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SENSITIVITY ANALYSIS FOR NON-LINEAR BEAMS AND FRAMES[1]

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A geometrically non-linear, elastic frame structure is considered and the effect of small variation of its parameters on structure deformation response is studied. Variations of cross-sectional stiffness, member length, orientation, and also variation of node positions of a discrete structure are considered. The explicit expressions for variation of a displacement functional in terms of primary and adjoint states and of design parameter variations are provided.

1. Introduction

The present paper is concerned with the derivation of the first order sensitivity expressions in bar or beam systems undergoing both cross-sectional and configuration changes. In particular, the variations of positions of nodes are considered and their effect on any global response functional is analysed. In this paper, only regular states are considered, for which the incremental stiffness matrix is not singular. The case of critical states (bifurcation or limit points) will be discussed separately [5]. The present analysis is related to previous treatises on sensitivity for linear and non-linear beams and plates by Bojczuk (1987), Mróz (1993), Mróz and Haftka (1988), (1994), Dems and Mróz (1989) and Haftka et al. (1990), where the direct and adjoint sensitivity

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methods were exposed. In this paper the adjoint method is used in order to generate sensitivity expressions.

2. Fundamental equations

Consider a structural system composed of n rectilinear beam elements connected elastically at joints. The structure is subjected to transverse distributed load $q_T(x_i)$ and axial distributed load $q_N(x_i)$ acting upon some members. Here (x_i, y_i) is the local cartesian system attached at the end of the ith member, so that x_i-axis coincides with the member axis and the coordinate values are positive, Fig.1.

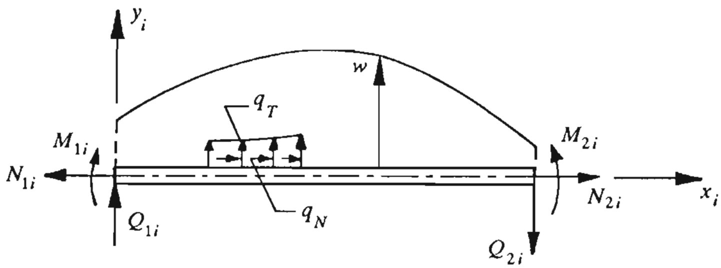


Fig. 1. Geometrically non-linear beam element – notation

Denote by $u(x_i)$ and $w(x_i)$ the axial and transverse member displacements, respectively. The generalized strains now are specified by the relations

$$
\begin{aligned}
e &= \varepsilon + \gamma = \frac{du}{dx_i} + \frac{1}{2}\Big(\frac{dw}{dx_i}\Big)^2 = u' + \frac{1}{2}(w')^2 \\
\kappa &= \frac{d^2 w}{dx_i^2} = w''
\end{aligned}
\tag{2.1}
$$

The equilibrium equations are

$$
\begin{aligned}
&\frac{dN}{dx_i} = N' = -q_N \\
&\frac{dM}{dx_i} - N\frac{dw}{dx_i} = M' - Nw' = Q \\
&\frac{dQ}{dx_i} = Q' = q_T
\end{aligned}
\tag{2.2}
$$

where N, M and Q denote the axial force, bending moment and the shear force, respectively. The usual decomposition of the shear force into two terms is useful, namely

$$Q = Q_M + Q_N \qquad Q_M = \frac{dM}{dx_i} \qquad Q_N = \frac{dw}{dx_i} \tag{2.3}$$

and after elimination of Q in Eqs (2.2), there is

$$\frac{d^2M}{dx_i^2} - \frac{d}{dx_i}\Big(N\frac{dw}{dx_i}\Big) = q_T \qquad \text{or} \qquad M'' - (Nw')' = q_T \tag{2.4}$$

The incremental problem with respect to the equilibrium state has the form

$$\begin{aligned}
&\delta e = \delta\varepsilon + \delta\gamma = \delta\Big(\frac{du}{dx_i}\Big) + \frac{dw}{dx_i}\delta\Big(\frac{dw}{dx_i}\Big) = \delta u' + w'\delta w' \\
&\delta\kappa = \delta\Big(\frac{d^2w}{dx_i^2}\Big) = \delta w'' \\
&\delta\Big(\frac{dN}{dx_i}\Big) = \delta N' = -\delta q_N \\
&\delta\Big(\frac{d^2M}{dx_i^2}\Big) - \frac{d}{dx_i}\Big[\delta N\frac{dw}{dx_i} + N\delta\Big(\frac{dw}{dx_i}\Big)\Big] = \delta M'' - (\delta N w' + N\delta w')' = \delta q_T
\end{aligned} \tag{2.5}$$

where δ denotes the small increment from the given state.

Parallely with the primary incremental problem, let us introduce *an adjoint structure* for which the static fields are $M^a(x_i)$, $N^a(x_i)$, $Q^a(x_i)$, $q^a(x_i)$ and the displacement and strain fields are denoted by $u^a(x_i)$, $w^a(x_i)$, $e^a(x_i)$, $\kappa^a(x_i)$. The strain-displacement relation and equilibrium conditions are identical to incremental equations (2.5), namely

$$\begin{aligned}
&e^a = \varepsilon^a + \gamma^a = \frac{du^a}{dx_i} + \frac{dw}{dx_i}\frac{dw^a}{dx_i} = u^{a\prime} + w'w^{a\prime} \\
&\kappa^a = \frac{d^2w^a}{dx_i^2} = w^{a\prime\prime} \\
&\frac{dN^a}{dx_i} = N^{a\prime} = -q_N^a \\
&\frac{d^2M^a}{dx_i^2} - \frac{d}{dx_i}\Big(N^a\frac{dw}{dx_i} + N\frac{dw^a}{dx_i}\Big) = M^{a\prime\prime} - (N^a w' + N w^{a\prime})' = q_T^a
\end{aligned} \tag{2.6}$$

where

$$Q^a = M^{a\prime} - (N^a w' + N w^{a\prime}) = Q_M^a + Q_N^a$$

The adjoint problem is characterized by the linear equations (2.6) similarly as the primary incremental problem. The loading and boundary conditions, however, will follow from the form of a functional whose sensitivity is analysed. The aim of the adjoint problem is to eliminate variations of primary state fields which should be determined in the direct approach (cf Dems and Mróz (1989), Mróz and Haftka (1994)).

The virtual work equations can be written for the static field of the adjoint structure and kinematic field variations of the primary structure. We have

$$\begin{aligned}
&\sum_{i=1}^{n}\Big[\int_0^{l_i} M^a\delta\kappa\;dx_i + \int_0^{l_i}(N^a\Theta + N\Theta^a)\delta\Theta\;dx_i\Big] = \\
&\qquad = \sum_{i=1}^{n}\Big[|M^a\delta\Theta|_0^{l_i} - |Q^a\delta w|_0^{l_i} + \int_0^{l_i} q_T^a\delta w\;dx_i\Big] \\
&\sum_{i=1}^{n}\int_0^{l_i} N^a\delta\varepsilon\;dx_i = \sum_{i=1}^{n}\Big[|N^a\delta u|_0^{l_i} + \int_0^{l_i} q_N^a\delta u\;dx_i\Big]
\end{aligned} \tag{2.7}$$

and after adding, we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\Big[\int_0^{l_i} M^a\delta\kappa\;dx_i + \int_0^{l_i}(N^a\Theta + N\Theta^a)\delta\Theta\;dx_i + \int_0^{l_i} N^a\delta\varepsilon\;dx_i\Big] = \\
&= \sum_{i=1}^{n}\Big[|M^a\delta\Theta|_0^{l_i} - |Q^a\delta w|_0^{l_i} + |N^a\delta u|_0^{l_i} + \int_0^{l_i} q_T^a\delta w\;dx_i + \int_0^{l_i} q_N^a\delta u\;dx_i\Big]
\end{aligned} \tag{2.8}$$

where $\Theta = w'$, $\Theta^a = w^{a'}$ are the deflection slopes.

Similarly, considering static field variations of the primary structure and the kinematic field of the adjoint structure, we can write

$$\begin{aligned}
&\sum_{i=1}^{n}\Big[\int_0^{l_i} \delta M\kappa^a\;dx_i + \int_0^{l_i}\delta(Nw')w^{a'}\;dx_i\Big] = \sum_{i=1}^{n}\Big[\int_0^{l_i} \delta M\kappa^a\;dx_i + \int_0^{l_i}\delta N\Theta\Theta^a\;dx_i + \\
&+ \int_0^{l_i} N\delta\Theta\Theta^a\;dx_i\Big] = \sum_{i=1}^{n}\Big[|\delta M\Theta^a|_0^{l_i} - |\delta Q w^a|_0^{l_i} + \int_0^{l_i}\delta q_T w^a\;dx_i\Big] \\
&\sum_{i=1}^{n}\int_0^{l_i}\delta N\varepsilon^a\;dx_i = \sum_{i=1}^{n}\Big[|\delta N u^a|_0^{l_i} + \int_0^{l_i}\delta q_N u^a\;dx_i\Big]
\end{aligned} \tag{2.9}$$

After adding two equalities in Eq (2.9), one has

$$\sum_{i=1}^{n}\Big[\int_0^{l_i} \delta M\kappa^a\,dx_i + \int_0^{l_i} \delta N\Theta\Theta^a\,dx_i + \int_0^{l_i} N\delta\Theta\Theta^a\,dx_i + \int_0^{l_i} \delta N\varepsilon^a\,dx_i\Big] =$$
$$= \sum_{i=1}^{n}\Big[|\delta M\Theta^a|_0^{l_i} - |\delta Q w^a|_0^{l_i} + |\delta N u^a|_0^{l_i} + \int_0^{l_i} \delta q_T w^a\,dx_i + \int_0^{l_i} \delta q_N u^a\,dx_i\Big] \tag{2.10}$$

Now, let us consider the sensitivity variations or derivatives for several types of structure modification.

3. Variation of cross-sectional stiffness

Consider the displacement functional

$$G = \sum_{i=1}^{n}\int_0^{l_i} F(w)\,dx_i \tag{3.1}$$

and assume the adjoint structure to be loaded by the transverse load

$$q_T^a = \frac{\partial F}{\partial w} \qquad\qquad q_N^a = 0 \tag{3.2}$$

The variation of G with respect to cross-section variation equals

$$\delta G = \sum_{i=1}^{n}\int_0^{l_i} \frac{\partial F}{\partial w}\delta w\,dx_i = \sum_{i=1}^{n}\int_0^{l_i} q_T^a\delta w\,dx_i \tag{3.3}$$

The virtual work equation (2.8), after using Eq (2.5) can be written as follows

$$\sum_{i=1}^{n}\int_0^{l_i} q_T^a\delta w\,dx_i = \sum_{i=1}^{n}\Big[\int_0^{l_i} M^a\delta\kappa\,dx_i + \int_0^{l_i} N\Theta^a\delta\Theta\,dx_i + \int_0^{l_i} N^a\delta e\,dx_i\Big] \tag{3.4}$$

where the boundary terms were assumed to vanish in view of the boundary support conditions and displacement continuity conditions in joints. Similarly, the virtual stress equation (2.10) can be presented in the form

$$\sum_{i=1}^{n}\Big[\int_0^{l_i} \delta M\kappa^a\,dx_i + \int_0^{l_i} N\Theta^a\delta\Theta\,dx_i + \int_0^{l_i} \delta N e^a\,dx_i\Big] = 0 \tag{3.5}$$

where in view of both the equilibrium conditions in joints and fixed loading for the primary structure, the boundary terms vanish.

Further, the following relations occur

$$\sum_{i=1}^{n}\int_{0}^{l_i} M^a\delta\kappa\, dx_i = \sum_{i=1}^{n}\int_{0}^{l_i} M^a\delta\Big(\frac{M}{EI}\Big)\, dx_i = \sum_{i=1}^{n}\Big[\int_{0}^{l_i}\delta M\kappa^a\, dx_i - \int_{0}^{l_i}\kappa\kappa^a\delta(EI)dx_i\Big]$$
$$\sum_{i=1}^{n}\int_{0}^{l_i} N^a\delta e\, dx_i = \sum_{i=1}^{n}\int_{0}^{l_i} N^a\delta\Big(\frac{N}{EA}\Big)\, dx_i = \sum_{i=1}^{n}\Big[\int_{0}^{l_i}\delta N e^a\, dx_i - \int_{0}^{l_i} ee^a\delta(EA)\, dx_i\Big] \tag{3.6}$$

Using Eqs (3.4), (3.5) and (3.6), the variation of G specified by Eqs (3.3) can now be expressed in the form

$$\delta G = -\sum_{i=1}^{n}\Big[\int_{0}^{l_i}\kappa\kappa^a\delta(EI)\, dx_i + \int_{0}^{l_i} ee^a\delta(EA)\, dx_i\Big] \tag{3.7}$$

that is by the primary and adjoint strain fields. Let us note that there is no need to specify the variations δw, δM and δN. For a stress functional, the identical sensitivity expression Eq (3.7) is obtained.

4. Variation of length of structural members

Consider now the sensitivity of the displacement functional Eq (3.1) due to length variation of an external structural member assuming its support conditions independent of this variation. Assume the first beam member of length l_1 to vary its length. We have

$$\delta G = \sum_{i=1}^{n}\int_{0}^{l_i}\frac{\partial F}{\partial w}\delta w\, dx_i + F(w)\Big|_{x_1=l_1}\delta l_1 = \sum_{i=1}^{n}\int_{0}^{l_i} q_T^a\delta w\, dx_i + F(w)\Big|_{x_1=l_1}\delta l_1 \tag{4.1}$$

The idea of referring the static and kinematic fields to the primary configuration is used, Fig.2. The proper equalities for the static fields in view of Eqs (2.2) ÷ (2.4) have the form

$$\begin{aligned}
\delta\bar{M} &= \delta M - \frac{dM}{dx_1}\Big|_{x_1=l_1}\delta l_1 = \delta M - (Q + N\Theta)\Big|_{x_1=l_1}\delta l_1 \\
\delta\bar{Q} &= \delta Q - \frac{dQ}{dx_1}\Big|_{x_1=l_1}\delta l_1 = \delta Q - q_T\Big|_{x_1=l_1}\delta l_1 \qquad (4.2)\\
\delta\bar{N} &= \delta N - \frac{dN}{dx_1}\Big|_{x_1=l_1}\delta l_1 = \delta N - q_N\Big|_{x_1=l_1}\delta l_1
\end{aligned}$$

where $\delta\bar{M}$, $\delta\bar{Q}$, $\delta\bar{N}$ are the variations referred to the primary configuration.

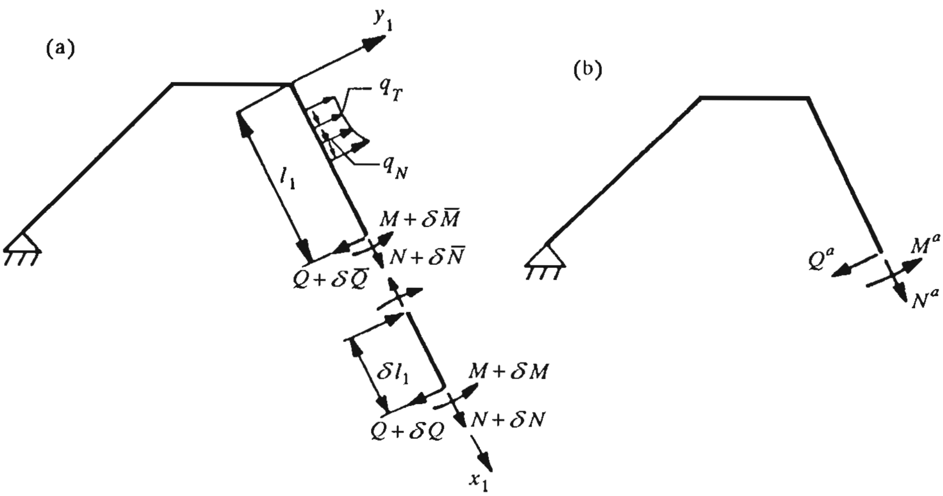


Fig. 2. (a) – primary frame – referring the static fields to the initial configuration; (b) – adjoint frame

Similar relations occur for the kinematic fields, thus

$$\begin{aligned}
\delta\bar{\Theta} &= \delta\Theta - \frac{d\Theta}{dx_1}\Big|_{x_1=l_1}\delta l_1 = \delta\Theta - \kappa\Big|_{x_1=l_1}\delta l_1 \\
\delta\bar{w} &= \delta w - \frac{dw}{dx_1}\Big|_{x_1=l_1}\delta l_1 = \delta w - \Theta\Big|_{x_1=l_1}\delta l_1 \qquad (4.3)\\
\delta\bar{u} &= \delta u - \frac{du}{dx_1}\Big|_{x_1=l_1}\delta l_1 = \delta u - \varepsilon\Big|_{x_1=l_1}\delta l_1
\end{aligned}$$

Writing the virtual work equation (2.8) and the virtual stress equation (2.10), by virtue of Eq (4.2) and (4.3), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{n}\Big(\int_0^{l_i} q_T^a \delta w dx_i + |M^a \delta\Theta|_0^{l_i} - |Q^a \delta w|_0^{l_i} + |N^a \delta u|_0^{l_i}\Big) - (M^a \kappa - Q^a \Theta + \\
&+N^a \varepsilon)\Big|_0^{l_i} \delta l_1 = \sum_{i=1}^{n}\Big(\int_0^{l_i} M^a \delta\bar{\kappa} dx_i + \int_0^{l_i} N\Theta^a \delta\bar{\Theta} dx_i + \int_0^{l_i} N^a \delta\bar{e} dx_i\Big)
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
&\sum_{i=1}^{n}\Big(|\delta M \Theta^a|_0^{l_i} - |\delta Q w^a|_0^{l_i} + |\delta N u^a|_0^{l_i}\Big) - \Big[(Q + N\Theta)\Theta^a - q_T w^a + \\
&-q_N u^a\Big]\Big|_0^{l_i} \delta l_1 = \sum_{i=1}^{n}\Big(\int_0^{l_i} \delta\bar{M} \kappa^a dx_i + \int_0^{l_i} N\Theta^a \delta\bar{\Theta} dx_i + \int_0^{l_i} \delta\bar{N} e^a dx_i\Big)
\end{aligned}
\tag{4.5}
$$

The following equalities for total variations are assumed to occur in view of the fixed boundary conditions, independent both of beam end position and of equilibrium and continuity conditions in joints

$$
\begin{aligned}
&\sum_{i=1}^{n}\Big(|M^a \delta\Theta|_0^{l_i} - |Q^a \delta w|_0^{l_i} + |N^a \delta u|_0^{l_i}\Big) = 0 \\
&\sum_{i=1}^{n}\Big(|\delta M \Theta^a|_0^{l_i} - |\delta Q w^a|_0^{l_i} + |\delta N u^a|_0^{l_i}\Big) = 0
\end{aligned}
\tag{4.6}
$$

In view of Eqs (4.4) ÷ (4.6), the variation δG specified by Eq (4.1) can be presented in the form

$$
\delta G = [M^a \kappa - (Q^a \Theta + Q\Theta^a) - N\Theta\Theta^a + N^a \varepsilon + q_T w^a + q_N u^a + F(w)|_0^{l_i}]\delta l_1 \tag{4.7}
$$

The sensitivity analysis of displacement functional due to length variation of an internal member is not considered here. This problem can be treated as a variational translation of a certain number of structural joints (see Section 6).

5. Rotation of a structure member

Consider now the sensitivity of the displacement functional (Eq (3.1)) due to rigid rotation of a plane structure. Denote by $\delta\varphi$ the infinitesimal variation of the orientation angle φ, Fig.3. Introduce global coordinate systems:

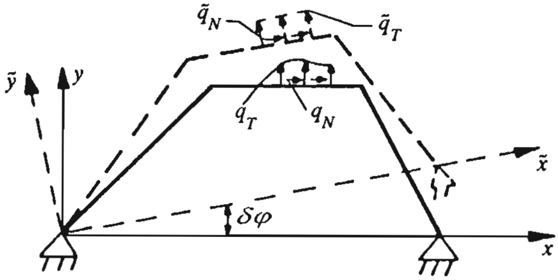


Fig. 3. Rotation of the structure

(x,y) related to the primary configuration and $(\tilde{x},\tilde{y})$ related to the rotated configuration. Similarly, introduce the local coordinate systems (x_i,y_i) and $(\tilde{x}_i,\tilde{y}_i)$ $i=1,2,...,n$ connected to the structural members. The variation of functional G now takes the form in terms of corotational components

$$\delta G = \sum_{i=1}^{n} \int_0^{l_i} \frac{\partial F}{\partial \tilde{w}} \delta\tilde{w} dx_i = \sum_{i=1}^{n} \int_0^{l_i} \tilde{q}_T^a \delta\tilde{w} dx_i \tag{5.1}$$

The virtual work equation (2.8) and virtual stress equation (2.10) can be written as follows

$$\sum_{i=1}^{n} \int_0^{l_i} \tilde{q}_T^a \delta\tilde{w} dx_i = \sum_{i=1}^{n} \Big(\int_0^{l_i} \widetilde{M}^a \delta\tilde{\kappa} dx_i + \int_0^{l_i} \tilde{N}\tilde{\Theta}^a \delta\tilde{\Theta} dx_i + \int_0^{l_i} \tilde{N}^a \delta\tilde{e} dx_i \Big) \tag{5.2}$$

$$\begin{aligned} &\sum_{i=1}^{n} \Big(\int_0^{l_i} \delta\tilde{q}_T \tilde{w}^a dx_i + \int_0^{l_i} \delta\tilde{q}_N \tilde{u}^a dx_i \Big) + \sum_{j=1}^{m} \Big(\delta\tilde{R}_{xj}\tilde{v}_{xj}^a + \delta\tilde{R}_{yj}\tilde{v}_{yj}^a \Big) = \\ &= \sum_{i=1}^{n} \Big(\int_0^{l_i} \delta\widetilde{M}\tilde{\kappa}^a dx_i + \int_0^{l_i} \tilde{N}\tilde{\Theta}^a \delta\tilde{\Theta} dx_i + \int_0^{l_i} \delta\tilde{N}\tilde{e}^a dx_i \Big) \end{aligned} \tag{5.3}$$

where tilde over a symbol denotes the quantity referred to the coordinate systems rotating with the structure. Moreover, m denotes the number of supports, R_{xj}, R_{yj} and v_{xj}^a, v_{yj}^a are the components of support reactions and corresponding displacement vectors, respectively.

Considering the problem at $\varphi = 0$, we have

$$\begin{aligned} w^a &= \tilde{w}^a & u^a &= \tilde{u}^a \\ v_{xj}^a &= \tilde{v}_{xj}^a & v_{yj}^a &= \tilde{v}_{yj}^a \end{aligned} \tag{5.4}$$

In view of Eqs (5.2) ÷ (5.4) the variation δG specified by Eq (5.1), can now be presented as follows

$$\delta G = \sum_{i=1}^{n} \Big(\int_0^{l_i} \delta \widetilde{q}_T w^a dx_i + \int_0^{l_i} \delta \widetilde{q}_N u^a dx_i \Big) + \sum_{i=1}^{n} \Big(\delta \widetilde{R}_{xj} v_{xj}^a + \delta \widetilde{R}_{yj} v_{yj}^a \Big) \tag{5.5}$$

If the loads $q_T(x_i)$, $q_N(x_i)$ preserve constant directions with respect to the beam elements and rotate with the structure (follower loads), then

$$\delta \widetilde{q}_N = 0 \qquad \qquad \delta \widetilde{q}_T = 0 \tag{5.6}$$

Now, we consider the case, when the loads $q_T(x_i)$, $q_N(x_i)$ are configuration-independent and remain constant in the global coordinate system (x, y) during the rotation of a structure. The relation between loads in the primary and rotated configurations has the form

$$\widetilde{q}_N = q_N \cos\varphi + q_T \sin\varphi \qquad \qquad \widetilde{q}_T = -q_N \sin\varphi + q_T \cos\varphi \tag{5.7}$$

Taking variations at $\varphi = 0$, we have

$$\delta \widetilde{q}_N = q_T \delta\varphi \qquad \qquad \delta \widetilde{q}_T = -q_N \delta\varphi \tag{5.8}$$

Similar formulae can also be provided for the variation of support reactions.

Other types of variations of loads and support reactions variations can be considered similarly to the case of rotation.

6. Translation of structural joints

Consider the plane structure modification by translation of its joints, Fig.4a. Denote by $\delta \boldsymbol{s}$ the vector of variation of joint position. Denote by $1, 2, ..., k$ $(k \leq n)$ the beam elements connected at the joint 1. Introduce the global coordinate system (x, y) and denote by β the angle between the vector $\delta \boldsymbol{s}$ and the x-axis, by α_i the angle between the ith element and the x-axis. Introduce also the local coordinate system (x_i, y_i) $(i = 1, 2, ..., k)$ with its origin at the joint 1 and with the x_i-axis following the member i, Fig.4b. The coordinates of the translation vector $\delta \boldsymbol{s}$ in the local coordinate system (x_i, y_i) are

$$\begin{aligned} \delta s_x^{(i)} &= \cos(\alpha_i - \beta)\delta s \\ \delta s_y^{(i)} &= -\sin(\alpha_i - \beta)\delta s \qquad \qquad i = 1, 2, ..., k \end{aligned} \tag{6.1}$$

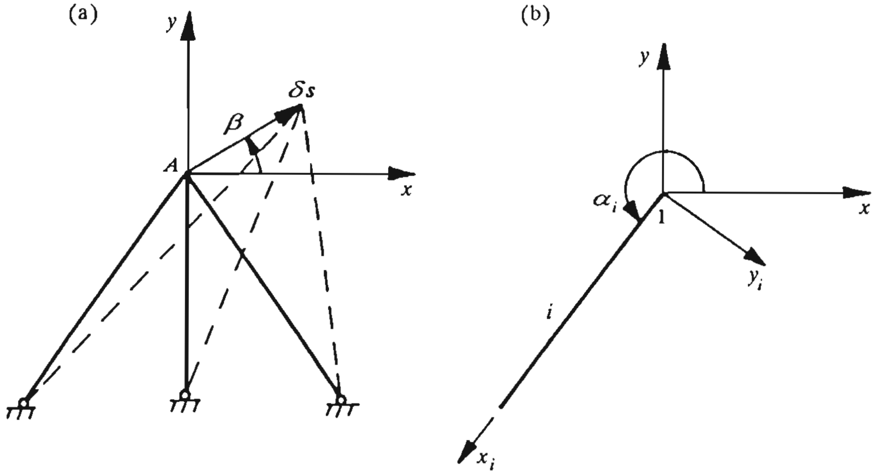


Fig. 4. Translation of the structure node: (a) – global coordinate system (x, y); (b) – local coordinate systems (x_i, y_i)

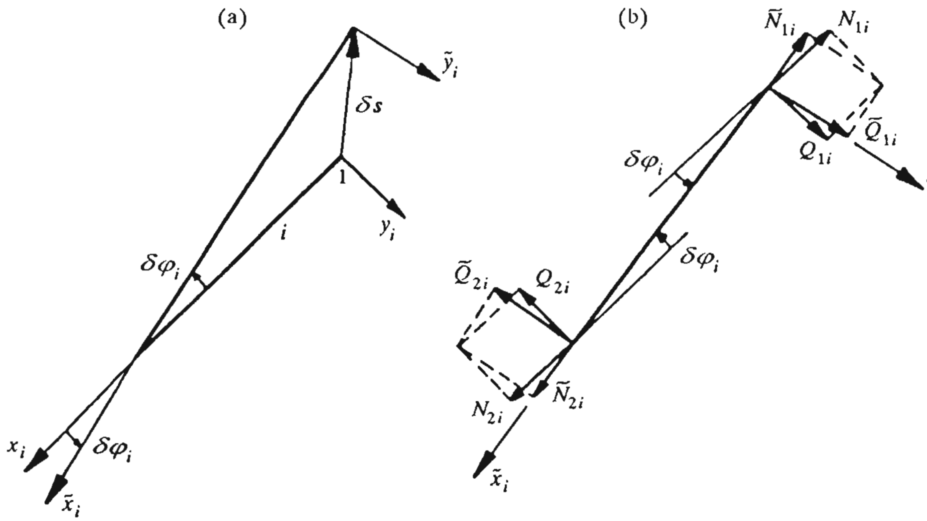


Fig. 5. (a) – local coordinate systems $(\tilde{x}_i, \tilde{y}_i)$ rotating with the member; (b) – transformation of the internal forces from the coordinate system $(\tilde{x}_i, \tilde{y}_i)$ to (x_i, y_i)

The variation of functional G now takes the form

$$\delta G = \sum_{i=1}^{n}\int_0^{l_i} \frac{\partial F}{\partial \widetilde{w}}\delta\widetilde{w}\,dx_i + \sum_{i=1}^{n} F(\widetilde{w})\Big|_{x_i=0}\delta l_i = \\ = \sum_{i=1}^{n}\int_0^{l_i} \widetilde{q}_T^a\delta\widetilde{w}\,dx_i + \sum_{i=1}^{n} F(\widetilde{w})\Big|_{x_i=0}\delta l_i \tag{6.2}$$

where as previously tilde over a symbol denotes the quantity referred to the local coordinate system $(\widetilde{x}_i, \widetilde{y}_i)$ rotating with the beam, Fig.5. Thus $\widetilde{u}$ and $\widetilde{w}$ denote axial and lateral displacements of a rotated member.

The virtual work equation expressed in terms of generalized stresses of the adjoint structure and of displacement variations in the rotated system can now be written as follows

$$\sum_{i=1}^{n}\int_0^{l_i} \widetilde{q}_T^a\delta\widetilde{w}dx_i + \sum_{i=1}^{k}\Big[(\widetilde{M}_{2i}^a\delta\widetilde{\Theta}_{2i} - \widetilde{M}_{1i}^a\delta\widetilde{\Theta}_{1i}) - (\widetilde{Q}_{2i}^a\delta\widetilde{w}_{2i} - \widetilde{Q}_{1i}^a\delta\widetilde{w}_{1i}) + \\ +(\widetilde{N}_{2i}^a\delta\widetilde{u}_{2i} - \widetilde{N}_{1i}^a\delta\widetilde{u}_{1i}) - (\widetilde{M}_{1i}^a\widetilde{\kappa}_{1i} - \widetilde{Q}_{1i}^a\widetilde{\Theta}_{1i} + \widetilde{N}_{1i}^a\widetilde{\varepsilon}_{1i})\delta l_i\Big] + \\ + \sum_{i=k+1}^{n}\Big(|\widetilde{M}^a\delta\widetilde{\Theta}|_0^{l_i} - |\widetilde{Q}^a\delta\widetilde{w}|_0^{l_i} + |\widetilde{N}^a\delta\widetilde{u}|_0^{l_i}\Big) = \\ = \sum_{i=1}^{n}\Big(\int_0^{l_i} \widetilde{M}^a\delta\widetilde{\kappa}\,dx_i + \int_0^{l_i} \widetilde{N}\Theta^a\delta\widetilde{\Theta}\,dx_i + \int_0^{l_i} \widetilde{N}^a\delta\widetilde{\varepsilon}\,dx_i\Big) \tag{6.3}$$

where elongation of members due to joint translation has accounted for.

Similarly, considering the generalized stress variation in the rotated member and displacement fields in the adjoint structure, we can write

$$\sum_{i=1}^{k}\Big\{(\delta\widetilde{M}_{2i}\widetilde{\Theta}_{2i}^a - \delta\widetilde{M}_{1i}\widetilde{\Theta}_{1i}^a) - (\delta\widetilde{Q}_{2i}\widetilde{w}_{2i}^a - \delta\widetilde{Q}_{1i}\widetilde{w}_{1i}^a) + (\delta\widetilde{N}_{2i}\widetilde{u}_{2i}^a - \delta\widetilde{N}_{1i}\widetilde{u}_{1i}^a) + \\ -\Big[(\widetilde{Q}_{1i} + \widetilde{N}_{1i}\widetilde{\Theta}_{1i})\widetilde{\Theta}_{1i}^a - \widetilde{q}_{T1i}\widetilde{w}_{1i}^a - \widetilde{q}_{N1i}\widetilde{u}_{1i}^a\Big]\delta l_i + \int_0^{l_i} \delta\widetilde{q}_T\widetilde{w}^a dx_i + \\ + \int_0^{l_i} \delta\widetilde{q}_N\widetilde{u}^a dx_i\Big\} + \sum_{i=k+1}^{n}\Big(|\delta\widetilde{M}\widetilde{\Theta}^a|_0^{l_i} - |\delta\widetilde{Q}\widetilde{w}^a|_0^{l_i} + |\delta\widetilde{N}\widetilde{u}^a|_0^{l_i}\Big) = \tag{6.4}$$

$$= \sum_{i=1}^{n} \Big(\int_0^{l_i} \delta\widetilde{M}\widetilde{\kappa}^a \, dx_i + \int_0^{l_i} \widetilde{N}\widetilde{\Theta}^a \delta\widetilde{\Theta} \, dx_i + \int_0^{l_i} \delta\widetilde{N}\widetilde{\varepsilon}^a \, dx_i \Big)$$

where $i = 1, 2, ..., k$ are the members undergoing rotation and/or elongation. Further, M_{1i}, Q_{1i}, N_{1i}, Θ_{1i}, w_{1i}, u_{1i} denote the generalized stresses and displacements of beam i at the joint 1, and M_{2i}, Q_{2i}, N_{2i}, Θ_{2i}, w_{2i}, u_{2i} are the generalized stresses and displacements of beam i at the joint 2.

Let us now express the nodal quantities in the rotated and initial local coordinate systems $(\widetilde{x}_i, \widetilde{y}_i)$ and (x_i, y_i). Denoting by φ_i the rotation angle of the member relative its initial configuration, we have

$$\widetilde{u}_{ji} = u_{ji} \cos\varphi_i + w_{ji} \sin\varphi_i \qquad \widetilde{w}_{ji} = -u_{ji} \sin\varphi_i + w_{ji} \cos\varphi_i \tag{6.5}$$

and

$$\widetilde{N}_{ji} = N_{ji} \cos\varphi_i - Q_{ji} \sin\varphi_i \qquad \widetilde{Q}_{ji} = N_{ji} \sin\varphi_i + Q_{ji} \cos\varphi_i \tag{6.6}$$

where $j = 1, 2$ and $i = 1, 2, ..., k$.

Obviously, there is $\widetilde{u}_{ji} = u_{ji}$, $\widetilde{w}_{ji} = w_{ji}$, $\widetilde{Q}_{ji} = Q_{ji}$, $\widetilde{N}_{ji} = N_{ji}$ for $\varphi_i = 0$. Considering variations of Eqs (6.5) and (6.6) at $\varphi_i = 0$, we have

$$\begin{aligned} \delta\widetilde{u}_{ji} &= \delta u_{ji} + w_{ji}\frac{\sin(\alpha_i - \beta)}{l_i}\delta s \qquad & \delta\widetilde{w}_{ji} &= \delta w_{ji} - u_{ji}\frac{\sin(\alpha_i - \beta)}{l_i}\delta s \\ \delta\widetilde{N}_{ji} &= \delta N_{ji} - Q_{ji}\frac{\sin(\alpha_i - \beta)}{l_i}\delta s \qquad & \delta\widetilde{Q}_{ji} &= \delta Q_{ji} + N_{ji}\frac{\sin(\alpha_i - \beta)}{l_i}\delta s \end{aligned} \tag{6.7}$$

and

$$\delta\widetilde{M}_{ji} = \delta M_{ji} \qquad \delta\widetilde{\Theta}_{ji} = \delta\Theta_{ji} \tag{6.8}$$

where the following geometric relations are used

$$\delta(\cos\varphi_i) = 0 \qquad \delta(\sin\varphi_i) = \frac{\sin(\alpha_i - \beta)}{l_i} = \delta s \tag{6.9}$$

and also

$$\delta l_i = -\cos(\alpha_i - \beta)\delta s \tag{6.10}$$

In view of Eqs (6.7) and (6.8) and (6.10), the virtual work equation (6.3) can now be written as follows

$$\begin{aligned}
&\sum_{i=1}^{n}\Big[\int_0^{l_i} \widetilde{q}_T^a \delta\widetilde{w} dx_i + \Big(|M^a\delta\Theta|_0^{l_i} - |Q^a\delta w|_0^{l_i} + |N^a\delta u|_0^{l_i}\Big)\Big] + \\
&+\sum_{i=1}^{k}\Big[(M_{1i}^a\kappa_{1i} - Q_{1i}^a\Theta_{1i} + N_{1i}^a\varepsilon_{1i})\cos(\alpha_i - \beta) + \\
&+(Q_{2i}^a u_{2i} + N_{2i}^a w_{2i} - Q_{1i}^a u_{1i} - N_{1i}^a w_{1i})\frac{\sin(\alpha_i - \beta)}{l_i}\Big]\delta s = \\
&= \sum_{i=1}^{n}\Big(\int_0^{l_i} \widetilde{M}^a\delta\widetilde{\kappa}\, dx_i + \int_0^{l_i} \widetilde{N}\widetilde{\Theta}^a\delta\widetilde{\Theta}\, dx_i + \int_0^{l_i} \widetilde{N}^a\delta\widetilde{\varepsilon}\, dx_i\Big)
\end{aligned} \tag{6.11}$$

and similarly the virtual stress equation (6.4) takes the form

$$\begin{aligned}
&\sum_{i=1}^{n}\Big(|\delta M\Theta^a|_0^{l_i} - |\delta Q w^a|_0^{l_i} + |\delta N u^a|_0^{l_i}\Big) + \sum_{i=1}^{n}\Big(\int_0^{l_i} \delta\widetilde{q}_T w^a dx_i + \int_0^{l_i} \delta\widetilde{q}_N u^a dx_i\Big) + \\
&+\sum_{i=1}^{n}\Big\{\Big[(Q_{1i} + N_{1i}\Theta_{1i})\Theta_{1i}^a - q_{T1i}w_{1i}^a - q_{N1i}u_{1i}^a\Big]\cos(\alpha_i - \beta) + \\
&+(-Q_{2i}^a u_{2i} - N_{2i}^a w_{2i} + Q_{1i}^a u_{1i} + N_{1i}^a w_{1i})\frac{\sin(\alpha_i - \beta)}{l_i}\Big\}\delta s = \\
&= \sum_{i=1}^{n}\Big(\int_0^{l_i} \delta\widetilde{M}\widetilde{\kappa}^a\, dx_i + \int_0^{l_i} \widetilde{N}\widetilde{\Theta}^a\delta\widetilde{\Theta}\, dx_i + \int_0^{l_i} \delta\widetilde{N}\widetilde{\varepsilon}^a\, dx_i\Big)
\end{aligned} \tag{6.12}$$

Eqs (4.5) occur in view of joint equilibrium and continuity conditions for the adjoint and primary structures. In view of Eqs (6.11), (6.12) and (4.5), the sensitivity of G is obtained in the form

$$\begin{aligned}
\delta G = &\sum_{i=1}^{k}\Big\{\Big[-M_{1i}^a\kappa_{1i} + Q_{1i}^a\Theta_{1i} - N_{1i}^a\varepsilon_{1i} + (Q_{1i} + N_{1i}\Theta_{1i})\Theta_{1i}^a + \\
&-q_{T1i}w_{1i}^a - q_{N1i}u_{1i}^a - F(\widetilde{w})\Big|_{x_i=0}\Big]\cos(\alpha_i - \beta) + (Q_{1i}^a u_{1i} + N_{1i}^a w_{1i} + \\
&+Q_{1i}^a u_{1i} + N_{1i}^a w_{1i} - Q_{2i}^a u_{2i} - N_{2i}^a w_{2i} - Q_{2i}^a u_{2i} - N_{2i}^a w_{2i})\frac{\sin(\alpha_i - \beta)}{l_i}\Big\}\delta s + \\
&+\sum_{i=1}^{k}\Big(\int_0^{l_i} \delta\widetilde{q}_T w^a\, dx_i + \int_0^{l_i} \delta\widetilde{q}_N u^a\, dx_i\Big)
\end{aligned} \tag{6.13}$$

where the first term in cubic brackets corresponds to length variation in the structural members and the second term corresponds to their rotation. The last term corresponds to load variation in the rotating reference system. The explicit relations for these variations can be obtained similarly, as it was described by Eqs (5.6) ÷ (5.8), separately for each structural member.

7. Application: sensitivity of a non-linear beam

The relations derived in the preceding sections can now be used in order to determine sensitivity derivatives for different cross-section and configuration variations.

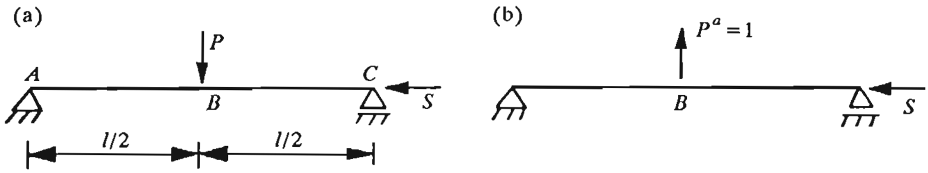


Fig. 6. (a) – primary beam; (b) – adjoint beam

Consider a simple example of a simply supported beam, loaded by the lateral force P at B and by the axial force S, Fig.6a. Let us calculate the sensitivity of the lateral deflection at B with respect to variation of cross-sectional stiffness EI, variation of beam length due to translation of support C, and rotation of beam about the support A. The displacement functional Eq (3.1) now is expressed as follows

$$G = \int_0^l \delta(x - x_B)w(x)\,dx = w_B \tag{7.1}$$

where $\delta(x)$ is the Dirac function.

The differential equilibrium equation of the beam is of the form

$$w'' + k^2 w = \begin{cases} \dfrac{P}{2EI}x & 0 \le x \le \dfrac{l}{2} \\ \dfrac{P}{2EI}(l - x) & \dfrac{l}{2} \le x \le l \end{cases} \tag{7.2}$$

where $k^2 = S/EI$. The beam deflection satisfying the boundary conditions $w(0) = w(l) = 0$ is expressed as follows

$$w(x) = \begin{cases} -\dfrac{P}{2Sk\cos(kl/2)}\sin kx + \dfrac{P}{2S}x & 0 \le x \le \frac{l}{2} \\ -\dfrac{P}{2Sk\cos(kl/2)}\sin k(l-x) + \dfrac{P}{2S}(l-x) & \frac{l}{2} \le x \le l \end{cases} \tag{7.3}$$

The deflection slope and curvature are calculated from Eq (7.3), namely

$$\Theta(x) = w'(x) = \begin{cases} -\dfrac{P}{2S\cos(kl/2)}\cos kx + \dfrac{P}{2S} & 0 \le x \le \frac{l}{2} \\ \dfrac{P}{2S\cos(kl/2)}\cos k(l-x) - \dfrac{P}{2S} & \frac{l}{2} \le x \le l \end{cases} \tag{7.4}$$

and

$$\kappa(x) = w''(x) = \begin{cases} \dfrac{Pk}{2S\cos(kl/2)}\sin kx & 0 \le x \le \frac{l}{2} \\ \dfrac{Pk}{2S\cos(kl/2)}\sin k(l-x) & \frac{l}{2} \le x \le l \end{cases} \tag{7.5}$$

The variation of functional Eq (7.1) is

$$\delta G = P^a \delta w_B \tag{7.6}$$

where P^a denotes the unit force applied at B to the adjoint beam, Fig.6b. The adjoint beam also carries the axial force S of the primary structure and the axial load $N^a = 0$, so the adjoint deflection, slope and curvature, respectively, are expressed by the same formulae as Eqs (7.3) ÷ (7.5) with $P = -P^a = -1$. Moreover, there is $e^a = N^a/(EA) = 0$.

Consider first the variation of cross-sectional stiffness. The formula (3.7) now takes the form

$$P^a \delta w_B = -\int\limits_0^l \kappa\kappa^a \, dx \delta(EI) \tag{7.7}$$

and in view of Eq (7.5), there is

$$\begin{aligned} \delta w_B &= \frac{Pk^2}{4S^2\cos^2(kl/2)}\Big[\int\limits_0^{\frac{l}{2}} \sin^2 kx \, dx + \int\limits_{\frac{l}{2}}^{l} \sin^2 k(l-x) \, dx\Big]\delta(EI) = \\ &= \frac{Pk}{8S^2\cos^2(kl/2)}(kl - \sin kl)\delta(EI) \end{aligned} \tag{7.8}$$

Consider next the translation of the beam end C. The formula (4.6) in this case has the form

$$P^a \delta w_B = -(Q^a \Theta + Q\Theta^a + N\Theta\Theta^a)\Big|_{x=l} \delta l \tag{7.9}$$

and since

$$\begin{aligned} \Theta(l) &= \frac{P}{2S\cos(kl/2)} - \frac{P}{2S} & \Theta^a(l) &= -\frac{1}{2S\cos(kl/2)} + \frac{1}{2S} \\ Q(l) &= -\frac{P}{2} \qquad N(l) = -S & Q^a(l) &= \frac{1}{2} \end{aligned} \tag{7.10}$$

we obtain

$$\delta w_B = \Big[-\frac{P}{4S\cos^2(kl/2)} + \frac{P}{4S}\Big]\delta l \tag{7.11}$$

Consider finally the beam rotation about the end A. When both loading and support orientation rotate with the beam, there is no variation of state fields and $\delta\tilde{w}_B = 0$.

In the case, when the loading P is beam orientation independent while the support reaction rotates with the structure, then from Eq (5.5) and in view of Eq (5.8), we have

$$\begin{aligned} \delta w_B &= -Pu^a_B \delta\varphi = P\int_0^{\frac{l}{2}} \Theta(x)\Theta^a(x)\,dx\delta\varphi = \\ &= \frac{P^2}{8S^2}\Big[-\frac{\tan(kl/2)}{\sin kl} + 3\frac{\tan(kl/2)}{kl} - 1\Big]l\delta\varphi \end{aligned} \tag{7.12}$$

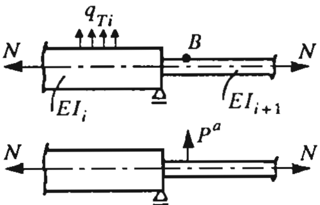
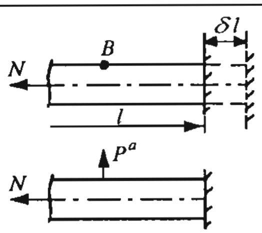
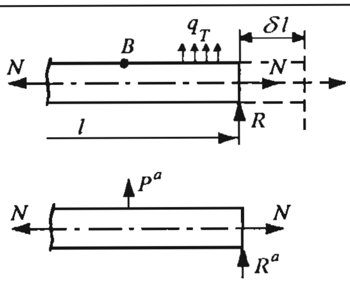
On the other hand, when the load P follows the beam transverse direction, and support reaction orientation is fixed, then we have

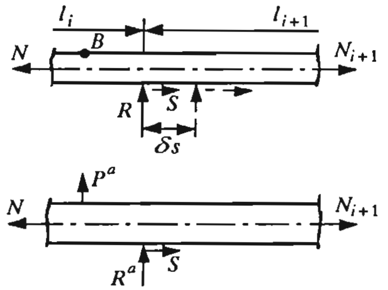
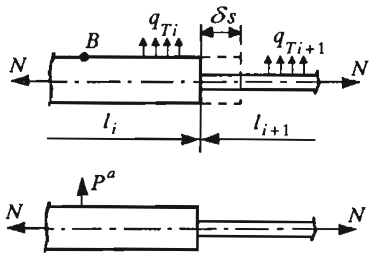
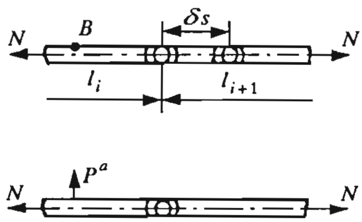
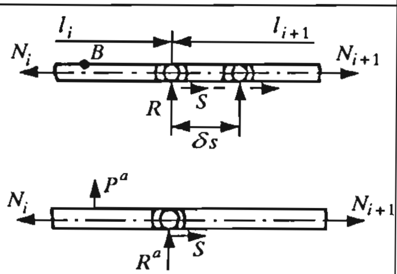
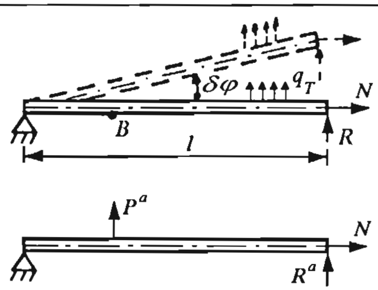
$$\begin{aligned} \delta w_B &= R_C u^a_C \delta\varphi = -\frac{P}{2}\int_0^{l} \Theta(x)\Theta^a(x)\,dx\delta\varphi = \\ &= -\frac{P^2}{8S^2}\Big[-\frac{\tan(kl/2)}{\sin kl} + 3\frac{\tan(kl/2)}{kl} - 1\Big]l\delta\varphi \end{aligned} \tag{7.13}$$

For the case of beam loaded by the axial force N at its ends, $q_N = 0$, and by concentrated or distributed transverse loading q_T, the expressions of sensitivity of the local transverse deflection at a selected point B are presented in Table 1. The adjoint beam has the same support conditions and is loaded by a concentrated lateral force P^a at B. It also carries the axial load of the primary beam.

Table 1 presents the sensitivity expressions for variation of different structure parameters, such as change of cross-sectional stiffness, translation of built-in or simply supported end, translation of internal support and variation of segment length, translation of supported or unsupported hinge and rotation of beam. The sensitivities are expressed in terms of local state values of primary and adjoint beams.

Table 1

	$P^a \delta w_B =$ $= -\sum_{i=1}^{n} \int_0^{l_i} \kappa\kappa^a \delta(EI) dx_i$	change of cross--sectional stiffness
	$P^a \delta w_B =$ $= (M^a \kappa)\Big\vert_{x=l} \delta l$	translation of built-in end
	$P^a \delta w_B =$ $= (R\Theta^a + R^a\Theta +$ $-N\Theta\Theta^a + q_T w^a)\Big\vert_{x=l} \delta l$	translation of simply supported end

	$P^a\delta w_B = (R\Theta^a + R^a\Theta + -S\Theta\Theta^a)\big\|_{x=l_i}\delta s,$ $S = N_i - N_{i+1}$	translation of internal rigid support
	$P^a\delta w_B =$ $= [M^a(\kappa_i - \kappa_{i+1}) + +(q_{Ti} - q_{Ti+1})w^a]\big\|_{x=l_i}\delta s$	translation of interface beetwen two segments of different stiffness
	$P^a\delta w_B =$ $= [Q(\Theta^a_{i+1} - Q^a_i) + +Q^a(\Theta_{i+1} - \Theta_i) + +N(\Theta_{i+1}\Theta^a_{i+1} + -\Theta_i\Theta^a_i)]\big\|_{x=l_i}\delta s$	translation of hinge
	$P^a\delta w_B = [(Q_{i+1} + +N_{i+1}\Theta_{i+1})\Theta_{i+1} + +Q^a_{i+1}\Theta_{i+1} + -(Q_i + N_i\Theta_i)\Theta^a_i + -Q^a_i\Theta_i]\big\|_{x=l_i}\delta s$	translation of supported hinge
	$P^a\delta w_B = \Big(\int_0^l q_T u^a dx + +Ru^a\big\|_{x=l}\Big)\delta\varphi$	rotation of simply supported beam with unchanged loading and support conditions

8. Concluding remarks

In this paper, the variational approach was used in deriving sensitivity derivatives with respect to cross-section or configuration parameters of frame or beam structures. The derived formulae could be useful in providing the assessment of structure response due to uncertainty or modification of its parameters. The sensitivity analysis can also be used directly in developing effective redesign and optimization algorithms.

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Analiza wrażliwości nieliniowych belek i ram

Streszczenie

W pracy są rozważane geometrycznie nieliniowe, sprężyste ramy, dla których analizuje się wpływ nieskończenie małych wariacji charakteryzujących je parametrów na wariację dowolnego funkcjonału przemieszczeniowego. Rozpatrzone są takie parametry jak sztywności elementów konstrukcji, ich długości, orientacja konstrukcji, a także położenie węzłów konstrukcji. Jawne wyrażenia na wariacje funkcjonału przemieszczeniowego są podane w zależności od rozwiązania konstrukcji podstawowej i sprzężonej oraz od wariacji parametru projektowania.

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