MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
4, 32, 1994

MATHEMATICAL MODEL OF AN ASTATIC AUTOPILOT OF A HELICOPTER INCLUDING COUPLINGS BETWEEN LONGITUDINAL AND LATERAL MOTIONS

GRZEGORZ KOWALECZKO

Military University of Technology, Warsaw

A method of determining the laws of control for an astatic autopilot has been considered. These laws have been formulated on the basis of the linearized equations of spatial motion of a single-rotor helicopter. All couplings between longitudinal and lateral motions have been taken into account. The numerical example for one flying speed and the variations of the compensation and amplification coefficients for the whole range of flying speeds have been presented.

1. Introduction

The instability of a helicopter appears for most ranges of flight. This is a reason of application of preventive measures which improve dynamic properties of the object. That is an automatic flight control system enhancing damping of natural motions of a helicopter-autopilot system.

The mathematical model of an astatic autopilot is presented in this work taking into account its dynamics. The laws of control have been determined on the basis of the method of separation of motions. Principles of that method were described by Kozhevnikov (1977). Kowaleczko (1992) and Dżygadło and Kowaleczko (1993b) developed that method denying some simplifying assumptions. A static model of autopilot was analyzed there. That model excluded all couplings between longitudinal and lateral motions. Kowaleczko (1994b) modelled the laws of control for the static autopilot which included the couplings. Kowaleczko (1994a) obtained those laws for the astatic autopilot without taking into account these couplings.

The most comprehensive model, including dynamics of the autopilot and couplings of motions, will be presented here.

2. Physical and mathematical model of the helicopter

Similarly to Kowaleczko (1992) and (1994a,b), Dżygadlo and Kowaleczko (1993a,b) the "aeroplane" model of the helicopter is taken as a base. The dynamics of this model is described by a simplified, linearized set of equations

$$\mathsf{C}\dot{x} + \dot{D}x + \mathsf{E}s = 0 \tag{2.1}$$

where

$x = [u, v, w, p, q, r, \vartheta, \varphi, \psi]$ – vector of increases in flight parameters:

- U, V, W – linear velocities of the centre of the fuselage mass in the coordinate a system $0x_k y_k z_k$ connected with the fuselage
- P, Q, R – angular velocities of the fuselage in the coordinate system $0x_k y_k z_k$
- Θ, Φ, Ψ – pitch, roll and yaw angles, respectively.

$s = [\Delta\theta_o, \Delta\kappa_s, \Delta\eta_s, \Delta\phi_{so}]$ – vector of the control parameters increases:

- θ_o – angle of collective pitch of the main rotor
- κ_s – control angle for longitudinal motion
- η_s – control angle for lateral motion
- ϕ_{so} – angle of collective pitch of the tail rotor.

The method of obtaining Eqs (2.1) was shown in detail by Kowaleczko (1992), Dżygadło and Kowaleczko (1993a). For further calculations this set will be transformed to

$$\dot{x} = \mathsf{F}x + \mathsf{G}s \tag{2.2}$$

where F is 9×9 matrix and is 9×4 matrix. These matrices can be written as, respectively

$$\mathsf{F} = -\mathsf{C}^{-1}\mathsf{D} \qquad\qquad \mathsf{G} = -\mathsf{C}^{-1}\mathsf{E} \tag{2.3}$$

where

$$\mathsf{F} = \begin{bmatrix} X^u & X^v & X^w & X^p & X^q & X^r & X^\vartheta & X^\varphi & X^\psi \\ Y^u & Y^v & Y^w & Y^p & Y^q & Y^r & Y^\vartheta & Y^\varphi & Y^\psi \\ Z^u & Z^v & Z^w & Z^p & Z^q & Z^r & Z^\vartheta & Z^\varphi & Z^\psi \\ L^u & L^v & L^w & L^p & L^q & L^r & L^\vartheta & L^\varphi & L^\psi \\ M^u & M^v & M^w & M^p & M^q & M^r & M^\vartheta & M^\varphi & M^\psi \\ N^u & N^v & N^w & N^p & N^q & N^r & N^\vartheta & N^\varphi & N^\psi \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{2.4}$$

$$\mathsf{G} = \begin{bmatrix} X^{\theta_o} & X^{\kappa_s} & X^{\eta_s} & X^{\phi_{so}} \\ Y^{\theta_o} & Y^{\kappa_s} & Y^{\eta_s} & Y^{\phi_{so}} \\ Z^{\theta_o} & Z^{\kappa_s} & Z^{\eta_s} & Z^{\phi_{so}} \\ L^{\theta_o} & L^{\kappa_s} & L^{\eta_s} & L^{\phi_{so}} \\ M^{\theta_o} & M^{\kappa_s} & M^{\eta_s} & M^{\phi_{so}} \\ N^{\theta_o} & N^{\kappa_s} & N^{\eta_s} & N^{\phi_{so}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.5}$$

Additionally it is assumed that the angles of pitch Θ, roll Φ and yaw Ψ of the helicopter are small and

$$p = \dot{\varphi} \qquad q = \dot{\vartheta} \qquad r = \dot{\psi} \tag{2.6}$$

According to the method of separation of motions, the vector of control $\boldsymbol{s}$ is composed of two parts. $\boldsymbol{s}_k$ is the compensation vector and the vector $\boldsymbol{s}_A$ ensures stabilization.

$$\boldsymbol{s} = \mathsf{A}^{-1}(\boldsymbol{s}_A + \boldsymbol{s}_k) \tag{2.7}$$

thus the compensation vector $\boldsymbol{s}_k$ can be expressed

$$\boldsymbol{s}_k = \mathsf{T}^k \boldsymbol{x} \tag{2.8}$$

The 4×4 matrix A and 4×9 matrix T^k have elements which are defined on the basis of the following compensation conditions

$$\begin{aligned} & Z^u u + Z^v v + Z^w w + Z^p p + Z^q q + Z^r r + Z^\vartheta \vartheta + Z^\varphi \varphi + Z^\psi \psi + \\ & + Z^{\kappa_s} \Delta\kappa_s + Z^{\eta_s} \Delta\eta_s + Z^{\phi_{so}} \Delta\phi_{so} + Z^{\theta_o} \Delta\theta_{ok} = 0 \\ & M^v v + M^w w + M^p p + M^r r + M^\varphi \varphi + M^\psi \psi + \\ & + M^{\theta_o} \Delta\theta_o + M^{\eta_s} \Delta\eta_s + M^{\phi_{so}} \Delta\phi_{so} + M^{\kappa_k} \Delta\kappa_k = 0 \\ & L^u u + L^w w + L^q q + L^r r + L^\vartheta \vartheta + L^\psi \psi + \\ & + L^{\theta_o} \Delta\theta_o + L^{\kappa_s} \Delta\kappa_s + L^{\phi_{so}} \Delta\phi_{so} + L^{\eta_s} \Delta\eta_k = 0 \\ & N^u u + N^v v + N^w w + N^p p + N^q q + N^\vartheta \vartheta + N^\varphi \varphi + \\ & + N^{\theta_o} \Delta\theta_o + N^{\kappa_s} \Delta\kappa_s + N^{\eta_s} \Delta\eta_s + N^{\phi_{so}} \Delta\phi_{sk} = 0 \end{aligned} \tag{2.9}$$

Thus the matrix $\mathbf{A}$ takes the form

$$\mathbf{A} = \begin{bmatrix} 1 & \dfrac{Z^{\kappa_s}}{Z^{\theta_o}} & \dfrac{Z^{\eta_s}}{Z^{\theta_o}} & \dfrac{Z^{\phi_{so}}}{Z^{\theta_o}} \\ \dfrac{M^{\theta_o}}{M^{\kappa_s}} & 1 & \dfrac{M^{\eta_s}}{M^{\kappa_s}} & \dfrac{M^{\phi_{so}}}{M^{\kappa_s}} \\ \dfrac{L^{\theta_o}}{L^{\eta_s}} & \dfrac{L^{\kappa_s}}{L^{\eta_s}} & 1 & \dfrac{L^{\phi_{so}}}{L^{\eta_s}} \\ \dfrac{N^{\theta_o}}{N^{\phi_{so}}} & \dfrac{N^{\kappa_s}}{N^{\phi_{so}}} & \dfrac{N^{\eta_s}}{N^{\phi_{so}}} & 1 \end{bmatrix} \tag{2.10}$$

On the other hand the matrix $\mathbf{T}^k$ is defined as follows

$$\mathbf{T} = -\begin{bmatrix} \dfrac{Z^u}{Z^{\theta_o}} & \dfrac{Z^v}{Z^{\theta_o}} & \dfrac{Z^w}{Z^{\theta_o}} & \dfrac{Z^p}{Z^{\theta_o}} & \dfrac{Z^q}{Z^{\theta_o}} & \dfrac{Z^r}{Z^{\theta_o}} & \dfrac{Z^\vartheta}{Z^{\theta_o}} & \dfrac{Z^\varphi}{Z^{\theta_o}} & \dfrac{Z^\psi}{Z^{\theta_o}} \\ 0 & \dfrac{M^v}{M^{\kappa_s}} & \dfrac{M^w}{M^{\kappa_s}} & \dfrac{M^p}{M^{\kappa_s}} & 0 & \dfrac{M^r}{M^{\kappa_s}} & 0 & \dfrac{M^\varphi}{M^{\kappa_s}} & \dfrac{M^\psi}{M^{\kappa_s}} \\ \dfrac{L^u}{L^{\eta_s}} & 0 & \dfrac{L^w}{L^{\eta_s}} & 0 & \dfrac{L^q}{L^{\eta_s}} & \dfrac{L^r}{L^{\eta_s}} & \dfrac{L^\vartheta}{L^{\eta_s}} & 0 & \dfrac{L^\psi}{L^{\eta_s}} \\ \dfrac{N^u}{N^{\phi_{so}}} & \dfrac{N^v}{N^{\phi_{so}}} & \dfrac{N^w}{N^{\phi_{so}}} & \dfrac{N^p}{N^{\phi_{so}}} & \dfrac{N^q}{N^{\phi_{so}}} & 0 & \dfrac{N^\vartheta}{N^{\phi_{so}}} & \dfrac{N^\varphi}{N^{\phi_{so}}} & 0 \end{bmatrix} \tag{2.11}$$

Substituting Eq (2.7) into Eq (2.2), and making use of Eq (2.8), we obtain the following set of equations

$$\dot{\boldsymbol{x}} = \mathbf{H}\boldsymbol{x} + \mathbf{K}\boldsymbol{s}_A \tag{2.12}$$

where the matrix $\mathbf{H}$ have size 9×9 and the $\mathbf{K}$ – size is 9×4. These matrices are respectively equal to

$$\mathbf{H} = \mathbf{F} + \mathbf{G}\mathbf{A}^{-1}\mathbf{T}^k \qquad\qquad \mathbf{K} = \mathbf{G}\mathbf{A}^{-1} \tag{2.13}$$

The set of Eqs (2.12) will be completed by the following laws of control which include the dynamics of automatic flight control system

$$T_{\theta_o}\Delta\dot{\theta}_{oA} + \Delta\theta_{oA} = \tau_w w \tag{2.14}$$

$$T_\kappa\Delta\dot{\kappa}_{sA} + \Delta\kappa_{sA} = \tau_u u + \tau_q q + \tau_\vartheta \vartheta \tag{2.15}$$

$$T_\eta\Delta\dot{\eta}_{sA} + \Delta\eta_{sA} = \tau_v v + \tau_p p + \tau_\varphi \varphi \tag{2.16}$$

$$T_\phi\Delta\dot{\phi}_{oA} + \Delta\phi_{sA} = \tau_r r + \tau_\psi \psi \tag{2.17}$$

It can be described by a general expression

$$\mathbf{T}^L\dot{\boldsymbol{s}}_A + \boldsymbol{s}_A = \mathbf{T}^g\boldsymbol{x} \tag{2.18}$$

According to Eqs (2.14) to (2.17) the matrix $\mathbf{T}^g$, which size is 4×9 has the form

$$\mathbf{T}^g = \begin{bmatrix} 0 & 0 & \tau_w & 0 & 0 & 0 & 0 & 0 & 0 \\ \tau_u & 0 & 0 & 0 & \tau_q & 0 & \tau_\vartheta & 0 & 0 \\ 0 & \tau_v & 0 & \tau_p & 0 & 0 & 0 & \tau_\varphi & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau_r & 0 & 0 & \tau_\psi \end{bmatrix} \tag{2.19}$$

T^L is the following diagonal matrix

$$\mathsf{T}^L = \begin{bmatrix} T_\theta & 0 & 0 & 0 \\ 0 & T_\kappa & 0 & 0 \\ 0 & 0 & T_\eta & 0 \\ 0 & 0 & 0 & T_\phi \end{bmatrix} \tag{2.20}$$

Eqs (2.12) and (2.18) can be analysed together rewritting them as

$$\mathsf{T}^*\dot{\boldsymbol{x}}^* = \mathsf{H}^*\boldsymbol{x}^* \tag{2.21}$$

where the vector $\boldsymbol{x}^* = [\boldsymbol{x}, \boldsymbol{s}_A]$ have the following components

$$\boldsymbol{x}^* = [u, v, w, p, q, r, \vartheta, \varphi, \psi, \Delta\theta_{oA}, \Delta\kappa_{sA}, \Delta\eta_{sA}, \Delta\phi_{sA}] \tag{2.22}$$

and the matrices T^* and H^*, respectively, are equal to

$$\mathsf{T}^* = \begin{bmatrix} \mathsf{I} & 0 \\ 0 & \mathsf{T}^L \end{bmatrix}_{13\times13} \qquad \mathsf{H}^* = \begin{bmatrix} \mathsf{H} & \mathsf{K} \\ \mathsf{T}^g & -\mathsf{I} \end{bmatrix}_{13\times13} \tag{2.23}$$

The symbol I stands for the identity matrix.

Next the set of ordinary differential equations (2.21) can be rearranged into a form

$$\dot{\boldsymbol{x}}^* = (\mathsf{T}^*)^{-1}\mathsf{H}^*\boldsymbol{x}^* \tag{2.24}$$

On the basis of this form we can determine the amplification coefficients responsible for stabilization (the elements of the matrix T^g) and the time constants of the stabilizing network (the elements of the matrix T^L).

To this end we have to determine a characteristic polynomial of Eqs (2.24). It can be described by a general expression

$$\begin{aligned} |\mathsf{I}\lambda - (\mathsf{T}^*)^{-1}\mathsf{H}^*| &= \lambda^{13} + D_{12}\lambda^{12} + D_{11}\lambda^{11} + D_{10}\lambda^{10} + D_9\lambda^9 + D_8\lambda^8 + \\ &+ D_7\lambda^7 + D_6\lambda^6 + D_5\lambda^5 + D_4\lambda^4 + D_3\lambda^3 + D_2\lambda^2 + D_1\lambda^1 + D_0 = 0 \end{aligned} \tag{2.25}$$

Then it is assumed that the spatial motion of the helicopter-autopilot system is a superposition of aperiodic and periodic motions, respectively. The time constants, the coefficients of damping and the periods of oscillations are assumed. On this basis the characteristic Eq (2.25) can be written in the form of a product of characteristic polynomials of the first order systems and the second order systems.

For further computations it is assumed that the helicopter-autopilot system realizes the following natural motions

– five aperiodic damping motions of the time constant

$$T^p_{I1}, \quad T^p_{I2}, \quad T^b_{I0}, \quad T^b_{I1}, \quad T^b_{I2}$$

– eight periodic damping motions; the damping coefficients and the periods of oscillations of them being, respectively

$$\begin{matrix} \zeta^p_{II1}, & \zeta^p_{II2}, & \zeta^b_{II1}, & \zeta^b_{II2} \\ T^p_{II1}, & T^p_{II2}, & T^b_{II1}, & T^b_{II2} \end{matrix}$$

Thus the characteristic polynomial (2.25) can be written in the following form

$$\begin{aligned} &(\lambda + P_1)(\lambda^2 + P_2\lambda + P_3)(\lambda + P_4)(\lambda^2 + P_5\lambda + P_6) \cdot \\ &\cdot(\lambda + B_0)(\lambda + B_1)(\lambda^2 + B_2\lambda + B_3)(\lambda + B_4)(\lambda^2 + B_5\lambda + B_6) = 0 \end{aligned} \tag{2.26}$$

where the symbols P_i and B_i denote

$$\begin{aligned} P_1 &= \frac{1}{T^p_{I1}} & P_2 &= \frac{2\xi^p_{II1}}{T^p_{II1}} & P_3 &= \frac{1}{(T^p_{II1})^2} \\ P_4 &= \frac{1}{T^p_{I2}} & P_5 &= \frac{2\xi^p_{II2}}{T^p_{II2}} & P_6 &= \frac{1}{(T^p_{II2})^2} \end{aligned}$$

$$\begin{aligned} B_0 &= \frac{1}{T^b_{I0}} & B_1 &= \frac{1}{T^b_{I1}} & B_2 &= \frac{2\xi^b_{II1}}{T^b_{II1}} & B_3 &= \frac{1}{(T^b_{II1})^2} \\ B_4 &= \frac{1}{T^b_{I2}} & B_5 &= \frac{2\xi^b_{II2}}{T^b_{II2}} & B_6 &= \frac{1}{(T^b_{II2})^2} \end{aligned} \tag{2.27}$$

From Eqs (2.26) and (2.27) it follows that, by prescribing the time constants T^p_{Ii} and T^b_{Ii}, the coefficients of damping ξ^p_{IIi} and ξ^b_{IIi} and the periods of oscillation T^p_{IIi} and T^b_{IIi}, we can obtain the following natural motions

$$\begin{aligned} \lambda^p_{I1} &= -\frac{1}{T^p_{I1}} & \lambda^p_{II1,2} &= -\frac{\xi^p_{II1}}{T^p_{II1}} \pm \frac{\sqrt{1-(\xi^p_{II1})^2}}{T^p_{II1}}\mathrm{i} \\ \lambda^p_{I2} &= -\frac{1}{T^p_{I2}} & \lambda^p_{II3,4} &= -\frac{\xi^p_{II2}}{T^p_{II2}} \pm \frac{\sqrt{1-(\xi^p_{II2})^2}}{T^p_{II2}}\mathrm{i} \\ \lambda^b_{I0} &= -\frac{1}{T^b_{I0}} \end{aligned} \tag{2.28}$$

$$\lambda^b_{I1} = -\frac{1}{T^b_{I1}} \qquad \lambda^b_{II1,2} = -\frac{\xi^b_{II1}}{T^b_{II1}} \pm \frac{\sqrt{1-(\xi^b_{II1})^2}}{T^b_{II1}}\mathrm{i}$$

$$\lambda^b_{I2} = -\frac{1}{T^b_{I2}} \qquad \lambda^b_{II3,4} = -\frac{\xi^b_{II2}}{T^b_{II2}} \pm \frac{\sqrt{1-(\xi^b_{II2})^2}}{T^b_{II2}}\mathrm{i}$$

By comparing Eq (2.25) with Eq (2.26), we can obtain, after some simple transformations the relations between the coefficients D_i and P_j and B_k.

Simultaneously the coefficients of the characteristic polynomial D_i ($i = 0, 12$) are the functions of the required values T_θ, T_κ, τ_w, τ_u, τ_q, τ_ϑ, T_η, T_ϕ, τ_v, τ_p, τ_φ, τ_r, τ_ψ which define the form of the laws of control (2.18).

In the end we obtain a set of thirteen nonlinear algebraic equations, of a general form

$$D_i(T_{\theta_o}, T_\kappa, T_\eta, T_\phi, \tau_u, \tau_q, \tau_\vartheta, \tau_w, \tau_v, \tau_p, \tau_\varphi, \tau_r, \tau_\psi) = D_i(P_j, B_k) \qquad i = 0, ..., 13 \tag{2.29}$$

The solution to this set gives values of the time constants and values of the amplification coefficients, which together with the compensation coefficients (2.11), characterize the automatic flight control system in the spatial motion.

3. Numerical example

Exemplary results of the numerical analysis are presented for the Polish "Sokol" helicopter. Detailed initial data will be presented together with the form of the laws of control for the flying speed of 100 km/h and the variations of the compensation and amplification coefficients of the autopilot over a range of flying speeds from 0 to 250 km/h. It has been assumed that the induced velocity in the plane of the main rotor is constant and the helicopter performs a steady horizontal flight, the parameters of which have been given by Kowaleczko (1992) and Dżygadlo and Kowaleczko (1991).

For a spatial flying speed of 100 km/h the matrices **F** and **G** have the following nonzero elements:

— for the matrix **F**

$X^u = -0.029283$ $X^v = -0.004664$ $X^w = 0.001854$ $X^p = 0.2095468$
$X^q = 1.5096089$ $X^r = 0.0100954$ $X^\vartheta = -9.80629$
$Y^u = -0.001183$ $Y^v = -0.095860$ $Y^w = 0.005634$ $Y^p = -1.473138$
$Y^q = 0.1856809$ $Y^r = -27.30430$ $Y^\vartheta = 0.005573$ $Y^\varphi = 9.804197$
$Z^u = -0.067603$ $Z^v = -0.002036$ $Z^w = -0.47314$ $Z^p = -0.355670$
$Z^q = 27.775872$ $Z^r = -0.522719$ $Z^\vartheta = 0.26978$ $Z^\varphi = -0.202519$
$L^u = 0.006338$ $L^v = -0.030324$ $L^w = 0.009394$ $L^p = -0.898412$
$L^q = -0.185780$ $L^r = 0.086907$
$M^u = 0.0076036$ $M^v = 0.002516$ $M^w = 0.002499$ $M^p = 0.017988$
$M^q = -0.444167$ $M^r = -0.006152$
$N^u = 0.013888$ $N^v = 0.055573$ $N^w = 0.028115$ $N^p = 0.030087$
$N^q = -0.123689$ $N^r = -0.754432$
$F_{75} = 1.0$ $F_{84} = 1.0$ $F_{96} = 1.0$

— for the matrix **G**

$X^{\theta_o} = -3.125898$ $X^{\kappa_s} = -10.8568$ $X^{\eta_s} = -0.30001$ $X^{\phi_{so}} = 0$
$Y^{\theta_o} = 1.377479$ $Y^{\kappa_s} = 0.36659$ $Y^{\eta_s} = -9.14644$ $Y^{\phi_{so}} = 12.2923$
$Z^{\theta_o} = -51.98020$ $Z^{\kappa_s} = -21.6915$ $Z^{\eta_s} = 1.91438$ $Z^{\phi_{so}} = 0$
$L^{\theta_o} = 1.475246$ $L^{\kappa_s} = 1.38278$ $L^{\eta_s} = -12.7172$ $L^{\phi_{so}} = 2.741854$
$M^{\theta_o} = 1.576425$ $M^{\kappa_s} = 5.40889$ $M^{\eta_s} = 0.40497$ $M^{\phi_{so}} = 0$
$N^{\theta_o} = -2.064799$ $N^{\kappa_s} = 0.83037$ $N^{\eta_s} = -0.44740$ $N^{\phi_{so}} = -19.1856$

On the basis of this data the following values of elements of the matrices **A** and $\mathbf{T}^g$ have been computed

— for the matrix **A**

$$\mathbf{A} = \begin{bmatrix} 1 & 0.4173050 & -0.0368291 & 0 \\ 0.2914512 & 1 & 0.0748709 & 0 \\ -0.1160039 & -0.1087329 & 1 & -0.2156018 \\ 0.1076225 & -0.0432810 & 0.0233195 & 1 \end{bmatrix}$$

— for the matrix $\mathbf{T}^k$

$$\mathbf{T}^k = \begin{bmatrix} -0.0013005 & -0.0000392 & -0.0091023 & -0.0068424 & 0.5343548 \\ 0 & -0.0004651 & -0.0004620 & -0.0033256 & 0 \\ 0.0004984 & 0 & 0.0007386 & 0 & -0.0146086 \\ 0.0007239 & 0.0028966 & 0.0014654 & 0.0015682 & -0.0064470 \end{bmatrix}$$

$$\begin{array}{cccc} -0.0100561 & 0.0051899 & -0.0038961 & 0 \\ 0.0011374 & 0 & 0 & 0 \\ 0.0068338 & 0 & 0 & 0 \\ 0.0068338 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

It has been assumed that the time constant and the damping coefficients are

$$\begin{array}{llll} T^p_{I1} = 1.2 & T^p_{II1} = 2.0 & T^p_{I2} = 1.6 & T^p_{II2} = 0.8 \\ \xi^p_{II1} = 0.9 & \xi^p_{II2} = 0.7 & & \\ T^b_{I0} = 0.5 & T^b_{I1} = 0.4 & T^b_{I2} = 1.0 & T^b_{II1} = 1.4 \\ T^b_{II2} = 1.8 & \xi^b_{II1} = 0.9 & \xi^b_{II2} = 0.9 & \end{array}$$

which corresponds to the following eigenvalues

$$\begin{array}{ll} \lambda^p_{I1} = -0.833 & \lambda^p_{II1,2} = -0.45 \pm 0.2179445i \\ \lambda^p_{I2} = -0.625 & \lambda^p_{II3,4} = -0.875 \pm 0.89269i \\ \lambda^b_{I0} = -2.0 & \lambda^b_{II1,2} = -0.64285714 \pm 0.31135i \\ \lambda^b_{I1} = -2.5 & \lambda^b_{II3,4} = -0.50 \pm 0.24216105i \\ \lambda^b_{I2} = -1.0 & \end{array}$$

The matrices of the time constant $\mathbf{T}^L$ and the matrix of the amplification coefficients $\mathbf{T}^g$ are respectively

$$\mathbf{T}^L = \begin{bmatrix} 1.4769608 & 0 & 0 & 0 \\ 0 & 0.361612 & 0 & 0 \\ 0 & 0 & 0.295133 & 0 \\ 0 & 0 & 0 & 0.3483328 \end{bmatrix}$$

$$\mathbf{T}^g = \left[\begin{array}{ccccc} 0 & 0 & 0.0053163 & 0 & 0 \\ -0.0008768 & 0 & 0 & 0 & -0.0608547 \\ 0 & 0.0026032 & 0 & 0.0966221 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right.$$

$$\left.\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & -0.0454142 & 0 & 0 \\ 0 & 0 & 0.1239504 & 0 \\ 0.0478454 & 0 & 0 & 0.0697777 \end{array}\right]$$

These matrices determine the laws of control (Eq (2.8)). They are used together with Eqs (2.12) to verify computations of the eigenvalues and eigenvectors on the basis of the Fade method (cf Fadeev (1960); Górecki (1980)).

The eigenvectors are made dimensionless and normalized according to the algorithm described by Kowaleczko (1992) and Dżygadło, Kowaleczko (1993a).

The linear velocities are divided by the blade tip velocity of the rotor while the angular ones by the angular velocity of the main rotor.

The Euclidean norm is used to normalize the eigenvectors.

The eigenvalues and the eigenvectors corresponding to them are as follows

	$\lambda^p_{I1} = -0.83333$	$\lambda^p_{II1,2} = -0.45 \pm 0.2179i$	$\lambda^p_{I2} = -0.625$
u^*	0.00595489	$0.0304009 \pm 0.000277192i$	0.0141017
v^*	0.0402704	$0.0219892 \mp 0.0527003i$	0.0497694
w^*	0.000168244	$-0.000157650 \pm 0.00177811i$	−0.0000421591
p^*	0.0304248	$-0.00538311 \mp 0.0160270i$	0.0222729
q^*	−0.00323212	$-0.00354191 \pm 0.00432597i$	−0.00413057
r^*	−0.00550777	$-0.00187519 \pm 0.00508494i$	−0.00543985
ϑ^*	0.103751	$0.271425 \mp 0.125698i$	0.176788
φ^*	−0.977005	$-0.114598 \pm 0.897575i$	−0.953641
ψ^*	0.176866	$0.208951 \mp 0.201186i$	0.232914
θ^*_o	−0.000813779	$0.00268383 \pm 0.00334289i$	−0.000612031
κ^*_s	−0.000782562	$-0.0145480 \mp 0.000284396i$	−0.00504025
η^*_s	−0.0271190	$-0.0149717 \pm 0.0484151i$	−0.0409915
ϕ^*_{so}	0.00745663	$0.0135306 \mp 0.0101482i$	0.0118752

	$\lambda^p_{II3,4} = -0.875 \pm 0.8927i$	$\lambda^b_{I0} = -2.0$	$\lambda^b_{I1} = -2.5$
u^*	$-0.0000439380 \mp 0.00103306i$	−0.00142359	0.0120800
v^*	$-0.0106382 \mp 0.0252528i$	−0.000474649	0.0366886
w^*	$-0.000787168 \mp 0.000106804i$	0.000160006	0.000941227
p^*	$-0.0316502 \pm 0.0330068i$	−0.0709595	−0.0693737
q^*	$0.00132148 \mp 0.000967241i$	0.00445464	−0.0224047
r^*	$0.00486908 \mp 0.00603308i$	0.0187547	0.0442234
ϑ^*	$-0.0345778 \mp 0.00570648i$	−0.0595808	0.239730
φ^*	$0.978922 \mp 0.0107475i$	0.949443	0.742580
ψ^*	$-0.165203 \pm 0.0159691i$	−0.250938	−0.473369
θ^*_o	$0.0000546653 \pm 0.00065439i$	−0.0000914186	−0.000390265
κ^*_o	$0.000458054 \pm 0.00274439i$	−0.0154769	0.243415
η^*_o	$0.0702026 \pm 0.0696704i$	−0.161032	−0.256352
ϕ^*_{so}	$-0.00988983 \pm 0.00508050i$	0.0214071	0.182470

	$\lambda^b_{I2} = -1.0$	$\lambda^b_{II1,2} = -0.64286 \pm 0.3113i$	$\lambda^b_{II3,4} = -0.50 \pm 0.2422i$
u^*	0.00363876	$0.00777833 \mp 0.00346443i$	$0.0210840 \mp 0.00378488i$
v^*	0.0339960	$0.00124297 \mp 0.0491639i$	$0.00950108 \mp 0.0550407i$
w^*	0.000288627	$-0.000681490 \pm 0.000110015i$	$-0.000536102 \pm 0.00110946i$
p^*	0.0367285	$-0.0184914 \mp 0.0183282i$	$-0.0104764 \mp 0.0163760i$
q^*	−0.00298533	$-0.000633506 \pm 0.00303494i$	$-0.00228528 \pm 0.00423995i$
r^*	−0.00589419	$0.00120515 \pm 0.00441103i$	$-0.000496670 \pm 0.00516315i$
ϑ^*	0.0798575	$0.0708949 \mp 0.0919512i$	$0.188021 \mp 0.135775i$
φ^*	−0.982860	$0.324183 \pm 0.919954i$	$0.110335 \pm 0.929883i$
ψ^*	0.157729	$0.0313982 \mp 0.168410i$	$0.129937 \mp 0.213402i$
θ^*_o	−0.000675552	$0.0000842846 \pm 0.00166368i$	$0.00145925 \pm 0.00274031i$
κ^*_s	0.000882015	$-0.00464144 \pm 0.000515767i$	$-0.0105061 \pm 0.00107534i$
η^*_s	−0.0117922	$-0.00294866 \pm 0.0494309i$	$-0.00537673 \pm 0.0507133i$
ϕ^*_{so}	0.00531288	$0.00363999 \pm 0.00837622i$	$0.00908988 \pm 0.0109578i$

Similar verifying computations have been made over the entire range of flying speed, the natural motions obtained occurace to be the same as assumed. Fig.1 ÷ Fig.4 show the variation of the compensation and amplification coefficients. Fig.5 show the variation of the time constants of the autopilot.

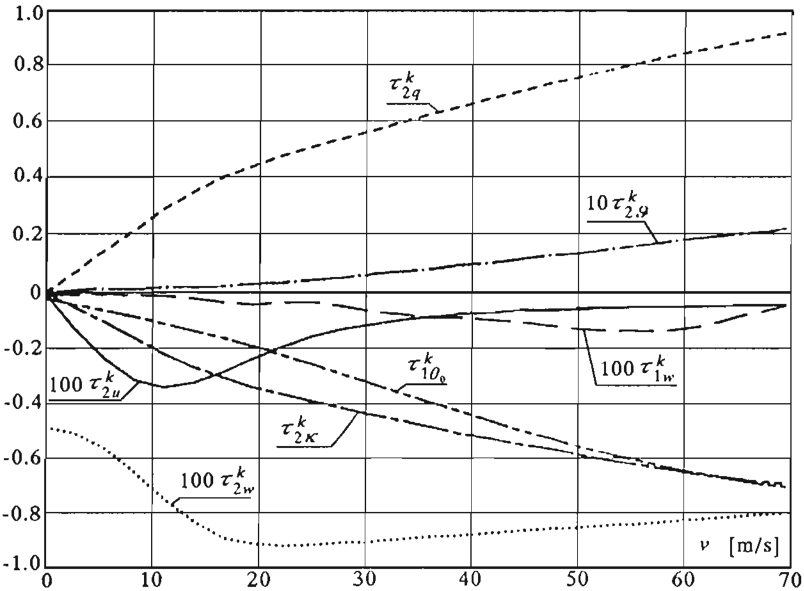

Fig. 1. Compensation coefficients for longitudinal motion

4. Concluding remarks

It can be seen, from Fig.1 ÷ Fig.5, that some of the coefficients and the time constants of the autopilot show considerable variation with the varying flying speed. This means that we cannot design an autopilot with constant parameters which would ensure the assumed performance to be realized over a wide range of the flying speed.

This remark is the same as remark which has been formulated by Dżygadło and Kowaleczko (1993b). But in this paper the couplings between longitudinal and lateral motions are included and we use the different model of the laws of control. Then we have other values of the amplification coefficients of autopilot.

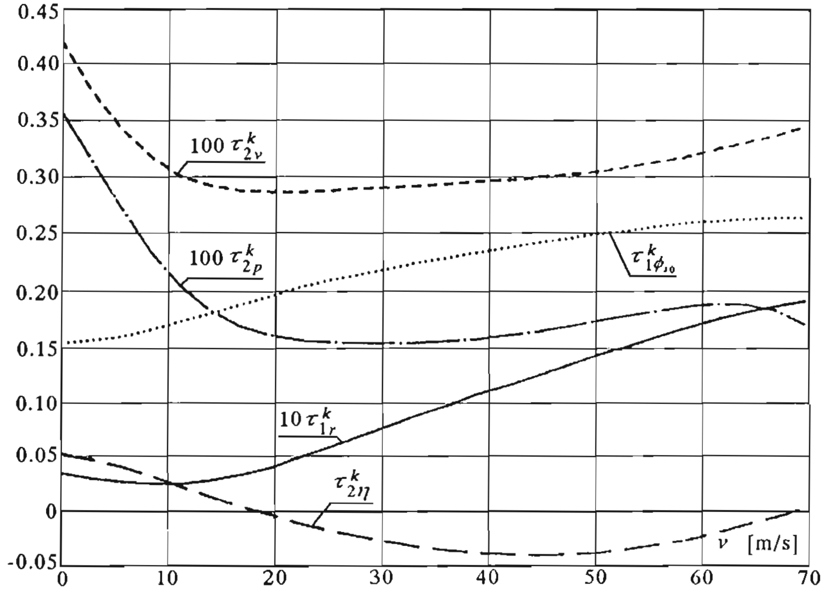

Fig. 2. Compensation coefficients for lateral motion

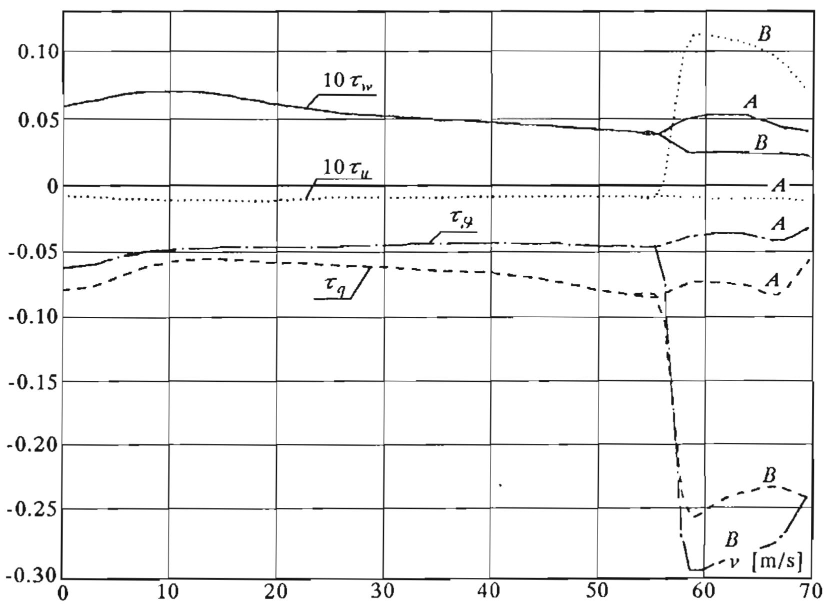

Fig. 3. Amplification coefficients for longitudinal motion

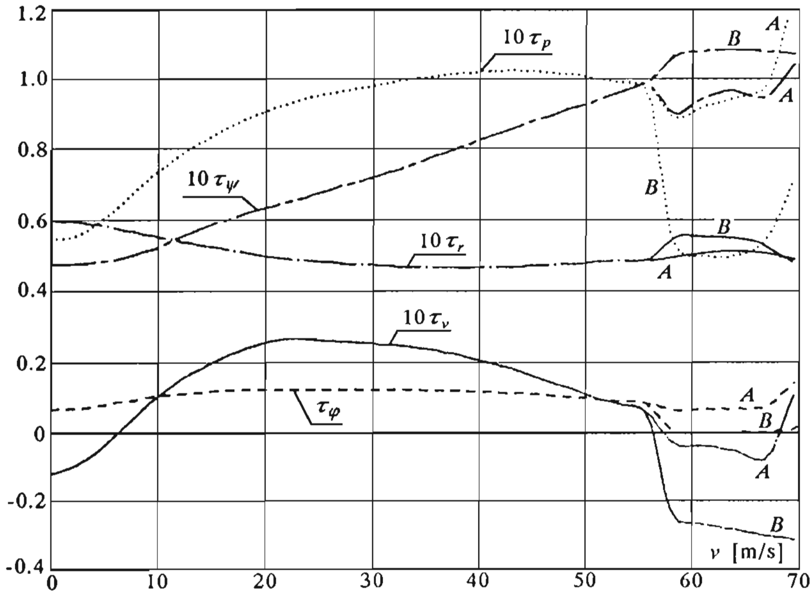

Fig. 4. Amplification coefficients for lateral motion

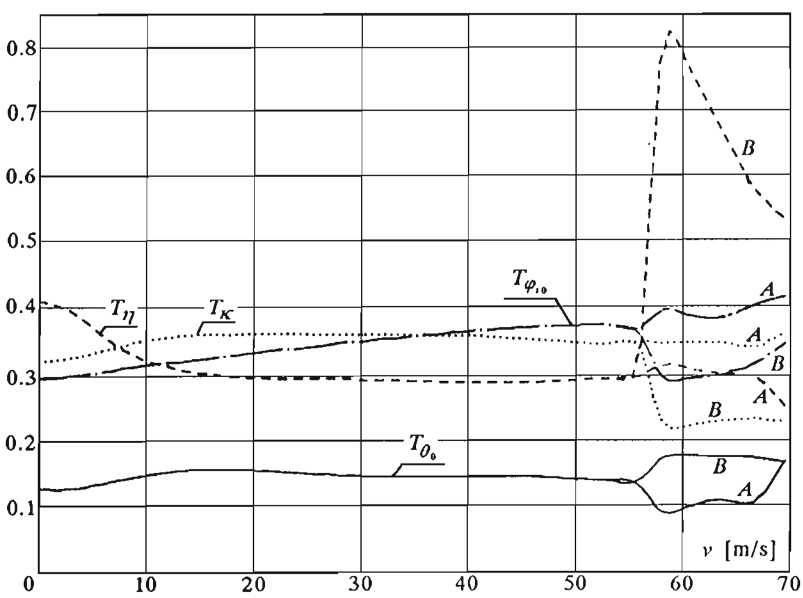

Fig. 5. Time constants of the autopilot

Numerical computations have been made employing the set of thirteen nonlinear algebraic equations (2.29) and the multidimensional Newton's method (cf Bjorek and Dahlquist (1983)). It requires good initial approximation of the searched variables. When we take as this approximation the solution which has been obtained for the astatic autopilot, excluding the longitudinal-lateral couplings, the algorithm is numerically unstable. The correct solution can be obtained on the basis of the "embedding" method (cf Bjorek and Dahlquist (1983)). In this method a solution to one problem is embedded into a wide family of problems. It has been made taking into account the coupling elements in the matrices $\mathbf{F}$ and $\mathbf{G}$, Eqs (2.3). In such a way we obtain successive approximations of the solution up to the final solution with full couplings.

The results of calculations show that for flight velocities higher then 200 km/h a family of solutions appear. Fig.3 ÷ Fig.5 show in this range only two branches (A and B) of this family. Existence of this family of solutions is in agreement with the theory of sets of nonlinear algebraic equations (cf Szmelter (1980)). In general way it is impossible to prejudge the existence and a number of those solutions.

In this range we observe considerable variation in the values of the amplification coefficients and the time constants of the autopilot. Each of the branches correspond, in theory, to assumed natural motions of the helicopter-autopilot system. However taking into account design limitations we have to realize laws of control making use of this branche only which can be obtained with the least variation of the coefficients and the time constants of the autopilot for the whole range of speeds. So it can be seen from Fig.3 ÷ Fig.5 that we have to take the branche A.

References

1. BJOREK A., DAHLQUIST G., 1983, *Numerical Methods,* (in Polish]), PWN, Warszawa
2. DŻYGADŁ Z., KOWALECZKO G., 1991, The Effect of the Model of the Induced Velocity Field on the Parameters of Steady Horizontal Fligh of a Single-Rotor Helicopter, (in Polish), *Biul. WAT,* 40, 9
3. DŻYGADŁO Z., KOWALECZKO G., 1993a, Analysis of Spatial Motion Dynamics of Helicopter for Various Models of Induced Velocity Field, *Journal of Technical Physics,* XXXIV, 2
4. DŻYGADŁO Z., KOWALECZKO G., 1993b, Dynamics Analysis of a Helicopter-Autopilot System, *Journal of Technical Physics,* XXXIV, 4
5. FADEEVA D., 1960, *Vychislitelnye metody lineĭnoĭ algebry,* GIF-ML, Moskva

6. GÓRECKI H., 1980, *Algorithms and Programs of Control,* (in Polish), WNT, Warszawa

7. KOWALECZKO G., 1992, Analysis of the Dynamics of Spatial Motion of a Helicopter, the Effect of an Autopilot Being Taken into Account, (in Polish), Doctor's Thesis, WAT, Warszawa

8. KOWALECZKO G., 1994, Model matematyczny astatycznego autopilota śmigłowca na bazie metody rozdzielenia ruchów, *Materiały konferencyjne VI Ogólnopolskiej Konferencji "Mechanika w Lotnictwie",* Warszawa

9. KOWALECZKO G., Model matematyczny autopilota statycznego przy uwzględnieniu sprzężeń ruchów podłużnych i bocznych śmigłowca, *Biuletyn WAT,* Warszawa (in print)

10. KOZHEVNIKOV V., 1977, *Avtomaticheckaya stabilizacya vertoletov,* Mashinostroenie, Moskva

11. SZMELTER J., 1980, *Metody komputerowe w mechanice,* (in Polish), PWN, Warszawa

Model matematyczny astetycznego autopilota dla śmigłowca uwzględniający sprzężenia pomiędzy ruchami podłużnymi i bocznymi

Streszczenie

W pracy przedstawiona została metoda określania praw sterowania dla autopilota astatycznego. Prawa te sformułowano na bazie zlinearyzowanych równań ruchu dla śmigłowca jednowirnikowego. Uwzględniono wszystkie sprzężenia pomiędzy ruchami podłużnymi i bocznymi. Pokazano przykład obliczeniowy dla konkretnej prędkości lotu oraz przebieg współczynników wzmocnienia autopilota w całym zakresie prędkości.

Manuscript received March 10, 1994; accepted for print May 17, 1994