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INTERNAL POINT TORQUE IN A TWO-PHASE MATERIAL. INTERFACE CRACK AND INCLUSION PROBLEMS

BOGDAN ROGOWSKI

Technical University of Łódź

The Green's function for two perfectly-bonded elastic, orthotropic, homogeneous half spaces with different elastic constants under concentrated internal torque on the interface parallel to the two-phase boundary is obtained by means of the Hankel transform.
Use is made of the obtained solution to the analysis of the interface crack and inclusion problems. The dual-integral equations of both problems are solved exactly to generate results of engineering interest: the stress intensity factor of Mode III at the crack tip and the rotation of a rigid disk inclusion.
Numerical calculations are carried out and presented graphically to illustrate results of engineering interest.

1. Introduction

The aim of this paper is to show a fundamental, exact solution to axisymmetric torsional problems of dissimilar elastic, orthotropic solids. The fundamental solution may be called the Green's function for axisymmetric body force problem of dissimilar elastic solids. The application of obtained solution to the interface crack and inclusion problems is presented.

Various numerical methods of solution were recently developed for engineering problems. Most of these methods, such as boundary element methods, charge simulation methods, eigenstrain methods, body force methods and so on, apply fundamental solutions to formulate integral equations for a problem. In order to obtain more accurate results efficiently, Green's functions are used by many investigators, because the Green's functions completely satisfy part of the boundary conditions of the problem.

Interfaces between materials are ubiquitous both in nature and in technological applications. This, coupled with the fact that the overall performance of a given component frequently is governed by the behaviour of such interfaces, has caused researchers to devote a great deal of attention to the study of interface failure mechanics.

Solutions to specific problems of cracks lying along bimaterial interfaces of isotropic media have been given by Cherepanov (1962), England (1965), Erdogan (1965), Rice and Sih (1965). More recently, Ting (1986) has presented a framework of determining the degree of singularity and the nature of the asymptotic fields for the general interfacial crack between two elastic anisotropic materials. Park and Earmme (1986), Hutchinson, Mear, and Rice (1987), and Suo and Hutchinson (1988) have obtained solutions for several elastic interfacial crack problems, and Rice (1988) has reexamined elastic fracture mechanics concepts for interface cracks. The linear elasticity solution to the displacement jumps across crack faces problem predicts that overlapping of crack faces always occurs. To redress this physically objectionable behaviour, investigators have proposed various models and approaches. Comninou (1977a,b) and Comninou and Schmueser (1978) reformulated the linear elasticity boundary value problem to allow a zone of contact to develop at the crack tip. Achenbach et al. (1979) introduced a Dugdale-Barenblatt strip yield zone at the crack tip which eliminated crack face overlapping as well as stress singularities altogether.

In the theory of micromechanics of materials presented by Mura (1982), when an eigenstrain was prescribed in a finite region in a homogeneous material, the finite region was called an inclusion. The elastic moduli of the inclusion are assumed to be the same as the matrix. If the finite region has elastic moduli different from those of the matrix, the region is called an inhomogeneity.

Defects raise stress concentrations. The theory of inclusions has been successfully applied to composite materials including fiber, precipitate, and martensite problems. A review of inclusion problems has been given by Mura (1982) and (1988). However, many results are not expressed in explicit form but are in the form of numerical solutions.

In this paper exact solutions are presented in closed forms for the stress intensity factor of Mode III around the crack contour and for relationship among the couple and rotation of an inclusion.

2. Basic equations

In this paper we use cylindrical coordinates and denote them by (r,ϑ,z). The stress-displacement relations for axisymmetric torsion problem are

$$\begin{aligned} \sigma_{r\vartheta} &= G_r\Big(\frac{\partial v}{\partial r}-\frac{v}{r}\Big)\\ \sigma_{\vartheta z} &= G_z\frac{\partial v}{\partial z} \end{aligned} \tag{2.1}$$

where $\sigma_{r\vartheta}$ and $\sigma_{\vartheta z}$ are the stress components, v is the displacement, G_r and G_z denote the shear moduli of the material where the subscripts r, z correspond to the directions chosen to coincide with the axes of materials orthotropy.

Substituting Eqs (2.1) into the equation of equilibrium

$$\frac{\partial \sigma_{r\vartheta}}{\partial r} + \frac{\partial \sigma_{\vartheta z}}{\partial z} + \frac{2}{r}\sigma_{r\vartheta} = 0 \tag{2.2}$$

one obtains

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{1}{\mu^2}\frac{\partial^2 v}{\partial z^2} = 0 \tag{2.3}$$

where μ^2 represents the ratio of elastic constants

$$\mu^2 = \frac{G_r}{G_z} \tag{2.4}$$

and $\mu = 1$ corresponds to an isotropic elastic solid.

3. The problem formulation

Two perfectly-bonded half-spaces 1 and 2: $0 \le r < \infty$, $(-1)^i z \le 0$ $(i = 1,2)$ are loaded by means of a concentrated point torque, which acts on the internal surface $z = z' \ge 0$.

The continuity and discontinuity conditions are

$$\begin{aligned}
&v^1(r,0) - v^2(r,0) = 0\\
&\sigma^1_{z\vartheta}(r,0) - \sigma^2_{z\vartheta}(r,0) = 0\\
&v^1(r,z'+0) - v^1(r,z'-0) = 0\\
&\sigma^1_{z\vartheta}(r,z'+0) - \sigma^1_{z\vartheta}(r,z'-0) = -\lim_{a\to 0}\frac{T\delta(r-a)}{2\pi r^2}
\end{aligned} \tag{3.1}$$

where $\delta(r-a)$ is the Dirac delta function and $v(r,z) = v^i(r,z)$, $\sigma_{z\vartheta}(r,z) = \sigma^i_{z\vartheta}(r,z)$ $(i = 1,2)$.

4. Application of integral transform

We use the Hankel transform defined as follows

$$\bar{v}(\xi,z) = \int_0^\infty v(r,z) r J_1(r\xi) dr \tag{4.1}$$

where ξ is the Hankel parameter $(r \to \xi)$ and $J_1(r\xi)$ is the Bessel function of the first kind and order one. Applying the foregoing transform to Eq (2.3) and Eqs (3.1) we obtain

$$\frac{\partial^2 \bar{v}^i}{\partial z^2} - \mu_i^2 \xi^2 \bar{v}^i = 0 \qquad i = 1,2 \tag{4.2}$$

$$\bar{v}^1(\xi,0) - \bar{v}^2(\xi,0) = 0 \tag{4.3}$$

$$G_z^1 \frac{\partial \bar{v}^1}{\partial z}\Big|_{z=0} - G_z^2 \frac{\partial \bar{v}^2}{\partial z}\Big|_{z=0} = 0 \tag{4.4}$$

$$\bar{v}^1(\xi, z'+0) - \bar{v}^1(\xi, z'-0) = 0 \tag{4.5}$$

$$G_z^1 \frac{\partial \bar{v}^1}{\partial z}\Big|_{z=z'+0} - G_z^1 \frac{\partial \bar{v}^1}{\partial z}\Big|_{z=z'-0} = -\frac{T}{4\pi}\xi \tag{4.6}$$

with $i = 1,2$ referring to bodies 1 and 2, respectively.

Applying the transform (4.1) to Eqs (2.1) we have

$$\begin{aligned} \hat{\sigma}^i_{r\vartheta} &= -G^i_r \xi \bar{v}^i \\ \bar{\sigma}^i_{\vartheta z} &= G^i_z \frac{\partial \bar{v}^i}{\partial z} \end{aligned} \tag{4.7}$$

where the symbol ^ over the stress $\sigma^i_{r\vartheta}$ denotes the second-order Hankel transform associated with $J_2(r\xi)$ function.

The elastic constants and other quantities of body i are denoted by corresponding superscripts $i = 1$ or $i = 2$.

The solutions of Eq (4.2) which satisfy the regularity conditions at infinity are

$$\begin{aligned} \bar{v}^1(\xi,z) &= A(\xi)e^{\xi\mu_1 z} + B(\xi)e^{-\xi\mu_1 z} & 0 \le z \le z' \\ \bar{v}^1(\xi,z) &= C(\xi)e^{-\xi\mu_1 z} & z \ge z' \\ \bar{v}^2(\xi,z) &= D(\xi)e^{\xi\mu_2 z} & z \le 0 \end{aligned} \tag{4.8}$$

The conditions (4.3) ÷ (4.6) yield

$$\begin{aligned} A(\xi) &= \frac{T}{8\pi G_z^1 \mu_1} e^{-\xi\mu_1 z'} & B(\xi) &= \kappa A(\xi) \\ C(\xi) &= \left(\kappa + e^{2\xi\mu_1 z'}\right) A(\xi) & D(\xi) &= (1+\kappa)A(\xi) \end{aligned} \tag{4.9}$$

where

$$\kappa = \frac{1-g_2}{1+g_2} \qquad g_2 = \frac{G_z^2 \mu_2}{G_z^1 \mu_1} \tag{4.10}$$

Here g_2 represents the ratio of geometric mean of elastic constants of materials 2 and 1.

The corresponding transforms of displacement and stress are

$$\begin{aligned}
\bar{v}^1(\xi,z) &= \frac{T}{8\pi G_z^1\mu_1}\left(e^{-\xi z_1}+\kappa e^{-\xi z_2}\right)\\
\bar{\sigma}^1_{\vartheta z}(\xi,z) &= -\frac{T}{8\pi}\xi\left(\alpha e^{-\xi z_1}+\kappa e^{-\xi z_2}\right)\\
\hat{\sigma}^1_{r\vartheta}(\xi,z) &= -\frac{T\mu_1}{8\pi}\xi\left(e^{-\xi z_1}+\kappa e^{-\xi z_2}\right) \qquad z\geq 0\\
\bar{v}^2(\xi,z) &= \frac{T}{4\pi G_z^1\mu_1}\frac{1}{1+g_2}e^{-\xi z_3}\\
\bar{\sigma}^2_{\vartheta z}(\xi,z) &= \frac{T}{4\pi}\frac{g_2}{1+g_2}\xi e^{-\xi z_3}\\
\hat{\sigma}^2_{r\vartheta}(\xi,z) &= -\frac{T\mu_2}{4\pi}\frac{g_2}{1+g_2}\xi e^{-\xi z_3} \qquad z\leq 0
\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}
&\alpha = 1 \quad \text{for} \quad z > z' \quad \text{and} \quad \alpha = -1 \quad \text{for} \quad z < z'\\
&z_1 = \mu_1|z-z'| \qquad z_2 = \mu_1(z+z')\\
&z_3 = \mu_1 z' - \mu_2 z \qquad (-1)^i z \leq 0
\end{aligned} \tag{4.12}$$

5. Determination of Green's functions

The Hankel transform is its own inverse

$$v(r,z) = \int_0^\infty \bar{v}(\xi,z)\xi J_1(r\xi)d\xi \tag{5.1}$$

Applying the integrals

$$\begin{aligned}
&\int_0^\infty \xi e^{-\xi z_k} J_1(r\xi)d\xi = \frac{r}{R_k^3}\\
&\int_0^\infty \xi^2 e^{-\xi z_k} J_1(r\xi)d\xi = \frac{3rz_k}{R_k^5}\\
&\int_0^\infty \xi^2 e^{-\xi z_k} J_2(r\xi)d\xi = \frac{3r^2}{R_k^5}\\
&R_k^2 = z_k^2 + r^2 \qquad k = 1,2,3
\end{aligned} \tag{5.2}$$

to invert Eq (4.11) give the original solutions

$$
\begin{aligned}
v^1(r,z) &= \frac{T}{8\pi G_z^1\mu_1} r\Big(\frac{1}{R_1^3}+\kappa\frac{1}{R_2^3}\Big)\\
\sigma_{\vartheta z}^1(r,z) &= -\frac{3T\mu_1}{8\pi} r\Big(\frac{z-z'}{R_1^5}+\kappa\frac{z+z'}{R_2^5}\Big)\\
\sigma_{r\vartheta}^1(r,z) &= -\frac{3T\mu_1}{8\pi} r^2\Big(\frac{1}{R_1^5}+\kappa\frac{1}{R_2^5}\Big) \qquad z\ge 0\\
v^2(r,z) &= \frac{T}{4\pi G_z^1\mu_1}\frac{1}{1+g_2}\frac{r}{R_3^3}\\
\sigma_{\vartheta z}^2(r,z) &= \frac{3T}{4\pi}\frac{g_2}{1+g_2} r\frac{\mu_1 z'-\mu_2 z}{R_3^5}\\
\sigma_{r\vartheta}^2(r,z) &= -\frac{3T\mu_2}{4\pi}\frac{g_2}{1+g_2}\frac{r^2}{R_3^5} \qquad z\le 0
\end{aligned}
\tag{5.3}
$$

where

$$
\begin{aligned}
&R_1^2 = \mu_1^2(z-z')^2+r^2 \qquad R_2^2=\mu_1^2(z+z')^2+r^2 \qquad z\ge 0\\
&R_3^2 = (\mu_1 z' - \mu_2 z)^2 + r^2 \qquad z\le 0
\end{aligned}
\tag{5.4}
$$

In the special cases we observe the following significant results.

(i) Two-phase interface torque $(z'=0)$

$$
\begin{aligned}
v^i(r,z) &= \frac{T}{4\pi G_z^1\mu_1}\frac{1}{1+g_2}\frac{r}{R_i^3}\\
\sigma_{\vartheta z}^i(r,z) &= -\frac{3T\mu_i}{4\pi}\frac{1}{1+g_{i\pm 1}}\frac{rz}{R_i^5}\\
\sigma_{r\vartheta}^i(r,z) &= -\frac{3T\mu_i}{4\pi}\frac{1}{1+g_{i\pm 1}}\frac{r^2}{R_i^5} \qquad (-1)^i z\le 0 \quad i=1,2\\
R_i^2 &= \mu_i^2 z^2 + r^2 \qquad g_1 = g_2^{-1}
\end{aligned}
\tag{5.5}
$$

(ii) Half-space with a free surface $(g_2=0)$

$$
\begin{aligned}
v^1(r,z) &= \frac{T}{8\pi G_z^1\mu_1} r\Big(\frac{1}{R_1^3}+\frac{1}{R_2^3}\Big)\\
\sigma_{\vartheta z}^1(r,z) &= -\frac{3T\mu_1}{8\pi} r\Big(\frac{z-z'}{R_1^5}+\frac{z+z'}{R_2^5}\Big)\\
\sigma_{r\vartheta}^1(r,z) &= -\frac{3T\mu_1}{8\pi} r^2\Big(\frac{1}{R_1^5}+\frac{1}{R_2^5}\Big) \qquad z\ge 0
\end{aligned}
\tag{5.6}
$$

(iii) Half-space with a rigidly fixed surface $(g_2 \to \infty)$

$$\begin{aligned} v^1(r,z) &= \frac{T}{8\pi G_z^1 \mu_1} r\Big(\frac{1}{R_1^3} - \frac{1}{R_2^3}\Big) \\ \sigma^1_{\vartheta z}(r,z) &= -\frac{3T\mu_1}{8\pi} r\Big(\frac{z-z'}{R_1^5} - \frac{z+z'}{R_2^5}\Big) \\ \sigma^1_{r\vartheta}(r,z) &= -\frac{3T\mu_1}{8\pi} r^2\Big(\frac{1}{R_1^5} - \frac{1}{R_2^5}\Big) \qquad z \geq 0 \end{aligned} \tag{5.7}$$

Green's function for a homogeneous orthotropic elastic solid is obtained for $g_2 = 1$ $(\kappa = 0)$.

The isotropic counterpart of the solution is obtained for $\mu_i = 1$.

6. Penny-shaped interface crack

Appropriate representations of the displacement v_C and stress $\sigma_{\vartheta z C}$ for cracked solid body can be given as

$$\begin{aligned} v_C^i(r,z) &= \frac{(-1)^{i-1}}{G_z^i \mu_i} \int_0^\infty E(\xi) \mathrm{e}^{(-1)^i \mu_i \xi z} J_1(\xi r) d\xi + v^i(r,z) \\ \sigma^i_{\vartheta z C}(r,z) &= -\int_0^\infty \xi E(\xi) \mathrm{e}^{(-1)^i \mu_i \xi z} J_1(\xi r) d\xi + \sigma^i_{\vartheta z}(r,z) \\ &\qquad\qquad (-1)^i z \leq 0 \end{aligned} \tag{6.1}$$

with $i = 1,2$ referring to bodies 1 and 2, respectively, and known displacement v^i and stress $\sigma^i_{\vartheta z}$; they are found from the uncracked solutions (5.3). The function $E(\xi)$ remains to be determined from the boundary condition associated with the two-phase interface $z = 0$, where the penny-shaped crack of radius a exists. The integral representations (6.1) are obtained by means of a well-known technique of a Hankel transform and by superposition, Sneddon and Lowengrub (1969).

Considering perfect bonding, matching of the displacement and the traction at the interface requires (Fig.1)

$$v_C^1(r,0) = v_C^2(r,0) \qquad r \geq a \tag{6.2}$$

$$\sigma^1_{z\vartheta C}(r,0) = \sigma^2_{z\vartheta C}(r,0) \qquad r > a \tag{6.3}$$

while over the crack domain

$$\sigma^i_{z\vartheta C}(r,0) = 0 \qquad r < a \qquad i = 1,2 \tag{6.4}$$

Eqs (6.1) and (5.3) show now that the boundary condition (6.3) is identically satisfied and that the boundary conditions (6.2) and (6.4) will be satisfied if

$$\int_0^\infty E(\xi)J_1(\xi r)d\xi = 0 \qquad r \geq a \tag{6.5}$$

$$\int_0^\infty \xi E(\xi)J_1(\xi r)d\xi = \tau(r) \qquad r < a \tag{6.6}$$

where

$$\tau(r) = \frac{3T}{4\pi}\frac{g_2}{1+g_2}\frac{\mu_1 z' r}{(\mu_1^2 z'^2 + r^2)^{5/2}} \tag{6.7}$$

is known traction for $z = 0$ corresponding to the concentrated point torque on the plane $z = z' > 0$ and $r = 0$.

The dual integral equations (6.5), (6.6) are converted to the Abel integral equation by employing the following integral representation for $E(\xi)$

$$E(\xi) = \xi^{1/2}\int_0^a x^{1/2}\psi(x)J_{3/2}(x\xi)dx \tag{6.8}$$

In this representation, the auxiliary function $\psi(x)$ is assumed to be continuous over the interval $[0, a]$ and is required to satisfy the condition

$$\lim_{x\to 0+}\left[x^{1/2}\psi(x)\right] = 0 \tag{6.9}$$

Using the Weber-Schaftheitlin integral, Watson (1966)

$$\int_0^\infty \xi^{1/2}J_{3/2}(x\xi)J_1(r\xi)d\xi = \begin{cases} 0 & 0 < x < r \\ \sqrt{\frac{2}{\pi}}\frac{r}{x^{3/2}\sqrt{x^2 - r^2}} & x > r \end{cases} \tag{6.10}$$

it is shown that the representation (6.8) of $E(\xi)$ identically satisfies Eq (6.5). Using trigonometric representation of the Bessel function $J_{3/2}(x\xi)$, integrating Eq (6.8) by parts and then substituting the resulting expression into Eq (6.6) leads to the following Abel integral equation

$$\sqrt{\frac{2}{\pi}}\int_0^r \frac{d[x\psi(x)]}{dx}\frac{dx}{\sqrt{r^2 - x^2}} = r\tau(r) \qquad r < a \tag{6.11}$$

Substituting the expression for $\tau(r)$ and applying Abel's solution method to Eq (6.11) results in the solution for $\psi(x)$

$$\psi(x) = \sqrt{\frac{2}{\pi}}\frac{1}{x}\int_0^x \frac{r^2\tau(r)}{\sqrt{x^2-r^2}}dr = \sqrt{\frac{2}{\pi}}\frac{T}{2\pi}\frac{g_2}{1+g_2}\frac{x^2}{(x^2+\mu_1^2 z'^2)^2} \tag{6.12}$$

$$z' > 0 \qquad x \in [0,a]$$

The crack shape after deformation is obtained from the following expression

$$v_C^i(r,0) = v_0(r) + \sqrt{\frac{2}{\pi}}\frac{(-1)^{i-1}}{G_z^i\mu_i}r\int_r^a \frac{\psi(x)}{x\sqrt{x^2-r^2}}dx =$$
$$= \frac{T}{4\pi}\frac{1}{G_z^1\mu_1+G_z^2\mu_2}\frac{r}{(r^2+\mu_1^2 z'^2)^{3/2}}\Big[1+\frac{2}{\pi}(-1)^{i-1}\frac{G_z^2\mu_2}{G_z^i\mu_i}\cdot$$
$$\cdot\Big(\cos^{-1}\sqrt{\frac{r^2+\mu_1^2 z'^2}{a^2+\mu_1^2 z'^2}}+\frac{\sqrt{(r^2+\mu_1^2 z'^2)(a^2-r^2)}}{a^2+\mu_1^2 z'^2}\Big)\Big] \tag{6.13}$$

$$r \le a \qquad i = 1,2$$

The jump of the displacement $v_C(r,0)$ on the crack surface $z = 0$, $r \le a$ is expressed by

$$v_C^1(r,0) - v_C^2(r,0) = \frac{T}{2\pi^2}\frac{1}{G_z^1\mu_1}\frac{r}{(r^2+\mu_1^2 z'^2)^{3/2}}\cdot$$
$$\cdot\Big(\cos^{-1}\sqrt{\frac{r^2+\mu_1^2 z'^2}{a^2+\mu_1^2 z'^2}}+\frac{\sqrt{(r^2+\mu_1^2 z'^2)(a^2-r^2)}}{a^2+\mu_1^2 z'^2}\Big) \tag{6.14}$$

The stress at $z = 0$ outside of the crack region is given by

$$\sigma_{\vartheta z C}^i(r,0) = \sqrt{\frac{2}{\pi}}\frac{1}{r}\Big[\frac{a\psi(a)}{\sqrt{r^2-a^2}} - \int_0^a \frac{d[x\psi(x)]}{dx}\frac{dx}{\sqrt{r^2-x^2}}\Big] + \tau(r) \tag{6.15}$$

$$r > a \qquad i = 1,2$$

Defining the mode III stress intensity factor as follows, Sneddon and Lowengrub (1969)

$$K_{III} = \lim_{r\to a+}\sqrt{2(r-a)}[\sigma_{\vartheta z C}^i(r,0)] \qquad r > a \tag{6.16}$$

we obtain

$$K_{III} = \frac{1}{\pi^2}Ta^{3/2}\frac{G_z^2\mu_2}{G_z^1\mu_1+G_z^2\mu_2}\frac{1}{(a^2+\mu_1^2 z'^2)^2} \tag{6.17}$$

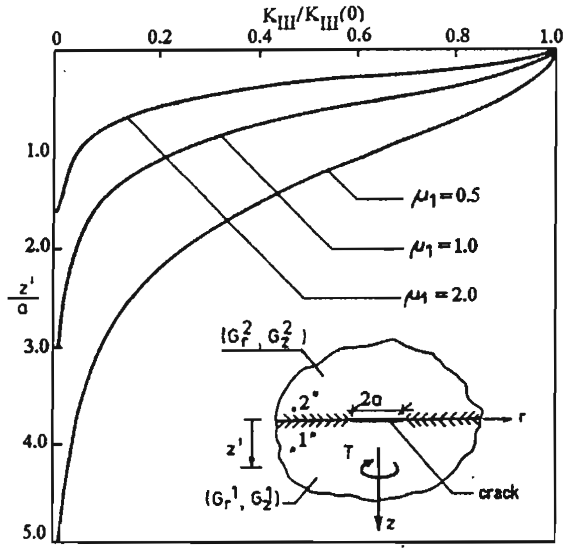

Fig. 1. Variation of the stress intensity factors ratio with z'/a for various values of $\mu_1 = \sqrt{G_r^1/G_z^1}$

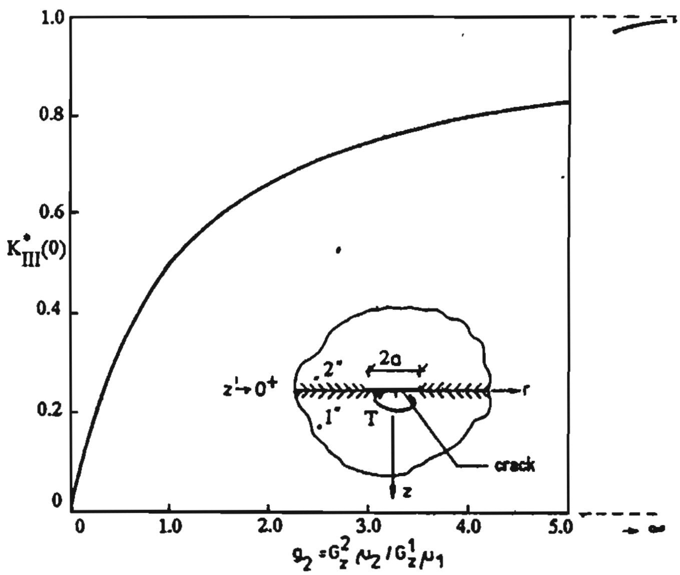

Fig. 2. Variation of dimensionless stress intensity factor $K^*_{III}(0) = \pi^2 a^{5/2} K_{III}(0)/T$ with $g_2 = G_z^2\mu_2/G_z^1\mu_1$

If the face of a penny-shaped crack is subjected to point torque $(z' \to 0^+)$ the stress intensity factor reduces to the limiting value

$$K_{III}(0) = \frac{1}{\pi^2} T a^{-5/2} \frac{G_z^2 \mu_2}{G_z^1 \mu_1 + G_z^2 \mu_2} \tag{6.18}$$

The intermediate results presented earlier are valid for $\mu_1 z' > 0$. For example the condition (6.9) is violated in the case of $\mu_1 z' \to 0$, since $\psi(x) \sim x^{-2}$ in this case. However, the basic parameter of fracture mechanics, $K_{III}(0)$, may be calculated exactly from the formula (6.18). The intermediate results for the case of loading of a crack surface may be calculated for $\mu_1 z' = \varepsilon$ $(\varepsilon \to 0^+)$.

Fig.1 shows the variations of $K_{III}(z')/K_{III}(0)$ with z'/a for various values of $\mu_1 = \sqrt{G_r^1/G_z^1}$. Fig.2 shows the variations of dimensionless quantity $K_{III}^*(0) = \pi^2 a^{5/2} K_{III}(0)/T$ with $g_2 = G_z^2\mu_2/G_z^1\mu_1$. From the figures, we can calculate the stress intensity factor K_{III} for given values of z'/a, μ_1, the bimaterial constant g_2 and the crack radius a, respectively.

7. Interface inclusion

We consider the problem of a rigid penny-shaped inclusion which is embedded in an elastic two-phase infinite medium and located on a plane of joint (Fig.3). Appropriate representations of the displacement v_I and the stress $\sigma_{\vartheta z I}$ for solid body with inclusion can be given as

$$\begin{aligned} v_I^i(r,z) &= \int_0^\infty F(\xi) e^{(-1)^i \mu_i \xi z} J_1(\xi r) d\xi + v^i(r,z) \\ \sigma_{\vartheta z I}^i(r,z) &= (-1)^i G_z^i \mu_i \int_0^\infty \xi F(\xi) e^{(-1)^i \mu_i \xi z} J_1(\xi r) d\xi + \sigma_{\vartheta z}^i(r,z) \end{aligned} \tag{7.1}$$

$$(-1)^i z \leq 0$$

with $i = 1,2$ referring to bodies 1 and 2, respectively, and known displacement v^i and stress $\sigma_{\vartheta z}^i$, which are given by Eqs (5.3). Function $F(\xi)$ remains to be determined from the boundary conditions on the interface $z = 0$, where the rigid penny-shaped inclusion with radius a exists. The integral representations (7.1) are obtained by means of a superposition and of a well-known technique of a Hankel transform. Considering perfect bonding at the interface, the continuity conditions at the plane of the inclusions are (Fig.3)

$$\sigma_{z\vartheta I}^1(r,0) = \sigma_{z\vartheta I}^2(r,0) \qquad r > a \tag{7.2}$$

$$v_l^1(r,0) = v_l^2(r,0) \qquad r \geq a \tag{7.3}$$

while over the inclusion domain

$$v_l^i(r,0) = \varphi r \qquad r \leq a \qquad i = 1,2 \tag{7.4}$$

where φ is an unknown constant twist angle of a rigid inclusion.

Eqs (7.1) and (5.3) now show that the boundary condition (7.3) is identically satisfied and that the boundary conditions (7.2) and (7.4) will be satisfied if

$$\int\limits_0^\infty \xi F(\xi) J_1(\xi r) d\xi = 0 \qquad r > a \tag{7.5}$$

$$\int\limits_0^\infty F(\xi) J_1(\xi r) d\xi = \varphi r - v_0(r) \qquad r \leq a \tag{7.6}$$

where

$$v_0(r) = \frac{T}{4\pi} \frac{1}{G_z^1 \mu_1 + G_z^2 \mu_2} \frac{r}{(r^2 + \mu_1^2 z'^2)^{3/2}} \tag{7.7}$$

is known displacement corresponding to the concentrated point torque on the plane $z = z' > 0$ and $r = 0$.

Let us now use Noble reduction (Noble (1963))

$$F(\xi) = \frac{2}{\pi} \int\limits_0^a \vartheta(x) \sin(\xi x) dx \tag{7.8}$$

and reduce Eqs (7.5) and (7.6) to the Abel integral equation for auxiliary function $\vartheta(x)$

$$\frac{2}{\pi} \int\limits_0^r \frac{x \vartheta(x)}{\sqrt{r^2 - x^2}} dx = \varphi r^2 - r v_0(r) \qquad r \leq a \tag{7.9}$$

The solution of this equation, for $v_0(r)$ given by Eq (7.7), is

$$\vartheta(x) = 2\varphi x - \frac{T}{2\pi} \frac{1}{G_z^1 \mu_1 + G_z^2 \mu_2} \frac{\mu_1 z' x}{(x^2 + \mu_1^2 z'^2)^2} \qquad x \in [0,a] \tag{7.10}$$

Apart from the displacement and stress there is one parameter which characterise the inclusion problem, namely the induced rigid rotation φ of the disc inclusion embedded in elastic medium.

Considering the equilibrium condition of the inclusion

$$M_z = -2\pi \int\limits_0^a r^2 \Big[\sigma_{z\vartheta}^1(r,0) - \sigma_{z\vartheta}^2(r,0)\Big] dr = 0 \tag{7.11}$$

and using Eqs (7.1) and (7.8), it can be shown that Eq (7.11) is equivalent to

$$\int_0^a x\vartheta(x)dx = 0 \tag{7.12}$$

Substitution of the solution (7.10) into Eq (7.12) and then integration yields the relationship between the induced rigid rotation φ of the disc inclusion and the internal torque T

$$\varphi = \frac{3T}{8\pi(G_z^1\mu_1 + G_z^2\mu_2)a^3}\left(\tan^{-1}\left(\frac{a}{\mu_1 z'}\right) - \frac{\mu_1 z' a}{a^2 + \mu_1^2 z'^2}\right) \tag{7.13}$$

For the case of homogeneous medium, the results (7.13) agree with the results given by Selvadurai (1982). If $z' \to 0$, then the solution (7.13) reduces to the classical result for a problem of Reissner-Sagoci type, Rogowski(1992), for two-phase infinite medium

$$\varphi \to \varphi_0 = \frac{3T}{16(G_z^1\mu_1 + G_z^2\mu_2)a^3} \tag{7.14}$$

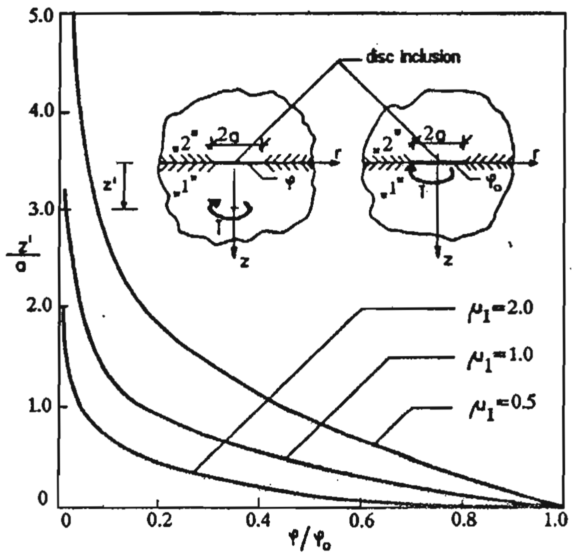

Fig. 3. Variation of the ratio φ/φ_0 with z'/a for various of $\mu_1 = \sqrt{G_r^1/G_z^1}$

Substituting the expression for φ, Eq (7.13), into the formula (7.10) results in the final solution for auxiliary function $\vartheta(x)$.

The stress and displacement fields in two-phase medium with interface inclusion may be easily obtained from Eqs (7.1) and (7.8), since the function $\vartheta(x)$ is determined analytically. The isotropic counterpart of the solution is obtained for $\mu_i = 1$, $i = 1,2$.

Fig.3 shows the variations of the ratio φ/φ_0 with z'/a for various values of $\mu_1 = \sqrt{G_r^1/G_z^1}$.

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Skupiony moment skręcający wewnątrz dwufazowego materiału. Zagadnienia szczeliny i inkluzji na powierzchni połączenia

Streszczenie

Otrzymano, za pomocą transformacji Hankela, funkcję Green'a dla dwóch idealnie połączonych sprężystych, ortotropowych, jednorodnych półprzestrzeni z różnymi sprężystymi stałymi poddanych działaniu skupionego wewnętrznego momentu skręcającego na płaszczyźnie równoległej do płaszczyzny połączenia materiałów. Wykorzystano otrzymane rozwiązanie do analizy zagadnień szczeliny i inkluzji na powierzchni połączenia materiałów. Dualne równania całkowe rozwiązano dokładnie. Rozwiązania te doprowadziły do przedstawienia w postaci prostych wzorów analitycznych interesujących, z punktu widzenia inżynierskiego, wielkości fizycznych. Są to: współczynnik intensywności naprężenia w wierzchołku szczeliny typu Mode III i obrót sztywnej, dyskowej inkluzji.

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