MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
4, 32, 1994

TRANSVERSE VIBRATIONS OF A BEAM WITH HINGE

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The note presents a method of determination of the solution (in a class of generalized functions) to equations describing small transverse vibrations of an elastic beam with a hinge.

1. Introduction

A practicable method of analysis of mechanical structures composed of a number of elastic beams connected by hinges is based on separating the structure at hinges into parts consisting of single beams, determining then the movement of each part and "glueing" together such obtained solutions to get the solution for the whole system (compare any textbook on mechanical engineering, e.g. Beer and Johnson (1977)). The disadvantage of such an approach consists in the fact that one is obliged to solve a number of problems and then to solutions corresponding to various parts of a system which needs much move time-consuming computations in comparison with those needed for solving a single beam.

The method of analysis of multi-beam structures vibrations presented in this paper does not require the partition of the initial structure and amounts to determining single beam vibrations. This is done by replacing a beam with hinge by the substite structure consisting of a continuous beam with a variable stiffness. The latter is obtained after passing to the limit as ε tends to 0 for the family of problems for continuous beams with stiffnesses depending on a parameter $\varepsilon > 0$, modelling a hinge. The solution for the original system is then obtained from the solution for the substitute one upon passing with the parameter ε to zero. Since the stiffness of the substitute system is not continuous, the solution will be sought for in a class of generalized functions.

Note that, in contrast to the classical method, the proposed approach permits to perform necessary computations only once.

The paper presents an extension of results Kasprzyk (1993) dealing with static problems concerning a beam with a hinge, to the study of vibrations of such beams.

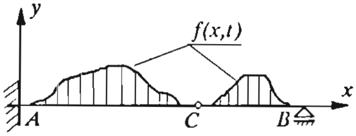

Fig. 1. Beam with a hinge

The method will be illustrated by considering the problem of determination of vibrations of the beam shown in Fig.1. We will carry out only this part of an analysis which differs from the standard considerations concerning vibrations of the beam having the form presented here but without the hinge.

2. Equation of motion

The cantilever elastic beam of length l is fixed at the point A, taken as the origin of coordinate system, and freely supported at the point B. It has a frictionless (ideal) hinge at the point C with an abscissa a, $0 < a < l$.

We assume that parameters of the system: density ρ, axial moment of inertia J and cross section F of a beam are constant. The Young modulus of material of a beam is denoted by E. Function $f(x,t)$ describes the distribution of the external load density. Moreover, suppose that there are no forces or moments acting upon beam in a neighbourhood of C.

Denote by $u(x,t)$ $(x \in [0,l],\ t \in [0,\infty))$ a transverse displacement of a point x of a beam at an instant t. Replacing the beam with a hinge by the continuous one, having the variable stiffness $EJa(x,\varepsilon_1,\varepsilon_2)$, where ε_1, ε_2 are positive parameters, the transverse vibrations of the latter are described by the equation

$$\frac{\partial^2}{\partial x^2}\Big[EJa(x,\varepsilon_1,\varepsilon_2)\frac{\partial^2 u}{\partial x^2}\Big] + \rho F\frac{\partial^2 u}{\partial t^2} = f(x,t) \tag{2.1}$$

The distribution of stiffness of the beam is characterized by the function

$$a(x,\varepsilon_1,\varepsilon_2) = \Big(1 - H(x-a^-)\Big) + H(x-a^+)$$

where $a^+ = a + \varepsilon_2$, $a^- = a - \varepsilon_1$, $H(x)$ is the Heaviside function

$$H(x) = \begin{cases} 1 & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 0 \end{cases}$$

Thus

$$EJa(x, \varepsilon_1, \varepsilon_2) = \begin{cases} 0 & \text{for} \quad x \in (a^-, a^+) \\ EJ & \text{for} \quad x \in (-\infty, a^-] \cup [a^+, \infty) \end{cases}$$

Passing with Eq (2.1) to a limit as $\varepsilon_1, \varepsilon_2 \to 0$ we get

$$EJ\frac{\partial^2}{\partial x^2}\Big(e(x)\frac{\partial^2 u}{\partial x^2}\Big) + \rho F\frac{\partial^2 u}{\partial t^2} = f(x,t) \tag{2.2}$$

where $e(x) = \lim\limits_{\substack{\varepsilon_1 \to 0 \\ \varepsilon_2 \to 0}} a(x, \varepsilon_1, \varepsilon_2)$.

The limit stiffness EJe satisfies $EJe(x) = EJ$ for $x \neq a$, $EJe(a) = 0$, thus Eq (2.2) represents vibrations of a beam presented in Fig.1.

We are looking for a solution u to Eq (2.2) satisfying the conditions:

$$u(x,\cdot) \in C^2 \qquad u(\cdot,t) \in C^0$$
$$\frac{\partial^4}{\partial x^4}u(\cdot,t) \in C^0 \qquad \text{for} \quad x \neq a$$

together with the geometric and mechanical boundary conditions

$$\begin{aligned} & u(0,t) = 0 && u(l,t) = 0 \\ & \frac{\partial u(0,t)}{\partial x} = 0 && \frac{\partial^2 u(l,t)}{\partial x^2} = 0 \\ & \frac{\partial^3 u(a_-,t)}{\partial x^3} = -\frac{\partial^3 u(a_+,t)}{\partial x^3} && \frac{\partial^2 u(a_-,t)}{\partial x^2} = \frac{\partial^2 u(a_+,t)}{\partial x^2} \\ & u(a_-,t) = u(a_+,t) \end{aligned} \tag{2.3}$$

and initial conditions

$$u(x,0) = \varphi_0(x) \qquad \frac{\partial u(x,0)}{\partial t} = \varphi_1(x) \tag{2.4}$$

We denote by $u(a_+,t)$ and $u(a_-,t)$ the left- and right-hand limits of $u(\cdot,t)$ at a, respectively. The similar convention is used for one-sided limits of partial derivatives of u or other functions depending on the x variable.

Eqs (2.2), (2.3) and (2.4) will be solved using the Galerkin method. Since the problem is linear, its solution is a sum of solutions representing free and forced vibrations of the beam, respectively.

3. Free vibrations of the system

Set

$$u(x,t) = X(x)T(t) \tag{3.1}$$

where X is assumed to be C^0 in the interval $[0,l]$, C^4 in subintervals $(0,a)$, (a,l) of $[0,l]$ and T is C^2 for $t \geq 0$.

Substituting expression (3.1) into Eq (2.2) we get the equation

$$[e(x)X'']''T(t) + b^2X(x)\ddot{T}(t) = 0$$

where $b^2 = (\rho F)/(EJ)$ which, after a separation of variables, can be rewritten as

$$\frac{\ddot{T}}{T} = -\frac{[e(x)X'']''}{b^2X} = -\omega^2 \tag{3.2}$$

We denote $\ddot{} = d^2/dt^2$, $'' = d^2/dx^2$. The derivative with respect to the variable x is understood in a sense of distributions on $(0,l)$.

From Eq (3.2) it follows that

$$\ddot{T} + \omega^2T = 0 \tag{3.3}$$

$$[e(x)X'']'' - \lambda^4X = 0 \qquad (\lambda^4 = b^2\omega^2) \tag{3.4}$$

Formula (3.1) and Eq (2.3) imply that X satisfies boundary conditions

$$\begin{aligned} &X(0) = X'(0) = 0 && X(l) = X''(l) = 0 \\ &X^{(j)}(a_-) = (-1)^jX^{(j)}(a_+) && j = 0,2,3 \end{aligned} \tag{3.5}$$

$X^{(j)}(x)$ denotes the jth derivative of $X(x)$. Set $\delta_a(x) = \delta(x-a)$, where δ is the Dirac distribution concentrated at the point $x = 0$ (i.e. δ_a is a Dirac measure concentrated at $x = a$).

Observe that if f is differentiable for $x < a$ and $x > a$ and one sided limits of its derivative exist at a, then the distributional derivative of f is given by (cf Schwartz (1965), Ch.II, §2)

$$f' = \{f'\} + \sigma_0\delta_a$$

where $\{f'\}$ denotes the derivative of f in a standard sense computed for $x < a$ or $x > a$ and $\sigma_0 = f(a_+) - f(a_-)$ denotes the jump of value of f while x passes a.

Using this observation one gets

$$[e(x)X'']'' = X^{(4)} - g(x,a_-,a_+)$$

with $g(x, a_-, a_+) = (X''(a_-) - X''(a_+))\delta'_a + (X^{(3)}(a_-) - X^{(3)}(a_+))\delta_a$ and Eq (3.4) can be written in the form

$$X^{(4)} - \lambda^4 X = g(x, a_-, a_+) \tag{3.6}$$

From Eq (3.5) it follows that $X''(a_-) - X''(a_+) = 0$, hence

$$g(x, a_-, a_+) = (X^{(3)}(a_-) - X^{(3)}(a_+))\delta_a \tag{3.7}$$

- *Remark 1.* A quantity $\gamma = X^{(3)}(a_-) - X^{(3)}(a_+)$ appearing in Eq (3.7) is considered as a parameter to be selected in a such a way that the boundary conditions are satisfied for solution X of Eq (3.6).

To solve (3.5) and (3.6) we will replace them by an equivalent boundary value problem for the first order system of differential equations.

Introducing new variables

$$X = y_1 \qquad y'_1 = \lambda y_2 \qquad y'_2 = \lambda y_3 \qquad y'_3 = \lambda y_4 \tag{3.8}$$

and putting $\gamma = y_4(a_-) - y_4(a_+)$ we obtain from Eq (3.6) the first order system of differential equations with a distributional right-hand side

$$y' - \mathbf{A}y = \boldsymbol{f}_0(x) \tag{3.9}$$

where

$$\boldsymbol{f}_0(x, a_-, a_+) = g_1(x, a_-, a_+)\boldsymbol{e}_4 \qquad g_1(x, a_-, a_+) = \gamma\delta_a$$

$$\mathbf{A} = \lambda \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \qquad \boldsymbol{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Note that $y_4(a_-) - y_4(a_+) = X^{(3)}(a_-) - X^{(3)}(a_+)$.

From Eq (3.5) it follows that

$$y_1(0) = 0 \qquad y_2(0) = 0 \qquad y_1(l) = 0 \qquad y_3(l) = 0 \tag{3.10}$$

$$\begin{aligned} &y_4(a_-) = -y_4(a_+) \qquad y_3(a_-) = y_3(a_+) = 0 \\ &y_1(a_-) = y_1(a_+) \end{aligned} \tag{3.11}$$

The conditions (3.10) are equivalent to

$$\mathbf{M}_1\boldsymbol{y}(0) + \mathbf{M}_2\boldsymbol{y}(l) = \mathbf{0} \tag{3.12}$$

where

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{M}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Eq (3.12) represents the boundary conditions, while Eq (3.11) corresponds to the mechanical and geometrical conditions imposed by a hinge.

It can be verified by a direct computation that the fundamental matrix solution $\mathbf{Y}(x)$ to the homogeneous equation (3.9) is given by

$$\mathbf{Y}(x) = \frac{1}{2}\cdot$$

$$\cdot\begin{bmatrix} \cosh\lambda x + \cos\lambda x & \sinh\lambda x + \sin\lambda x & \cosh\lambda x - \cos\lambda x & \sinh\lambda x - \sin\lambda x \\ \sinh\lambda x - \sin\lambda x & \cosh\lambda x + \cos\lambda x & \sinh\lambda x + \sin\lambda x & \cosh\lambda x - \cos\lambda x \\ \cosh\lambda x - \cos\lambda x & \sinh\lambda x - \sin\lambda x & \cosh\lambda x + \cos\lambda x & \sinh\lambda x + \sin\lambda x \\ \sinh\lambda x + \sin\lambda x & \cosh\lambda x - \cos\lambda x & \sinh\lambda x - \sin\lambda x & \cosh\lambda x + \cos\lambda x \end{bmatrix}$$

Since $\mathbf{Y}(0) = \mathbf{I}$ is the unit matrix, the general solution to Eq (3.9) can be written in the form

$$\boldsymbol{y}(x) = \mathbf{Y}(x)\boldsymbol{c} + \mathbf{Y}(x)\int\limits_0^x \mathbf{Y}(-s)\boldsymbol{f}_0(s)\,ds \tag{3.13}$$

where $\boldsymbol{c} \in \mathcal{R}^4$ denotes a vector of integration constants and $\int_0^x \mathbf{Y}(-s)\boldsymbol{f}_0(s)\,ds$ is understood as a primitive of a distribution $\boldsymbol{h}(s) = \mathbf{Y}(-s)\boldsymbol{f}_0(s)$. Applying to each component of the vector $\mathbf{Y}(-s)e_4$ the formula for a primitive of a distribution $\delta_a\varphi(s)$ (φ being a C^1-function) $\int_0^x \delta_a\varphi(s)\,ds = H_a(x)\varphi(a)$ we obtain

$$\boldsymbol{f}_{10}(x) = \int\limits_0^x \mathbf{Y}(-s)\boldsymbol{f}_0(s)\,ds = \gamma\int\limits_0^x \mathbf{Y}(-s)e_4\delta_a\,ds = \gamma H_a(x)\mathbf{Y}(-a)e_4 \tag{3.14}$$

From Eqs (3.12) and (3.13) it follows that $\boldsymbol{y}(x) = \mathbf{Y}(x)[\boldsymbol{c} + \gamma H_a(x)\mathbf{Y}(-a)\boldsymbol{e}_4]$. Since $\boldsymbol{y}(a_-) = \mathbf{Y}(a)\boldsymbol{c}$ and $\boldsymbol{y}(a_+) = \mathbf{Y}(a)\boldsymbol{c} + \gamma\boldsymbol{e}_4$, it is clear that $y_i(a_-) = y_i(a_+)$, $i = 1,2,3$, hence the last two boundary conditions of Eq (3.11) are satisfied.

From the first one we get $\gamma = -2e_4^\top \mathsf{Y}(a)\boldsymbol{c}$ which yields the formula for general solution to Eq (3.9) satisfying Eq (3.11)

$$\boldsymbol{y}(x) = \mathsf{Y}(x)[\mathsf{I} - 2H_a(x)\mathsf{Y}(-a)e_4e_4^\top \mathsf{Y}(a)]\boldsymbol{c} \tag{3.15}$$

- *Remark 2.* Using Eq (3.15) one can show that $y_3(x) \leq 0$ for $x \in (0,a)$ and $y_3(x) \geq 0$ for $x \in (a,l)$. Thus from the second condition of Eq (3.11) it follows that $y_3(a) = 0$, which proves the correctness of the assumed mathematical model describing beam with the hinge.

The formulae (3.15) and (3.12) result in a system of linear equations for $\boldsymbol{c}$

$$\Big(\mathsf{M}_1 + \mathsf{M}_2\mathsf{Y}(l)[\mathsf{I} - 2\mathsf{Y}(-a)e_4e_4^\top \mathsf{Y}(a)]\Big)\boldsymbol{c} = \boldsymbol{0} \tag{3.16}$$

Note that the coefficient matrix of Eq (3.16)

$$\begin{aligned}\mathsf{B}(\lambda,a) &= \mathsf{M}_1 + \mathsf{M}_2\mathsf{Y}(l)[\mathsf{I} - 2\mathsf{Y}(-a)e_4e_4^\top \mathsf{Y}(a)]\\ &= \mathsf{M}_1 + \mathsf{M}_2\mathsf{Y}(l-a)(\mathsf{I} - 2e_4e_4^\top)\mathsf{Y}(a)\end{aligned}$$

is a (transcendental) function of λ, because $\mathsf{Y}(x) = \mathsf{Y}(x,\lambda)$ is transcendental with respect to λ.

Eq (3.16) has a nontrivial solution, provided $\mathsf{B}(\lambda,a)$ is singular, i.e.

$$\det \mathsf{B}(\lambda,a) = 0 \tag{3.17}$$

- *Remark 3.* It can be proved that Eq (3.17) has an infinite set $\{\lambda_n\}$ of eigenvalues satisfying

  $$0 < \lambda_1 \leq \lambda_2 \leq \dots \qquad \lim_{n\to\infty} \lambda_n = \infty$$

  If c_n, $y_{0n}(a)$ are solutions to Eq (3.16) corresponding to an eigenvalue λ_n, then

  $$\boldsymbol{y}_n(x) = \mathsf{Y}(x,\lambda_n)[\mathsf{I} - 2H_a(x)\mathsf{Y}(-a,\lambda_n)e_4e_4^\top \mathsf{Y}(a,\lambda_n)]c_n \tag{3.18}$$

  is a solution to Eq (3.9) such that $X_n(x) = e_1^\top \boldsymbol{y}_n(x)$, $e_1^\top = [1,0,0,0]$, satisfies Eq (3.4) with $\lambda = \lambda_n$, moreover the family $\{X_n\}$ fulfills orthogonality conditions

  $$\int\limits_0^l X_n(s)X_m(s)\,ds = \begin{cases} 0 & \text{for} \quad m \neq n \\ \kappa_n \neq 0 & \text{for} \quad m = n \end{cases} \tag{3.19}$$

The proof of Remark 3 follows very closely the proof presented by Kasprzyk (1983), where the similar eigenproblem has been considered. The formula (3.18) can be verified by a direct computation. Condition (3.19) follows from the orthogonality condition proved by Kasprzyk (1983).

- *Remark 4.* One can get an implicit equation in λ_n only in the case of beams with constant stiffness. For beams having the variable stiffness the application of numerical approach is needed to obtain eigenvalues and eigenfunctions of the problem.

- *Remark 5.* It follows from Eq (3.17) that the solution λ_n depends on a and l or, more precisely, on the ratio a/l. The investigation of the dependence of eigenvalues on parameters of the system lies beyond the scope of this note and will be carried out in the next paper.

The desired formula for free vibrations of the system under considerations is given by

$$u(x,t)=\sum_{n=1}^{\infty} X_n(x)(a_n \sin\omega_n t + b_n \cos\omega_n t)$$

where $\omega_n^2 = \lambda_n^4/b^2$ and a_n, b_n are coefficients of the Fourier expansions of initial data $\varphi_0(x), \varphi_1(x)$ relative to the orthogonal family $\{X_n\}$ of eigenfunctions corresponding to Eq (3.4). Details can be found in Kasprzyk (1983).

4. Forced vibrations

Determination of the forced vibrations of a system is carried in a standard way and does not involve any additional problems. It is necessary to keep in mind that the Fourier expansion of $f(x,t)$ needed in the course of solving the problem has to be done relative to the orthogonal basis $\{X_n\}$. We will not pursue this question here.

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Drgania poprzeczne belki z przegubem

Streszczenie

W pracy podano metodę wyznaczania rozwiązania (w klasie funkcji uogólnionych) równania opisującego małe drgania poprzeczne belki sprężystej z przegubem.

Manuscript received January 11, 1994; accepted for print May 23, 1994