \\
\Phi^1{}_{,3}(x_i,t)\Big|_{x_3\to-\frac{h}{2}} - \frac{h}{4}\Big(\Phi^{2+}_{,\alpha\alpha} + \Phi^{1-}_{,\alpha\alpha}\Big) &= f - L[\varphi]^- g
\end{aligned} \tag{3.24}
$$

4. Further simplifications. Boundary and initial conditions

In course of developing the plate equations in Section 3 no assumptions but (3.1) are admitted. Now we submit some further simplifications due to the thinness of the plate. First, under the assertion that the last three terms are negligible in comparison to the remainder, Eq (3.8) reduces to

$$j_\alpha(x_i,t) = \frac{1}{\mu}\varepsilon_{\alpha\beta}\Big[f(x_\alpha,t),_\beta - \frac{1}{h}b^r_\beta(x_\alpha,t) - \varphi(x_3),_3 l_\beta(x_\alpha,t)\Big] \tag{4.1}$$

Eq (3.16) has to be simplified in the same way. As a consequence Eqs (3.12), (3.13) and (3.18) are to be replaced respectively by

$$\begin{aligned} p^e_\alpha &= \varepsilon_{\beta\alpha}\frac{P_\beta}{\mu}\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\varphi}{h}dx_3 s - \frac{P_3}{\mu}(hf,_\alpha - b^r_\alpha) \\ p^e_3 &= \frac{P_\alpha}{\mu}(hf,_\alpha - b^r_\alpha) \qquad m^e_\alpha = -\frac{P_3}{\mu}\Big(\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \varphi dx_3 l_\alpha\Big) \end{aligned} \tag{4.2}$$

$$\begin{aligned} F^e_\alpha &= \varepsilon_{\beta\alpha}T_\beta\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\varphi}{h}x_3 s - T_3(hf,_\alpha - b^r_\alpha) \\ F^e_3 &= T_\alpha(hf,_\alpha - b^r_\alpha) - T_3\Big(\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\varphi}{h}dx_3 g\Big) \end{aligned} \tag{4.3}$$

$$\begin{aligned} & h^2\Big[f,_{\alpha\alpha} - \mu\lambda\dot{f} + \mu\lambda(P_\alpha\dot{u}_3,_\alpha - P_3\dot{u}_\alpha,_\alpha)\Big] - a^r - \mu\lambda\frac{h^2}{12}\dot{a}^r = 0 \\ & \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\varphi}{h}dx_3 g + \mu\lambda\Big(\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} L[\varphi]x_3 dx_3\dot{g} + P_3\frac{h^3}{12}\dot{u}_3,_{\alpha\alpha} + \frac{h^2}{24}\dot{a}^s\Big) = 0 \end{aligned} \tag{4.4}$$

Moreover, in Eq $(4.4)_2$ we disregard the last term in bracket in comparison to the first term. Thus, Eq $(4.4)_2$ changes to

$$\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\varphi}{h}dx_3 g + \mu\lambda\Big(\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} L[\varphi]x_3 dx_3\dot{g} + P_3\frac{h^3}{12}\dot{u}_3,_{\alpha\alpha}\Big) = 0 \tag{4.5}$$

By virtue of Eq (4.1) the term involving $g_{,\alpha\alpha}$ does not appear in Eq $(4.4)_2$. Removing from Eq (3.23) the term involving $s_{,\alpha\alpha}$ we find

$$\left(\varphi_{,3}\Big|_{x_3\to\frac{h}{2}} - \varphi_{,3}\Big|_{x_3\to-\frac{h}{2}}\right)s - \mu\lambda\left(\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \varphi dx_3\dot{s} + \varepsilon_{\alpha\beta}h^2(P_\beta\dot{u}_\alpha)_{,\gamma\gamma}\right) = 0 \tag{4.6}$$

As regards the plate equations of the theory derived in the present work there are two groups of the two-dimensional unknowns, i.e., basic unknowns: u_i, f, g, s and l_α, and extra unknowns: b_α^+, b_α^-, b_α^r, b_α^s, a^r and a^s. The former ones may be found from the equations: (2.5), $(4.4)_1$, (4.5), (4.6) and (3.11), respectively. The quantities a^r and a^s are expressed in terms of rates of b_α^r and b_α^s by means of Eqs $(3.7)_2$ and $(3.7)_3$, however, b_α^r and b_α^s are defined in terms of $b_\alpha^\pm$ by means of Eqs (3.2). The quantities $b_\alpha^\pm$ in turn may be deduced from the continuity conditions (2.14) afterwards Eqs (2.11) have been solved under the conditions (2.12) and the conditions at infinity. As was mentioned previously, the quantities $b_\alpha^\pm$ cannot be found with the aid of Eqs (3.1).

Summarizing, equations of motion (2.5) with F_i^e being defined by Eqs (4.3), and E-M equations $(4.4)_1$, (4.5) and (4.6) constitute a tenth-order set of six differential equations on unknowns: u_i, f, g and s. Thus, at the curve $\partial\Omega$ we have to formulate five boundary conditions. The boundary conditions of mechanical origin are assumed in the form of the first four equations of the set (2.15). Integration of Eq $(2.15)_6$ with respect to x_3 between the limits $-h/2$ and $h/2$ supplies the missing fifth condition. If the curve $\partial\Omega$ coincides with the line $x_3 = 0$ and x_α = const (with α being fixed), after using Eq (4.1), the reduced form of Eq $(2.15)_6$ reads

$$b_\alpha^r - hf_{,\alpha} = 0 \tag{4.7}$$

The initial conditions consist of

$$\begin{aligned} &u_i(x_\alpha,t)\Big|_{t=0} = \overset{0}{u}_i(x_\alpha) && \dot{u}_i(x_\alpha,t)\Big|_{t\to0} = \overset{1}{u}_i(x_\alpha) \\ &s(x_\alpha,t)\Big|_{t=0} = \overset{0}{s}(x_\alpha) && \\ &f(x_\alpha,t)\Big|_{t=0} = \overset{0}{f}(x_\alpha) && g(x_\alpha,t)\Big|_{t=0} = \overset{0}{g}(x_\alpha) \end{aligned} \tag{4.8}$$

where $x_\alpha \in \Omega$.

After deducing the unknowns g and s the quantities l_α may be determined by means of two second-order differential equations (3.11). The corresponding boundary conditions are to be derived from Eqs $(2.15)_5$ and $(2.15)_7$. The condition $(2.15)_7$ is met at any point of the lateral surface, provided that

$$s = 0 \tag{4.9}$$

Taking Eq (3.10) into account we change Eq (4.9) to the form

$$\varepsilon_{\alpha\beta} l_{\alpha,\beta} = 0 \tag{4.10}$$

The condition $(2.15)_5$ may be fulfilled in the average sense. If the lateral surface coincides with the plane x_α = const (α being fixed), then integration of Eq $(2.15)_5$ ($b_n = b_\alpha$) with respect to x_3 between the limits $-h/2$ and $h/2$ leads to

$$b^s_\alpha + \frac{2}{h}\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \varphi dx_3 l_\alpha = 0 \tag{4.11}$$

Let us note that at the boundary x_α = const Eq $(4.2)_3$, after the use of Eq (4.11), becomes

$$m^e_\alpha = \frac{h}{2}\frac{P_3}{\mu} b^s_\alpha \tag{4.12}$$

5. Final equations in the case $\varphi = \varphi^1$

Some equations derived in the present work, for example $(3.18)_1$, (3.7) and (3.10), do not depend on the approximating function $\varphi(x_3)$. However, many equations of the introduced theory strictly depend on this function. Here we shall be not dealing with the question of the right choice of $\varphi(x_3)$. Nevertheless, it seems important to examine the special case when $\varphi(x_3)$ is the same as $\varphi^1(x_3)$ in Eq (3.6) (cf Rudnicki, [16]). For the sake of brevity we investigate below the final governing equations only.

Taking Eqs (3.6) and (3.4) into account we find

$$\begin{aligned}
&\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \varphi^1 dx_3 = \frac{2}{3}h & & L[\varphi^1] = \frac{x_3}{h}\Big[1 - \frac{4}{3}\Big(\frac{x_3}{h}\Big)^2\Big] \\
&\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} L[\varphi^1] x_3 dx_3 = \frac{h^2}{15} & & \\
&\varphi_{,3}\Big|_{x_3 \to \frac{h}{2}} - \varphi_{,3}\Big|_{x_3 \to -\frac{h}{2}} = -\frac{8}{h} & & L[\varphi^1]^{\pm} = \pm\frac{1}{3}
\end{aligned} \tag{5.1}$$

with the aid of which Eqs (4.3), (4.5), (4.6), (3.24) and (4.11), respectively, simplify to

$$
\begin{aligned}
F^e_\alpha &= \varepsilon_{\beta\alpha} T_\beta \frac{2}{3} s - T_3 (h f_{,\alpha} - b^r_\alpha) \\
F^e_3 &= T_\alpha (h f_{,\alpha} - b^r_\alpha) - T_3 \frac{2}{3} g
\end{aligned}
\tag{5.2}
$$

$$
\begin{aligned}
&\frac{2}{3}\left(g + \mu\lambda \frac{h^2}{10} \dot{g}\right) + \mu\lambda P_3 \frac{h^3}{12} \dot{u}_{3,\alpha\alpha} = 0 \\
&\frac{2}{3}\left(s + \mu\lambda \frac{h^2}{12} \dot{s}\right) + \varepsilon_{\alpha\beta}\mu\lambda \frac{h^3}{12} (P_\beta \dot{u}_\alpha)_{,\gamma\gamma} = 0
\end{aligned}
\tag{5.3}
$$

$$
\begin{aligned}
&\Phi^2{}_{,3}(x_i,t)\Big|_{x_3 \to \frac{h}{2}} + \frac{h}{4}\left(\Phi^{2+}_{,\alpha\alpha} + \Phi^{1-}_{,\alpha\alpha}\right) = f - \frac{g}{3} \\
&\Phi^1{}_{,3}(x_i,t)\Big|_{x_3 \to -\frac{h}{2}} - \frac{h}{4}\left(\Phi^{2+}_{,\alpha\alpha} + \Phi^{1-}_{,\alpha\alpha}\right) = f + \frac{g}{3}
\end{aligned}
\tag{5.4}
$$

$$
b^s_\alpha + \frac{4}{3} l_\alpha = 0 \tag{5.5}
$$

Using Eqs (5.1) one may easily transform the remaining equations (3.1), (3.5), (4.1), (3.9), (3.16) (when simplified in accord and with Eq (4.1) too), and Eq (3.20) to the special form regarding the case $\varphi = \varphi^1$ (see also Rudnicki [16]).

6. Conclusions

In the theories based on the hypothesis of magnetoelasticity of thin bodies as well as its modification proposed by Rudnicki (1984) the quantities b_3 and e_α are those which undergo some restrictions imposed on their distributions in the thickness direction within the plate region. As a result no further step could be made unless the appropriate displacement assumptions were employed. The current density components were expressed by some E-M unknowns as well as midsurface displacement components. By contrast to those theories the hypothesis (3.1) enables us to establish the approximate thickness distribution of the secondary magnetic field regardless of any displacement assumptions. Thereby, we have expressed the current density components entirely by appropriate E-M quantities (cf Eqs (3.9) and (4.1)). Despite above-mentioned differences the final equations regarding the case $\varphi = \varphi^1$ are nearly the same as the corresponding equations due to the hypothesis formulated by Rudnicki (1984).

References

1. ALPHER R.A., RUBIN R.J,, 1954, *Magnetic dispersion and attenuation of sound in conducting fluids and solids,* J.Acoust.Soc.Amer., **26**, 3
2. AMBARTSUMIAN S.A., 1979, *On the problem of vibrations of electroconductive plate in transverse magnetic field,* (in Russian), Izv. AN SSSR, MTT, **3**
3. AMBARTSUMIAN S.A., 1987, *Theory of anisotropic plates,* (in Russian), Nauka, Moscow
4. AMBARTSUMIAN S.A., BAGHDASARIAN G.E., BELUBEKIAN M.V., 1971, *On three-dimensional problem of magnetoelastic vibrations of a plate,* (in Russian), PMM, **35**, 2
5. AMBARTSUMIAN S.A., BAGHDASARIAN G.E., BELUBEKIAN M.V., 1973, *On magnetoelasticity of thin shells and plates,* (in Russian), PMM, **37**, 1
6. AMBARTSUMIAN S.A., BELUBEKIAN M.V., MINASSIAN M.M., 1984, *On the problem of vibrations of non-linear elastic electroconductive plates in transverse and longitudinal magnetic fields,* Int.J.Non-linear Mech., **19**, 2
7. BAGHDASARIAN G.E., 1983, *Equations of magnetoelastic vibrations of thin perfectly conducting plates,* (in Russian), PM, **19**, 12 .
8. CAGNIARD L., 1952, *Sur la nature des ondes seismiques capables de traverser le Noyau terrestre,* (in French), Compt.Rend., **234**, 17
9. ERINGEN A.C., 1989, *Theory of electromagnetic elastic plates,* Int.J.Engn.Sci., **27**, 4
10. INGARDEN R.S., JAMIOŁKOWSKI A., 1980, *Elektrodynamika klasyczna,* (in Polish), PWN, Warsaw
11. KALISKI S., 1962, *Magnetoelastic vibration of perfectly conducting plates and bars assuming the principle of plane sections,* Proc.Vibr.Probl., **3**, 4
12. MAUGIN G.A., 1988, *Continuum mechanics of electromagnetic solids,* North-Holland Series in Applied Mathematics and Mechanics, **33**, Elsevier, Amsterdam
13. RADOVINSKII A.L., 1987, *On a spectrum structure of the problems of magnetoelasticity of thin plates,* (in Russian), Izv.AN SSSR, MTT, **1**
14. ROGULA D., 1982, *Pola sprzężone i nieklasyczne ośrodki ciągłe,* (in Polish), Mech.Teor.i Stos., **20**, 1/2
15. RUDNICKI M., 1984, *A modified hypothesis of magnetoelasticity of plates,* Bull. Polon.Acad.Sci., Techn.Sci., **32**, 3/4
16. RUDNICKI M., *On the electromagnetic field within magnetoelastic plate,* accepted for print in the Int.J.Engn.Sci.
17. RYMARZ C., 1986, *Fale w ośrodkach sprężystych i niesprężystych,* (in Polish), Mechanika techniczna, vol 3, Drgania i fale (ed. S.Kaliski), PWN, Warsaw

Teoria płyt magnetosprężystych

Streszczenie

Praca dotyczy jednorodnych izotropowych liniowo sprężystych i nie idealnie przewodzących płyt o stałej grubości. Uwzględniono początkowe pole magnetostatyczne niejednorodne w kierunkach stycznych. Punktem wyjścia są związki określające zmienność przemieszczeń na grubości płyty oraz równania ruchu teorii płyt Kirchhoffa uwzględniające siłę Lorentza, uzupełnione o trójwymiarowe równania elektrodynamiki. Wzajemnie sprzężone równania dwuwymiarowe wyprowadzono na mocy częściowo nowych założeń elektromagnetycznych. Końcowy układ równań uproszczono do postaci niewiele odbiegającej od równań opartych na zmodyfikowanej hipotezie magnetosprężystości cial cienkich.

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