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NONLINEAR DESCRIPTION OF FIBRE-REINFORCED ELASTIC MATERIALS

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The constitutive relationships proposed elsewhere Jemioło et al. (1990) are here generalized and discussed in more detail. Nonlinear elasticity equations are considered and modified in such a manner as to suitably describe a composite consisting of an isotropic matrix reinforced with three orthogonal curvilinear families of fibres, each having different mechanical properties. As an alternative, a canonical form of the equations is also formulated. Isotropy, transversal isotropy and local orthotropy are considered as three important practical situations. In each case the reinforcement is described by a positive-valued second order tensor. In addition, the constitutive relationships are derived for such specific situations as plane stress, plane strain and antiplane stress. Two variants of simplified physical relations for linear elasticity are given and compared against the standard Hooke's law. Relations are given enabling material constants to be found with the use of standard tests for orthotropic material.

1. Introduction

The present paper is a sequel to a number of earlier papers Jemioło (1991a,b), Jemiolo et al. (1990), Jemiolo and Kwieciński (1991), Jemiolo, Kwieciński and Wojewódzki (1990) and (1992), in which the elasticity and plasticity problems were dealt with by using the theory of nonpolynomial representations of tensor-valued functions to formulate appropriate constitutive equations for fibre-reinforced materials. Polynomial representations were used by Spencer (1972) and (1984) for formulation of the constitutive relationships for materials equipped with either one or two families of fibres.

Primary aim of this paper is to generalize and discuss in detail one of the physical relations proposed elsewhere Jemiolo et al. (1990). The generalization consists in a different definition of the reinforcement tensor from that introduced in previous papers Jemiolo et al. (1990) and (1992). It was in Jemiolo et al. (1990)

that the general – in the framework of tensor representation theory, constitutive relationships were formulated for matrices reinforced in three orthogonal directions with rectilinear families of fibres each having exactly the same mechanical properties. Now, three orthogonal curvilinear families of fibres are dealt with, each having different mechanical characteristics.

The description is confined to nonlinearly elastic materials under small strains and to the materials of Green's type.

Depending on the type of a symmetric second-order reinforcement tensor, three cases of local orthotropy, transversal isotropy and full isotropy are considered. For each particular situation an irreducible set of invariants and generators is determined to enter the constitutive relationships. Since the obtained sets of invariants constitute so-called functional basis Boehler (1987) and, at the same time, a polynomial basis (the integrity basis), a straightforward linearization of the general equations readily leads to suitable linear relationships between the stress and the strain tensors. Next, these equations can be also linearized with respect to the reinforcement tensor to obtain the simplest expression possible. On comparing these equations with Hooke's law for orthotropic and transverselly isotropic situations, the number of independent material constants to be necessarily found by tests turns out to be smaller. It is also pointed out that under certain restrictions the proposed equations reduce to these known for a composite with averaged material constants (cf, for instance, Dąbrowski, 1989).

Relations among material constants are also established enabling the standard procedures for determining material parameters to be employed both in the case of equations linear in the strain tensor and in the case of equations bilinear in the strain and in the reinforcement tensor.

2. Description of reinforcement

An isotropic material of a matrix is assumed to be reinforced with three families of curvilinear, orthogonal and evenly spaced fibres. On the macroscale level a representative spatial element can be distinguished as a particle of the considered body. Macroscale-wise the composite is a locally orthotropic material. Full bond between fibres and the surrounding matrix is assumed to be present. Microscale description of reinforcement is made by means of a symmetric second-order tensor, Jemioło et al. (1990) and (1992), as follows

$$\mathsf{R} = k_1 R_1 e_1 \otimes e_1 + k_2 R_2 e_2 \otimes e_2 + k_3 R_3 e_3 \otimes e_3 \tag{2.1}$$

where R_i, $i = 1,2,3$ are the intensities of reinforcement embedded in the directions e_i ($R_i = A_{Ri}/A_i$, $A_i = A_{Mi} + A_{Ri}$, A_{Ri} denotes the cross-sectional area of reinforcing fibres in the direction e_i whereas A_{Mi} is the area of matrix, perpendicular to e_i and belonging to the fibre of area A_{Ri}), k_i are the ratios of corresponding anisotropy of the families of fibres (for example, $k_1 = 1, k_2 = E_2/E_1, k_3 = E_3/E_1$, where E_i denotes Young modulus of the ith family, compare a remark in subsection 5.3), e_i are base vectors of a local, orthogonal frame of reference.

When the tensor $\mathbf{R}$ is formally treated as a parametric tensor to describe a group of material symmetry, then $\mathbf{R}$ (since $R_i > 0$) characterizes a cristal of a dipyramidal class from the rhombic system, Lokhin and Sedov (1963). According to Schoenflies' designation, Penkala (1977) the cristal belongs to the class $\mathbf{D}_{2h}$. It is worth noting that, for the considered function, I-Shih Liu (1982) maintains that the dipyramidal class can be described with the use of the same parametric tensor as in the case of non-crystallic orthotropic material.

When $k_1R_1 = k_2R_2 = k_3R_3$, the composite under consideration has certain additional symmetries that escape to be described by any second-order tensor. The equivalent monocrystal belongs to the 48-faced crystal from regular system (according to Schoenflies it belongs to the class $\mathbf{O}_h$). The material symmetry group is in this case completely characterized by a parametric tensor having the form, Lokhin and Sedov (1963) or Zhang and Rychlewski (1990)

$$\mathbf{O}_h = e_1 \otimes e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 \otimes e_2 + e_3 \otimes e_3 \otimes e_3 \otimes e_3 \tag{2.2}$$

3. Nonlinear elasticity

Nonlinear constitutive equations for elasticity in Green's sense will now be established for the composite described above. As mentioned before, orthotropy, transversal isotropy and perfect isotropy will be dealt with. Suitable constraints will also be imposed to obtain relationships appropriate for plane stress, plane strain and antiplane stress states.

As well known, elastic models derived via energy formulation are insensitive to the loading path and the whole deformation process is reversible. Geometrical interpretation of a specific elastic energy W and a specific complementary energy Ω is shown in Fig.1 whereas

$$\mathrm{tr}\mathbf{TE} = W + \Omega > 0 \qquad W(0,\mathrm{R}) = 0 \qquad \Omega(0,\mathrm{R}) = 0 \tag{3.1}$$

Fig.1 clearly shows that two equivalent descriptions of the constitutive relationships are possible, namely

$$\mathbf{T} = \frac{\partial W}{\partial \mathbf{E}} \qquad \text{or} \qquad \mathbf{E} = \frac{\partial \Omega}{\partial \mathbf{T}} \tag{3.2}$$

where $\mathbf{T}$ is the Cauchy stress tensor and $\mathbf{E}$ is a small strain tensor.

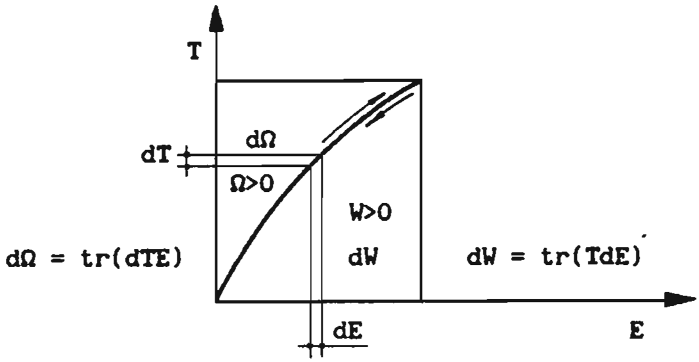

Fig. 1.

3.1. Orthotropy

When $k_1R_1 \neq k_2R_2 \neq k_3R_3$, a constitutive equation of the type (3.2), derived by Jemiolo et al. (1990), has the form

$$\mathsf{T} = \alpha_1\mathsf{I} + \alpha_2\mathsf{R} + \alpha_3\mathsf{R}^2 + 2\alpha_4\mathsf{E} + \alpha_5(\mathsf{ER} + \mathsf{RE}) + \alpha_6(\mathsf{ER}^2 + \mathsf{R}^2\mathsf{E}) + 3\alpha_7\mathsf{E} \tag{3.3}$$

where I is a unit tensor

$$\alpha_m = \frac{\partial W}{\partial I_m} \qquad \text{and} \qquad \frac{\partial \alpha_m}{\partial I_p} = \frac{\partial \alpha_p}{\partial I_m} \qquad m,p = 1,...,7$$

and, in turn

$$W = f(\mathrm{tr}\mathsf{E}, \mathrm{tr}\mathsf{ER}, \mathrm{tr}\mathsf{ER}^2, \mathrm{tr}\mathsf{E}^2, \mathrm{tr}\mathsf{E}^2\mathsf{R}, \mathrm{tr}\mathsf{E}^2\mathsf{R}^2, \mathrm{tr}\mathsf{E}^3, \mathrm{tr}\mathsf{R}^i) = f(I_n)$$
$$n = 1,...,10 \quad , \qquad i = 1,2,3$$

In the above expression for a specific elastic energy we recognize 7 variables (invariants E), 3 parameters $\mathrm{tr}\mathsf{R}^i$ (or k_iR_i) as well as other dimensional material constants which cannot be seen in $(3.3)_3$ explicitly. It should be emphasized that the invariants I_m constitute a polynomial basis for the function $(3.3)_3$ as well as its functional basis, Boehler (1979) and (1987) in which an alternative system of invariants was employed to describe a scalar-valued orthotropic function.

Eq (3.3) can be expressed in the canonical form (after Betten, cf Boehler, 1987)

$$\mathsf{T} = \mathsf{C}^{(0)}\mathsf{I} + \mathsf{C}^{(1)}\mathsf{E} + \mathsf{C}^{(2)}\mathsf{E}^2 \tag{3.4}$$

where $\mathsf{C}^{(q)}$, $q = 0,1,2$ are some fourth-order tensors depending on α_m and R and having the forms

$$\mathsf{C}^{(0)} = \alpha_1 1 + \frac{1}{2}\alpha_2\mathsf{C}_R^{(1)} + \frac{1}{2}\alpha_3\mathsf{C}_R^{(2)}$$

$$
\begin{aligned}
\mathsf{C}^{(1)} &= 2\alpha_4\mathbf{1} + \alpha_5\mathsf{C}_R^{(1)} + \alpha_6\mathsf{C}_R^{(2)} \\
\mathsf{C}^{(2)} &= 3\alpha_7\mathbf{1}
\end{aligned}
$$

In $(3.4)_2$ the following notation is used

$$
\mathbf{1} = \mathsf{I}\diamond\mathsf{I} \qquad\qquad \mathsf{C}_R^{(\alpha)} = \mathsf{I}\diamond\mathsf{R}^\alpha + \mathsf{R}^\alpha\diamond\mathsf{I} \tag{3.5}
$$

The operation $\diamond$ for two arbitrary symmetric second-order tensors A, B (introduced by Sadegh et al. (1991)), means the following

$$
(\mathsf{A}\diamond\mathsf{B})_{ijkl} = \frac{1}{4}(A_{il}B_{jk} + A_{jk}B_{il} + A_{jl}B_{ik} + A_{ik}B_{jl}) \tag{3.6}
$$

Derivation of Eq $(3.2)_2$ is similar and will not be shown here for brevity.

3.2. Transversal isotropy

When $k_1R_1 = k_2R_2 = R \geq 0$ and $R_3 > 0$, e_3 being a preferred direction, the composite in question has locally transversal isotropy. Eq (3.3) becomes much simpler since among invariants and generators the following identities apply, respectively

$$
\begin{aligned}
\mathrm{tr}\mathsf{R}^i &= 2R^i + k_3^iR_3^i \\
\mathrm{tr}\mathsf{E}^\alpha\mathsf{R}^\beta &= R^\beta\mathrm{tr}\mathsf{E}^\alpha + (k_3^\beta R_3^\beta - R^\beta)\mathrm{tr}\mathsf{E}^\alpha\mathsf{M}
\end{aligned} \tag{3.7}
$$

where $\mathsf{M} = e_3\otimes e_3$, $\alpha,\beta = 1,2$

$$
\begin{aligned}
\mathsf{R}^\alpha &= R^\alpha\mathsf{I} + (k_3^\alpha R_3^\alpha - R^\alpha)\mathsf{M} \\
\mathsf{E}\mathsf{R}^\alpha + \mathsf{R}^\alpha\mathsf{E} &= 2R^\alpha\mathsf{E} + (k_3^\alpha R_3^\alpha - R^\alpha)(\mathsf{EM} + \mathsf{ME})
\end{aligned} \tag{3.8}
$$

It results in Eq (3.3) having the form

$$
\mathsf{T} = \beta_1\mathsf{I} + \beta_2\mathsf{M} + 2\beta_3\mathsf{E} + \beta_4(\mathsf{EM} + \mathsf{ME}) + 3\beta_5\mathsf{E}^2 \tag{3.9}
$$

where

$$
\beta_n = \frac{\partial W}{\partial I_n} \qquad\qquad \frac{\partial\beta_n}{\partial I_p} = \frac{\partial\beta_p}{\partial I_n} \qquad\qquad n,p = 1,...,5
$$

and, in turn

$$
W = f(\mathrm{tr}\mathsf{E}, \mathrm{tr}\mathsf{EM}, \mathrm{tr}\mathsf{E}^2, \mathrm{tr}\mathsf{E}^2\mathsf{M}, \mathrm{tr}\mathsf{E}^3, R, k_3R_3) = f(I_m) \qquad\qquad m = 1,...,7
$$

The canonical equation (3.4) contains the following tensors $\mathbf{C}^{(q)}$

$$\mathbf{C}^{(0)} = \beta_1 1 + \frac{1}{2}\beta_2 \mathbf{C}_M^{(1)}$$
$$\mathbf{C}^{(1)} = 2\beta_3 1 + \beta_4 \mathbf{C}_M^{(1)}$$
$$\mathbf{C}^{(2)} = 3\beta_5 1$$

where

$$\mathbf{C}_M^{(1)} = \mathbf{I} \diamond \mathbf{M} + \mathbf{M} \diamond \mathbf{I}$$

From Boehler's papers (1979) and (1987) it follows that, for Cauchy's material, the constitutive equation (3.9) contains an extra generator $\mathbf{E}^2\mathbf{M} + \mathbf{M}\mathbf{E}^2$ supplied with an additional scalar-valued function β_6 whose arguments are the invariants shown in $(3.9)_3$. Expressions $(3.9)_2$ are no longer satisfied. The conclusion is that for transversally isotropic Cauchy's material the energy dissipated over a closed cycle results from the fact that the relationship $(3.9)_2$ is violated and that there exists an extra generator in the constitutive equations.

3.3. Isotropy

As pointed out in section 2, $k_1R_1 = k_2R_2 = k_3R_3 > 0$ means that the composite under consideration has the same local symmetries as in monocrystals of the 48-faced class belonging to the regular system. Suitable constitutive equation for such a material with the parametric tensor (2.2) was derived by Basista (1985).

If the principal magnitudes of the parametric tensor $\mathbf{R}$ in Eq (3.3) are equal, the description fits an isotropic material whose matrix is provided with very short segments of fibres scattered in random directions (e.g. ferroconcrete).

When $k_1R_1 = k_2R_2 = k_3R_3 = R$, the identities among invariants and generators are, respectively

$$\mathrm{tr}\mathbf{R}^i = 3R^i \qquad \mathrm{tr}\mathbf{E}^\alpha\mathbf{R}^\beta = R^\beta \mathrm{tr}\mathbf{E}^\alpha \tag{3.10}$$

$$\mathbf{R}^\alpha = R^\alpha \mathbf{I} \qquad \mathbf{E}\mathbf{R}^\alpha + \mathbf{R}^\alpha\mathbf{E} = 2R^\alpha\mathbf{E} \tag{3.11}$$

Eq (3.3) simply becomes

$$\mathbf{T} = \gamma_1\mathbf{I} + 2\gamma_2\mathbf{E} + 3\gamma_3\mathbf{E}^2 \tag{3.12}$$

where

$$\gamma_i = \frac{\partial W}{\partial I_i} \qquad \frac{\partial \gamma_i}{\partial I_j} = \frac{\partial \gamma_j}{\partial I_i}$$

and, in turn

$$W = f(\mathrm{tr}\mathbf{E}, \mathrm{tr}\mathbf{E}^2, \mathrm{tr}\mathbf{E}^3, R) = f(I_i)$$

The tensors $\mathbf{C}^{(q)}$ entering the canonical form of (3.4) are also very simple

$$\mathbf{C}^{(0)} = \gamma_1 \mathbf{1} \qquad \mathbf{C}^{(1)} = 2\gamma_3 \mathbf{1} \qquad \mathbf{C}^{(2)} = 3\gamma_5 \mathbf{1}$$

4. Plane problem in nonlinear elasticity

Let us specify the constitutive equation (3.3) for two-dimensional situations. Eq (3.9) and (3.12) can be transformed in a very similar way so none of those results will be shown here.

For simplicity and with no loss of generality the local Cartesian coordinate system will be used to express relevant tensor components.

4.1. Plane strain

The plane strain state can be neatly shown as

$$\mathbf{E} = \begin{bmatrix} \multicolumn{2}{c}{\bar{\mathbf{E}}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 \quad 0 & 0 \end{bmatrix} \tag{4.1}$$

where a bar above $\mathbf{E}$ indicates that $\mathbf{E}$ is a plane strain tensor. Under the notation (4.1) it follows from Eq (3.3) that $T_{13} = T_{23} = 0$. The constitutive equation (3.3) assumes the form

$$\begin{aligned} \bar{\mathbf{T}} &= \bar{\alpha}_1 \bar{\mathbf{I}} + \bar{\alpha}_2 \bar{\mathbf{R}} + 2\bar{\alpha}_3 \bar{\mathbf{E}} \\ T_{33} &= \bar{\alpha}_1 + \bar{\alpha}_2 k_3 R_3 \end{aligned} \tag{4.2}$$

The functions $\bar{\alpha}_i$ satisfy Eq $(3.3)_2$ and the elastic energy is

$$W = f(\mathrm{tr}\bar{\mathbf{E}}, \mathrm{tr}\bar{\mathbf{E}}\bar{\mathbf{R}}, \mathrm{tr}\bar{\mathbf{E}}^2, R_i) \tag{4.3}$$

4.2. Plane stress

The stress tensor has now the form

$$\mathbf{T} = \begin{bmatrix} \multicolumn{2}{c}{\bar{\mathbf{T}}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 \quad 0 & 0 \end{bmatrix} \tag{4.4}$$

and from Eq (3.3) it follows that $E_{13} = E_{23} = 0$. Eq (3.3) can be readily shown to have the form

$$\mathsf{T} = \alpha_1'\mathsf{I} + \alpha_2'\mathsf{R} + \alpha_3'\mathsf{R}^2 + 2\alpha_4'\mathsf{E} \tag{4.5}$$

where α_m', $m = 1, ..., 4$ are defined as in Eq (3.3). The specific energy has the form

$$W = f(\mathrm{tr}\mathsf{E}, \mathrm{tr}\mathsf{ER}, \mathrm{tr}\mathsf{ER}^2, \mathrm{tr}\mathsf{E}^2, R_i) \tag{4.6}$$

Since $T_{33} = 0$, Eq (4.5) provides an equation in E_{33} in the form

$$\alpha_1' + \alpha_2' k_3 R_3 + \alpha_3' k_3^2 R_3^2 + 2\alpha_4' E_{33} = 0 \tag{4.7}$$

Using Eq (4.7) in Eq (4.5) and employing the Cayley-Hamilton principle for plane tensors, we eventually obtain

$$\begin{aligned} \bar{\mathsf{T}} &= \Big[\alpha_1' - \frac{1}{2}\alpha_3'(\mathrm{tr}^2\bar{\mathsf{R}} - \mathrm{tr}\bar{\mathsf{R}}^2)\Big]\bar{\mathsf{I}} + (\alpha_2' + \alpha_3'\mathrm{tr}\bar{\mathsf{R}})\bar{\mathsf{R}} + 2\alpha_4'\bar{\mathsf{E}} = \\ &= \bar{\alpha}_1'\bar{\mathsf{I}} + \bar{\alpha}_2'\bar{\mathsf{R}} + 2\bar{\alpha}_3'\bar{\mathsf{E}} \end{aligned} \tag{4.8}$$

From Eq (4.7) it follows that the E_{33} component of the strain tensor is a function of the invariants $\mathrm{tr}\bar{\mathsf{E}}$, $\mathrm{tr}\bar{\mathsf{E}}\bar{\mathsf{R}}$ and $\mathrm{tr}\bar{\mathsf{E}}^2$. Therefore the specific elastic energy (4.6) depends on the very same invariants as for the plane strain (see Eq (4.3)).

4.3. Antiplane stress state

This state is characterized by the absence of the diagonal components T_{11}, T_{22}, T_{33} and, additionaly, T_{12}. This type of stress state takes place when the displacement functions, according, to the known kinematic de Saint-Venant assumption (cf e.g. Hearmon (1961)) are as follows

$$\begin{aligned} u_1 &= -\vartheta x_2 x_3 \\ u_2 &= -\vartheta x_1 x_3 \\ u_3 &= -\vartheta \varphi(x_1, x_2) \end{aligned} \tag{4.9}$$

Then the strain tensor components E_{11}, E_{22}, E_{33}, E_{12} vanish and the only nonzero strain invariants $(3.3)_3$ are $\mathrm{tr}\mathsf{E}^2$ and $\mathrm{tr}\mathsf{E}^2\mathsf{R}^\alpha$. The constitutive equation (3.3) reduces to the form

$$\mathsf{T} = 2\tilde{\alpha}_4\mathsf{E} + \tilde{\alpha}_5(\mathsf{ER} + \mathsf{RE}) + \tilde{\alpha}_6(\mathsf{ER}^2 + \mathsf{R}^2\mathsf{E}) \tag{4.10}$$

5. Linear elasticity

Particular cases of the constitutive equation (3.3), linear with respect to the strain tensor $\mathbf{E}$, will be now dealt with.

5.1. Physical relationships linear in strain tensor

Linearization of Eq (3.3) under $W(0,\mathbf{R}) = 0 \Rightarrow \mathbf{T} = \mathbf{0}$ leads to the equations with the following functions α_m

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ \text{sym.} & & a_{33} \end{bmatrix} \begin{bmatrix} \mathrm{tr}\mathbf{R} \\ \mathrm{tr}\mathbf{ER} \\ \mathrm{tr}\mathbf{ER}^2 \end{bmatrix} \tag{5.1}$$

$$\alpha_4 = a_{44} \qquad \alpha_5 = a_{55} \qquad \alpha_6 = a_{66} \qquad \alpha_7 = 0$$

Nine coefficients a_{ij}, a_{44}, a_{55}, a_{66} (their number coincides with the number of material constants for a standard orthotropic Hooke's material) may by functions of $k_i R_i$. The specific elastic energy is expressible by the formula

$$\begin{aligned} W &= \frac{1}{2}\mathrm{tr}\mathbf{TE} = \frac{1}{2}(a_{11}\mathrm{tr}^2\mathbf{E} + a_{22}\mathrm{tr}^2\mathbf{ER} + a_{33}\mathrm{tr}^2\mathbf{ER}^2) + a_{12}\mathrm{tr}\mathbf{E}\mathrm{tr}\mathbf{ER} + \\ &+ a_{13}\mathrm{tr}\mathbf{E}\mathrm{tr}\mathbf{ER}^2 + a_{23}\mathrm{tr}\mathbf{ER}\mathrm{tr}\mathbf{ER}^2 + a_{44}\mathrm{tr}\mathbf{E}^2 + a_{55}\mathrm{tr}\mathbf{E}^2\mathbf{R} + a_{66}\mathrm{tr}\mathbf{E}^2\mathbf{R}^2 \end{aligned} \tag{5.2}$$

Allowing for the identities (3.7), (3.8) in Eq (3.3) and Eq (5.1) or linearizing Eq (3.9) with respect to $\mathbf{E}$ yields a linear equation for a locally transversal isotropic material with the functions β_n as follows

$$\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ \text{sym.} & b_{22} \end{bmatrix} \begin{bmatrix} \mathrm{tr}\mathbf{E} \\ \mathrm{tr}\mathbf{EM} \end{bmatrix} \tag{5.3}$$

$$\beta_3 = b_{33} \qquad \beta_4 = b_{44} \qquad \beta_5 = 0$$

Five different $k_i R_i$ – dependent coefficients are present in Eq (5.3), similarly as for the transversally isotropic Hooke's material.

Full isotropy takes place when in Eq (3.12) the following functions are inserted:

$$\gamma_1 = c_1\mathrm{tr}\mathbf{E} \qquad \gamma_2 = c_2 \qquad \gamma_3 = 0 \tag{5.4}$$

Using above functions in suitable tensors $\mathbf{C}^{(q)}$, canonical equations can be arrived at for a given type of material symmetry.

5.2. Bilinear physical relationships with respect to the strain and the reinforcement tensors

Linearizing Eq (3.3) with the help of the functions (5.1) of $\mathbf{R}$, we get

$$
\begin{aligned}
&\alpha_1 = a_{11}\mathrm{tr}\mathbf{E} + a_{12}\mathrm{tr}\mathbf{ER} \\
&\alpha_2 = a_{12}\mathrm{tr}\mathbf{E} \qquad \alpha_3 = 0 \qquad \alpha_4 = a_{44} \\
&\alpha_5 = a_{55} \qquad \alpha_6 = 0 \qquad \alpha_7 = 0
\end{aligned}
\tag{5.5}
$$

Very simple equation (3.3) with the functions (5.5) was also derived by Jemiolo et al. (1990) in a different manner. It was a fourth-order tensor-valued function linear in the tensor $\mathbf{R}$. Constants a_{11} and a_{44} (see Jemiolo et al. (1990)) were interpreted as the Lame's constants λ_M, μ_M for the matrix. It should be stressed here that such an interpretation is true only in the specific case when mechanical properties of the matrix suffer no changes during the formation of the composite.

Under the previous assumption plus a similar assumption of constancy of the mechanical properties of fibres, the values of constants in Eq (5.5) can be predicted. In the case of $\mathbf{R} = \mathbf{0}$, Eq (3.3) with the functions (5.5) leads to the Hooke's law for a matrix characterized by the constants

$$
a_{11} = \lambda_M \qquad a_{44} = \mu_M \tag{5.6}
$$

When $\mathbf{R} = \mathbf{I}$, Eq (3.3) together with Eq (5.5) results in Hooke's law for the reiforcing material provided

$$
a_{11} + 2a_{12} = \lambda_R \qquad a_{44} + a_{55} = \mu_R \tag{5.7}
$$

Accounting Eq (5.6) and (5.7) in Eq (5.5) yields a specific form of Eq (3.3), namely

$$
\begin{aligned}
\mathbf{T} &= \Big(\lambda_M \mathrm{tr}\mathbf{E} + \frac{\lambda_R - \lambda_M}{2}\mathrm{tr}\mathbf{ER}\Big)\mathbf{I} + \\
&+ \frac{\lambda_R - \lambda_M}{2}(\mathrm{tr}\mathbf{E})\mathbf{R} + 2\mu_M\mathbf{E} + (\mu_R - \mu_M)(\mathbf{ER} + \mathbf{RE})
\end{aligned}
\tag{5.8}
$$

It is worth emphasizing that the physical equation (3.3) with functions (5.5) as well as a simplified relation (5.8) have been derived under very restrictive assumptions and can only be treated within the framework of linear elasticity as an approximation of Eq (3.3) with functions (5.1). The above reasoning clearly shows the physical sense of the constants entering Eq (5.5).

On allowing in the constitutive equation (5.8) the assumptions made in the subsection (3.3), i.e. $\mathbf{R} = R\mathbf{I} = A_R\mathbf{I}/A = V_R\mathbf{I}/V$, the following equation is obtained

$$
\mathbf{T} = \lambda_{av}(\mathrm{tr}\mathbf{E})\mathbf{I} + 2\mu_{av}\mathbf{E} \tag{5.9}
$$

where

$$\lambda_{av} = \frac{\lambda_M V_M + \lambda_R V_R}{V} \qquad \mu_{av} = \frac{\mu_M V_M + \mu_R V_R}{V}$$

and, in turn, $V = V_M + V_R$, V_M and V_R being the volumes of the matrix and the reinforcement, respectively. The constants $(5.9)_2$ are identical with those that would be obtained from the law of mixtures by using Voight's averaging process (cf for example Dąbrowski, 1989) for an isotropic matrix with spheroidal isotropic inclusions.

5.3. Comparison with standard Hooke's law

A matrix form of Hooke's law for orthotropic material is the following

$$\mathsf{T}_{6\times1} = \mathsf{C}_{6\times6}\mathsf{E}_{6\times1} \tag{5.10}$$

where

$$\mathsf{C}_{6\times6} = \begin{bmatrix} e_1 & f_3 & f_2 & 0 & 0 & 0 \\ & e_2 & f_1 & 0 & 0 & 0 \\ & & e_3 & 0 & 0 & 0 \\ & & & g_3 & 0 & 0 \\ & & & & g_2 & 0 \\ \text{sym.} & & & & & g_1 \end{bmatrix} \qquad \mathsf{T}_{6\times1} = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{12} \\ T_{23} \\ T_{13} \end{bmatrix} \qquad \mathsf{E}_{6\times1} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{23} \\ 2E_{13} \end{bmatrix}$$

Nine elasticity constants e_i, f_i, g_i depend on the constants a_{ij}, a_{44}, a_{55}, a_{66} and k_iR_i in the following maner

$$\begin{aligned} e_i &= (a_{11} + 2a_{44}) + 2k_iR_i(a_{12} + a_{55}) + k_i^2R_i^2\Big[a_{22} + 2(a_{13} + a_{66})\Big] + \\ &+ 2a_{23}k^3R^3 + a_{33}k^4R^4 \\ f_i &= a_{11} + a_{12}(k_jR_j + k_kR_k) + k_jR_jk_kR_k\Big[a_{22} + a_{23}(k_jR_j + k_kR_k) + \\ &+ a_{33}k_jR_jk_kR_k)\Big] + a_{13}(k_j^2R_j^2 + k_k^2R_k^2) \\ g_i &= a_{44} + \frac{1}{2}a_{55}(k_jR_j + k_kR_k) + \frac{1}{2}a_{66}(k_j^2R_j^2 + k_k^2R_k^2) \end{aligned} \tag{5.11}$$

(no summation over $i,j,k = 1,2,3$), $(i,j,k) = (1,2,3); (2,3,1); (3,1,2)$.

The constants e_i, f_i, g_i can be determined from standard tests for an orthotropic material (cf, for instance, Hearmon, 1961). Since the constants a_{11} and a_{44} are not modified by the expressions k_iR_i, it seems reasonable to interpret them as the Lame's constants for the matrix. Formally, the constants k_i, $i = 2,3$ and the remaining a_{ij}, a_{55}, a_{66} are calculated from nine equations. Thus the obtained results for k_i can confirm that their definition in Eq (2.1) is correct. It is evident that it is the experimental verification only that can supply an answer how rational the simplification of the function (5.1) to become (5.5) – or a neglect of certain expression in (5.11) – really is.

In the case of the function (5.5), although nine different constants appear in the standard Hooke's law, they actually depend on only four constants a_{11}, a_{12}, a_{44}, a_{55}. Similar relationship is valid for transversally isotropic material. It is now five magnitudes of standard Hooke's law that depend on the constants listed above. Substituting Eq (5.5) into Eq (5.11) and making use of Eq (3.7), we obtain

$$\begin{aligned}
e_1 &= e_2 = (a_{11} + 2a_{44}) + 2R(a_{12} + a_{55}) \\
e_3 &= (a_{11} + 2a_{44}) + 2k_3R_3(a_{12} + a_{55}) \\
f_1 &= f_2 = a_{11} + a_{12}(R + k_3R_3) \\
f_3 &= a_{11} + a_{12}R \\
g_1 &= g_2 = a_{44} + a_{55}(R + k_3R_3) \\
g_3 &= \frac{e_1 - f_3}{2} = a_{44} + a_{55}R
\end{aligned} \tag{5.12}$$

For the perfect isotropy it is obviously only two constants that enter the picture

$$\begin{aligned}
e_1 &= e_2 = e_3 = (a_{11} + 2a_{44}) + 2R(a_{12} + a_{55}) \\
f_1 &= f_2 = f_3 = a_{11} + 2a_{12}R \\
g_1 &= g_2 = g_3 = \frac{e_1 - f_1}{2} = a_{44} + a_{55}R
\end{aligned} \tag{5.13}$$

The above survey makes it possible to select, for the considered composite, the known solutions of boundary value problems for orthotropic, transversally isotropic and perfectly isotropic bodies.

As far as practical applications are concerned, an inverse of Eq (5.10) is necessary

$$\mathbf{E}_{6\times1} = \mathbf{C}^{-1}_{6\times6}\mathbf{T}_{6\times1} \tag{5.14}$$

where

$$\mathbf{C}^{-1}_{6\times6} = \begin{bmatrix} p_1 & r_3 & r_2 & 0 & 0 & 0 \\ & p_2 & r_1 & 0 & 0 & 0 \\ & & p_3 & 0 & 0 & 0 \\ & & & s_3 & 0 & 0 \\ & & & & s_2 & 0 \\ \text{sym.} & & & & & s_1 \end{bmatrix}.$$

and, in turn

$$dp_i = e_je_k - f_i^2 \qquad dr_i = f_jf_k - e_if_i \qquad \text{(no summation over } i\text{)}$$

$$s_i = \frac{1}{g_i} \qquad d = e_1e_2e_3 + f_1f_2f_2 - e_1f_1^2 - e_2f_2^2 - e_3f_3^2$$
$$(i,j,k) = (1,2,3);\ (2,3,1);\ (3,1,2)$$

Substituting Eq (5.11) into Eq $(5.14)_3$ we obtain relationships between the constants in the inverted Hooke's law and the constants a_{ij}, a_{44}, a_{55} and a_{66}.

5.4. Relationships among material constants

The constants entering Eq $(5.14)_2$ can be determined with the use of standart tests on uniaxial and biaxial compression and on pure torsion. Following Hayes' paper (1972), the following relations can be established

— generalized Young moduli for an arbitrary direction

$$\begin{aligned}\frac{1}{E(\mathbf{n})} &= p_1n_1^4 + p_2n_2^4 + p_3n_3^4 + 2(r_1+2s_3)n_2^2n_3^2 + \\ &+ 2(r_2+2s_2)n_1^2n_3^2 + 2(r_3+2s_1)n_1^2n_2^2\end{aligned} \tag{5.15}$$

where $\mathbf{n}$ is an arbitrary versor with the componets n_i,

— generalized Poisson ratios for an arbitrary plane

$$\begin{aligned}\nu(\mathbf{m},\mathbf{n}) &= -E(\mathbf{n})(p_1m_1^4 + p_2m_2^4 + p_3m_3^4 + 2r_1m_2^2m_3^2 + 2r_2m_1^2m_3^2 + \\ &+ 2r_3m_1^2m_2^2 + 4s_1m_1m_2n_1n_2 + 4s_2m_1m_3n_1n_3 + 4s_3m_2m_3n_2n_3)\end{aligned} \tag{5.16}$$

where $\mathbf{m}$ is a versor with the components m_i,

— generalized Kirchhoff moduli for an arbitrary plane

$$\begin{aligned}\frac{1}{\mu(\mathbf{m},\mathbf{n})} &= 4\Big[p_1n_1^2m_1^2 + p_2n_2^2m_2^2 + p_3n_3^2m_3^2 + 2r_1m_2m_3n_2n_3 + 2r_2m_1m_3n_1n_3 + \\ &+ 2r_3m_1m_2n_1n_2 + s_1(n_2m_3+m_2n_3) + s_2(n_1m_3+m_1n_3) + \\ &+ s_3(n_1m_2+m_1n_2)\Big]\end{aligned} \tag{5.17}$$

Next procedure to follow is simple enough: by proper choice of the directions $\mathbf{n}$ as well as $\mathbf{n}$ and $\mathbf{m}$ we consecutively determine from standard tests $E(\mathbf{n})$, $\nu(\mathbf{m},\mathbf{n})$, $\mu(\mathbf{m},\mathbf{n})$ and, making use of Eqs (5.15) ÷ (5.17), calculate the constants $(5.14)_2$. Use of Eqs $(5.14)_3$ and (5.11) leads to the constants of Eq (5.1) – or even simpler by using the inverted equation $(3.2)_2$. For example, performing an uniaxial tensile test in the direction $\mathbf{n}$ we calculate $E(\mathbf{n})$ and $\nu(\mathbf{m},\mathbf{n})$, since $\mathsf{T} = T\mathbf{n}\otimes\mathbf{n}$, where T stands for the tensile stress

$$E(\mathbf{n}) = \frac{T}{E} \qquad \nu(\mathbf{m},\mathbf{n}) = -\frac{\mathrm{tr}(\mathsf{E}\mathbf{m}\otimes\mathbf{m})}{E} \tag{5.18}$$

where

$$E = \mathrm{tr}(\mathsf{E}\boldsymbol{n} \otimes \boldsymbol{n})$$

Simple torsion test suplies $\mu(\boldsymbol{m}, \boldsymbol{n})$

$$\mu(\boldsymbol{m}, \boldsymbol{n}) = \frac{S}{2}\mathrm{tr}(\mathsf{E}\boldsymbol{n} \otimes \boldsymbol{m}) \tag{5.19}$$

where S is a shearing stress and $\mathsf{T} = S(\boldsymbol{n} \otimes \boldsymbol{m} + \boldsymbol{m} \otimes \boldsymbol{n})$, $\boldsymbol{n} \cdot \boldsymbol{m} = 0$.

The above procedure makes it possible to assess quantitatively an influence of a nonlinear reinforcement tensor R in the expression (5.11). In addition, an answer will be given on the applicability of the simplified constitutive equations with functions (5.5).

6. Unconstraint torsion of an elliptic bar

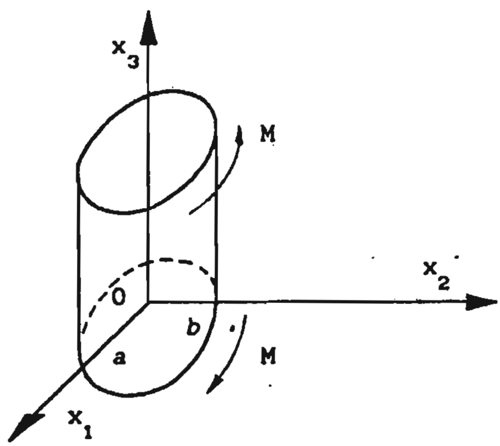

Fig. 2.

This type of test is frequently used to establish Kirchhoff modulus for a material under investigation. Consider a free torsion of a prismatic bar with an elliptical cross-section (Fig.2). Let the longitudinal x_3-axis coincide with one of the reinforcement families. M, a, b denote a twisting moment, major and minor semiaxes of the elliptical cross-section, respectively. Solution to this problem for an orthotropic bar can be found in Hearmon (1961). Remembering Eq (3.3) with the functions (5.1) and assuming the displacement function in the form (4.9), the constitutive equation (4.10) takes the form

$$\mathsf{T} = 2a_{44}\mathsf{E} + a_{55}(\mathsf{ER} + \mathsf{RE}) + a_{66}(\mathsf{ER}^2 + \mathsf{R}^2\mathsf{E}) \tag{6.1}$$

Adopting the commonly known results, we arrive at the following expressions for a unit angle of twist ϑ and a deplanation function

$$\vartheta = M\frac{a^2g_2 + b^2g_1}{2\pi a^3b^3g_1g_2} \qquad \varphi = \frac{b^2g_1 - a^2g_2}{a^2g_2 + b^2g_1}x_1x_2 \tag{6.2}$$

where g_1 and g_2 denote material constants defined by Eq $(5.11)_3$. The cross-section of the bar remains plane only if its shape is circular, $a = b$, and the material is transversely isotropic with a preferred direction e_3, $k_1R_1 = k_2R_2$. Using Eq (6.2) in Eq (4.9), remembering Eq (6.1) and standard elasticity relationships for strain, we finally arrive at the strains

$$E_{13} = -\frac{M}{\pi ab^3g_1}x_2 \qquad E_{23} = \frac{M}{\pi a^3bg_2}x_1 \tag{6.3}$$

and the stresses

$$T_{13} = -\frac{2Mg_2}{\pi ab^3g_1}x_2 \qquad T_{23} = \frac{2Mg_1}{\pi a^3bg_2}x_1 \tag{6.4}$$

Constants a_{55} and a_{66} can be calculated from the expressions (6.3) and $(5.11)_3$. Assuming, for instance, $a = b$, $x_1 = x_2 = a$, $a_{44} = \mu_M$, we obtain

$$\begin{aligned} a_{55} &= -\frac{2}{\bar{r}_2\bar{s}_1 - \bar{r}_1\bar{s}_2}\Big[\frac{M}{\pi a^3}\Big(\frac{\bar{r}_2}{E_{13}^0} + \frac{\bar{s}_2}{E_{23}^0}\Big) + \mu_M(k_1^2R_1^2 - k_2^2R_2^2)\Big] \\ a_{66} &= \frac{2}{\bar{r}_2\bar{s}_1 - \bar{r}_1\bar{s}_2}\Big[\frac{M}{\pi a^3}\Big(\frac{\bar{r}_1}{E_{13}^0} + \frac{\bar{s}_1}{E_{23}^0}\Big) + \mu_M(k_1R_1 - k_2R_2)\Big] \\ \bar{r}_\alpha &= k_1^\alpha R_1^\alpha + k_3^\alpha R_3^\alpha \qquad \bar{s}_\alpha = k_2^\alpha R_2^\alpha + k_3^\alpha R_3^\alpha \end{aligned} \tag{6.5}$$

where E_{13}^0 and E_{23}^0 are the deformations on the generators of a bar, known form the test.

7. Concluding remarks

Since the reinforcement tensor R, Eq (2.1), is a positive definite second-order tensor, other interpretations of the derived constitutive equations are also possible. For instance, an internal structure of bones can be represented by a symmetric second rank tensor (the so-called fabric tensor), Cowin (1986). Compact (cortical) and cancellous (spongy or trabecular) bones can be treated in a number of problems, excluding bone fracture mechanics; as linearly elastic transverselly isotropic materials and orthotropic materials, respectively, Uklejewski (1992). Assuming, instead of Eq (2.1), different definitions of the structural (fabric) tensor such as proposed by Cowin (1986), Kubik (1981), Litewka (1985), the constitutive relations for other materials are obtained such as bones, porous media and bodies with regular system of cracks. Once again, in all the above mentioned situations the structural tensors are positive definite symmetric tensors of the second order.

It is worth emphasizing that the simplest equations (bilinear with respect to the strain tensor and the structural (reinforcement) tensor contain fewer independent material constants (provided the structural tensor is specified) than those corresponding to an anisotropic Hooke's material (isotropic situation excluded).

Appendix

Representations of generators and invariants in the Cartesian frame of reference whose axes coincide with the local directions of orthotropy are as follows

$$\mathsf{E}^\alpha = \begin{bmatrix} E_{11}^{(\alpha)} & E_{12}^{(\alpha)} & E_{13}^{(\alpha)} \\ & E_{22}^{(\alpha)} & E_{23}^{(\alpha)} \\ \text{sym.} & & E_{33}^{(\alpha)} \end{bmatrix} \qquad \alpha = 1,2$$

$$E_{11}^{(1)} = E_{11} \qquad \text{etc}$$

$$E_{11}^{(2)} = E_{11}^2 + E_{12}^2 + E_{13}^2$$

$$E_{12}^{(2)} = E_{11}E_{12} + E_{12}E_{22} + E_{13}E_{23}$$

$$E_{13}^{(2)} = E_{11}E_{13} + E_{12}E_{23} + E_{13}E_{33}$$

$$E_{22}^{(2)} = E_{12}^2 + E_{22}^2 + E_{23}^2$$

$$E_{23}^{(2)} = E_{12}E_{13} + E_{22}E_{23} + E_{23}E_{33}$$

$$E_{33}^{(2)} = E_{13}^2 + E_{23}^2 + E_{33}^2$$

$$\mathrm{tr}\mathsf{E} = E_{11} + E_{22} + E_{33}$$

$$\mathrm{tr}\mathsf{E}^2 = E_{11}^2 + E_{22}^2 + E_{33}^2 + 2(E_{12}^2 + E_{13}^2 + E_{23}^2)$$

$$\mathrm{tr}\mathsf{E}^3 = E_{11}^3 + E_{22}^3 + E_{33}^3 + 3\Big[E_{11}(E_{12}^2 + E_{13}^2) + E_{22}(E_{12}^2 + E_{23}^2) + +E_{33}(E_{13}^2 + E_{23}^2)\Big] + 6E_{12}E_{13}E_{23}$$

$$\mathsf{R}^\alpha = \begin{bmatrix} k_1^\alpha R_1^\alpha & 0 & 0 \\ & k_2^\alpha R_2^\alpha & 0 \\ \text{sym.} & & k_3^\alpha R_3^\alpha \end{bmatrix}$$

$$\mathrm{tr}\mathbf{R}^i = k_1^i R_1^i + k_2^i R_2^i + k_3^i R_3^i \qquad i = 1,2,3 \quad \text{(no summation over } i)$$

$$\mathbf{E}\mathbf{R}^\alpha + \mathbf{R}^\alpha\mathbf{E} = \begin{bmatrix} 2k_1^\alpha R_1^\alpha E_{11} & (k_1^\alpha R_1^\alpha + k_2^\alpha R_2^\alpha)E_{12} & (k_1^\alpha R_1^\alpha + k_3^\alpha R_3^\alpha)E_{13} \\ & 2k_2^\alpha R_2^\alpha E_{22} & (k_2^\alpha R_2^\alpha + k_3^\alpha R_3^\alpha)E_{23} \\ \text{sym.} & & 2k_3^\alpha R_3^\alpha E_{33} \end{bmatrix}$$

$$\mathrm{tr}\mathbf{E}\mathbf{R}^\alpha = k_1^\alpha R_1^\alpha E_{11} + k_2^\alpha R_2^\alpha E_{22} + k_3^\alpha R_3^\alpha E_{33}$$

$$\mathrm{tr}\mathbf{E}^2\mathbf{R}^\alpha = k_1^\alpha R_1^\alpha E_{11}^2 + k_2^\alpha R_2^\alpha E_{22}^2 + k_3^\alpha R_3^\alpha E_{33}^2 + (k_1^\alpha R_1^\alpha + k_2^\alpha R_2^\alpha)E_{12}^2 + \\ +(k_1^\alpha R_1^\alpha + k_3^\alpha R_3^\alpha)E_{13}^2 + (k_2^\alpha R_2^\alpha + k_3^\alpha R_3^\alpha)E_{23}^2$$

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Nieliniowy opis sprężystych materiałów wzmocnionych włóknami

Streszczenie

W pracy szczegółowo przedyskutowano i uogólniono równania konstytutywne zaproponowane w artykule Jemioło i inni (1990). Rozpatrzono związki fizyczne nieliniowej sprężystości kompozytu składającego się z izotropowej matrycy zbrojonej trzema ortogonalnymi krzywoliniowymi rodzinami włókien o różnych własnościach mechanicznych. Sformułowano alternatywne równania w tzw. postaci kanonicznej. Przedyskutowano przypadki lokalnej ortotropii, transwersalnej izotropii i izotropii. Zbrojenie w każdym przypadku opisano dodatnio określonym symetrycznym tensorem drugiego rzędu. Wyprowadzono odpowiednie równania konstytutywne przy dodatkowych założeniach: płaskiego stanu naprężenia, płaskiego stanu odkształcenia i antypłaskiego stanu naprężenia. Zaproponowano dwa warianty uproszczonych związków fizycznych liniowej sprężystości. Porównano otrzymane równania z klasycznym prawem Hooke'a. Podano zależności pozwalające wyznaczyć stałe materiałowe z klasycznych testów dla materiału ortotropowego.

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