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PLANE CONTACT PROBLEM INVOLVING HEAT GENERATION AND RADIATION

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The stationary problem of an elastic heat conducting punch sliding over the rigid insulated half-space is investigated. The problem is solved under the assumption that the frictional heating inside the contact region and the heat radiation outside the contact zone are occured. The problem is reduced to a system of two singular integral equations which is solved numerically. The influence of the heat radiation on the contact zone and the temperature distribution in the punch is analyzed.

1. Introduction

For investigation of the thermoelastic proccesses in the rubbing contact it is necessary to have the solutions of the contact problems modelling these proccesses. The Hertz theory can not be accepted because it does not involve the thermal processes. It is known that the heat generation within the contact area has the great influence on interaction between the bodies. The problems involving frictional heating are usually solved with ideal boundary condition connected with the perfect insulation of the surface outside the contact zone (cf Barber (1976), Hills and Barber (1985), (1986)). This assumption, though simplifying the mathematical calculations is not realistic for the majority of frictional pairs, esspecially when coolling the body on free surface. Thus, the investigation into heat radiation is very important problem engineering point of view. In this work the influence of heat radiation has been analyzed in the case when one body is rigid and insulated, and heat is convected into moving body.

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2. Formulation

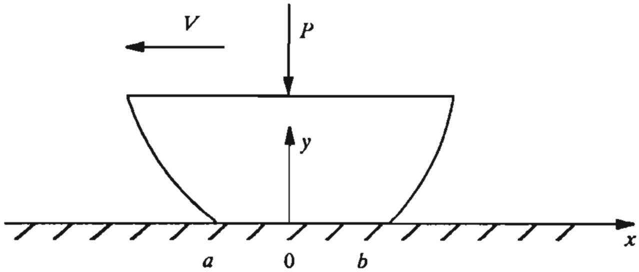

Fig. 1. Geometry of the contact

The geometry of contact problem is shown in Fig.1. The elastic heat conducting cylinder upon the load P is moving at a constant velocity V over the surface of the rigid insulated half-space. It is assumed that the friction forces $\sigma_{xy}(x)$ generate the heat flux $q(x)$ which is directed towards the punch. The thermoelastic proccesses in it are assumed to be stationary. Outside the contact zone (a,b) the heat radiation is described by the Newton's law. Mathematically, the problem is resolved to the solution of the thermoelasticity equations

$$\begin{aligned} 2(1-\nu)\frac{\partial^2 u}{\partial x^2} + (1-2\nu)\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} &= 2(1+\nu)\alpha_t \frac{\partial T}{\partial x} \\ (1-2\nu)\frac{\partial^2 v}{\partial x^2} + 2(1-\nu)\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} &= 2(1+\nu)\alpha_t \frac{\partial T}{\partial y} \end{aligned} \tag{2.1}$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{2.2}$$

with the boundary conditions at $y = 0$

$$K\frac{\partial T}{\partial y} = \begin{cases} -q(x) & \text{for} \quad x \in (a,b) \\ hT(x) & \text{for} \quad x \notin (a,b) \end{cases} \tag{2.3}$$

$$\sigma_{yy}(x) = \begin{cases} -p(x) & \text{for} \quad x \in (a,b) \\ 0 & \text{for} \quad x \notin (a,b) \end{cases} \tag{2.4}$$

$$\sigma_{xy}(x) = \begin{cases} fp(x) & \text{for} \quad x \in (a,b) \\ 0 & \text{for} \quad x \notin (a,b) \end{cases} \tag{2.5}$$

$$q(x) = -fVp(x) \qquad \text{for} \quad x \in (a,b) \tag{2.6}$$

$$\frac{dv}{dx} = -\frac{x}{R} \qquad \text{for} \quad x \in (a,b) \tag{2.7}$$

where

(x, y)	–	the Cartesian coordinate system
u, v	–	displacements
$\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$	–	stresses
T	–	temperature
p	–	contact pressure
ν	–	Poisson ratio
α_t	–	thermal expansion
K	–	thermal conductivity
h	–	radiation coefficient
f	–	coefficient of the Coulomb friction
R	–	radius of the cylinder.

3. Integral equations

Using the Fourier transforms to the solution of the problem, Eqs (2.2) and (2.3), we find that the temperature in punch has to satisfy the integral equation

$$T(x,y) - \frac{h}{\pi K}\int_a^b T(x',0)K(x-x',y)\,dx' = \frac{1}{\pi K}\int_a^b q(x')K(x-x',y)\,dx' \tag{3.1}$$

where

$$K(z,y) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\exp(-i\xi z)}{|\xi| + \frac{h}{K}} \exp(-y|\xi|)\,d\xi$$

In the case of $h = 0$ the well known result presented by Carslaw and Jaeger (1959) is obtained for $y = 0$

$$T(x) = \frac{1}{\pi K}\int_a^b q(x')(-\ln|x-x'| + c)\,dx' \tag{3.2}$$

where c is a constant.

Applying the Fourier transforms to the solution of the problem represented by Eqs (2.1), (2.4), (2.5) we obtain the formula for the displacements

$$\frac{dv}{dx} = \frac{dv_e}{dx} + \frac{dv_{th}}{dx} \tag{3.3}$$

Elastic displacements of the punch surface are determined by the equation

$$\frac{dv_e}{dx} = \frac{1-\nu}{\pi\mu}\int\limits_a^b \frac{p(x')}{x-x'}\,dx' - \frac{1-2\nu}{2\mu}fp(x) \tag{3.4}$$

where μ is the shear modulus.

Thermal displacements of the surface have the forms

$$\frac{dv_{th}}{dx} = \frac{\delta}{\pi}\int\limits_a^b [q(x') + hT(x')]R(x-x')\,dx' \tag{3.5}$$

where $\delta = (1+\nu)\alpha_t/K$ is the thermal distortivity and

$$R(z) = \frac{\mathrm{i}}{2}\int\limits_{-\infty}^{\infty} \frac{\exp(-\mathrm{i}\xi z)}{|\xi| + \frac{h}{K}}\mathrm{sgn}(\xi)\,d\xi$$

In the case of $h = 0$ the well known result given by Barber (1973) is obtained

$$\frac{dv_{th}}{dx} = \frac{\delta}{2}\int\limits_a^b q(x')\mathrm{sgn}(x-x')\,dx'$$

By satisfying the boundary condition (2.6) and (2.7) with the help of representations (3.3) ÷ (3.5) and using Eq (3.1) for $y = 0$ we obtain the following system of singular integral equations

$$-\frac{1-2\nu}{2\mu}fp(x) + \frac{1-\nu}{\pi\mu}\int\limits_a^b \frac{p(x')}{x-x'}\,dx' - \frac{\delta}{\pi}fV\int\limits_a^b p(x')R(x-x')\,dx' +$$

$$+\frac{\delta h}{\pi}\int\limits_a^b T(x')R(x-x')\,dx' = -\frac{x}{R} \qquad x \in (a,b) \tag{3.6}$$

$$T(x) - \frac{h}{\pi K}\int\limits_a^b T(x')K(x-x')\,dx' + \frac{fV}{\pi K}\int\limits_a^b p(x')K(x-x')\,dx' = 0$$

The contact pressure must satisfy the equilibrium condition

$$\int_a^b p(x)\, dx = -P \tag{3.7}$$

To obtain the system (3.6) and (3.7) expressed in terms of dimensionless variables the folloving notations are introduced

$$x = a_0 s + b_0 \qquad x' = a_0 r + b_0 \qquad y = a_0 t$$

$$a_0 = \frac{b-a}{2} \qquad b_0 = \frac{b+a}{2}$$

$$p(x') = \frac{P}{a_0} p^*(r) \qquad T(x') = \frac{fPV}{K} T^*(r)$$

Then the system of integral equations (3.6) and (3.7) may be rewritten in the form (the asterisks are omitted)

$$\begin{aligned} -f\mathrm{B}p(s) + \frac{1}{\pi}\int_{-1}^{1} \frac{p(r)}{r-s}\, dr - \frac{2f\mathrm{Pe}H}{\pi}\int_{-1}^{1} p(r)R(s-r)\, dr + \\ -\frac{2f\mathrm{Pe}H\mathrm{Bi}}{\pi}\int_{-1}^{1} T(r)R(s-r)\, dr = -\frac{A^2 s + C}{2\pi\alpha\beta}\frac{P_H}{P} \qquad |s| \le 1 \\ T(s) - \frac{\mathrm{Bi}}{\pi}\int_{-1}^{1} T(r)K(s-r)\, dr + \frac{1}{\pi}\int_{-1}^{1} p(r)K(s-r)\, dr = 0 \end{aligned} \tag{3.8}$$

$$\int_{-1}^{1} p(r)\, dr = -1 \tag{3.9}$$

where

$$H = \frac{\mu\delta k}{1-\nu} \qquad A = \frac{a_0}{a_H} \qquad C = \frac{b_0}{b_H}(1-2\alpha)A$$

$$\tan\pi\alpha = \frac{1}{f\mathrm{B}} \qquad \beta = 1-\alpha$$

and

B – Dundurs' parameter, $\mathrm{B} = (1-2\nu)/[2(1-\nu)]$

Bi – Biot's number, $\mathrm{Bi} = ha_0/K$

Pe – Peclet's number, $\mathrm{Pe} = Va_0/(2k)$
k – thermal diffusivity
a_H – half-width in corresponding isothermal problem
b_H – contact zone location in corresponding isothermal problem
P_H – load in corresponding isothermal problem.

The following relationship holds true (cf Timoshenko and Goodier (1934))

$$b_H = (1-2\alpha)a_H \qquad a_H^2 = \frac{P_H R}{2\pi\alpha\beta}\frac{1-\nu}{\mu}$$

The kernels $K(z)$ and $R(z)$ of integral equations (3.8) take the forms

$$K(z) = \frac{1}{2}\int\limits_{-\infty}^{\infty} \frac{\exp(-\mathrm{i}\zeta z)}{|\zeta| + \mathrm{Bi}}\,d\zeta = \int\limits_{0}^{\infty} \frac{\cos\zeta z}{\zeta + \mathrm{Bi}}\,d\zeta$$

$$R(z) = \frac{\mathrm{i}}{2}\int\limits_{-\infty}^{\infty} \frac{\exp(-\mathrm{i}\zeta z)\mathrm{sgn}\zeta}{|\zeta| + \mathrm{Bi}}\,d\zeta = \int\limits_{0}^{\infty} \frac{\sin\zeta z}{\zeta + \mathrm{Bi}}\,d\zeta$$

which can be calculated using the representations in terms of the special functions according to Abramowitz and Stegun (1964)

$$K(z) = -\mathrm{Ci}(z\mathrm{Bi})\cos(z\mathrm{Bi}) - \mathrm{si}(z\mathrm{Bi})\sin(z\mathrm{Bi})$$
$$R(z) = \mathrm{Ci}(z\mathrm{Bi})\sin(z\mathrm{Bi}) - \mathrm{si}(z\mathrm{Bi})\cos(z\mathrm{Bi})$$

where $\mathrm{si}(\cdot)$ and $\mathrm{Ci}(\cdot)$ are, respectivelly, sine and cosine integrals, i.e.

$$\mathrm{si}(x) = -\int\limits_{x}^{\infty} \frac{\sin t}{t}\,dt \qquad \mathrm{Ci}(x) = -\int\limits_{x}^{\infty} \frac{\cos t}{t}\,dt$$

The function $R(z)$ is regular and $K(z)$ displays the logarithmical singularity.

4. Discretization

The first equation of the system (3.8) is a Cauchy-type singular integral equation with index equal to -1 for an unknown function $p(r)$ expressed in terms of $T(r)$ and certain given functions. The second equation of Eq (3.8) is a Fredholm-type integral equation of the second kind in temperature $T(r)$, expressed in terms of $p(r)$ and certain known functions. It is well known that the contact pressure $p(r)$ can be assumed in the following form

$$p(r) = (1-r)^{\alpha}(1+r)^{\beta}\varphi(r) \qquad r \le 1 \qquad 0 < \alpha, \beta < 1$$

where $\varphi(r)$ is a regular function neither vanishing at $r = \pm 1$ not tending to infinity as $r \to \pm 1$.

Using the Gauss-Jacobi quadrature method (cf Erdogan et al. (1973); Belocerkovskiĭ and Lifanov (1985)) the discretized form of the system (3.8), (3.9) is obtained

$$\gamma_{0n} + \frac{1}{\pi}\sum_{k=1}^{n}\varphi(r_k)w_k\Big[\frac{1}{s_m - r_k} - 2f\mathrm{Pe}H\,R(s_m - r_k)\Big] + $$
$$-\frac{2f\mathrm{Pe}H\,\mathrm{Bi}}{\pi}\sum_{k=1}^{n}\frac{2}{n}T(\rho_k)R(s_m - \rho_k) = -\frac{A^2 s_m + C}{2\pi\alpha\beta} \qquad m = 1,\dots,n+1$$
$$(4.1)$$
$$T(\rho_m) - \frac{\mathrm{Bi}}{\pi}\sum_{k=1}^{n}T(\rho_k)\Theta_{km} + \frac{1}{\pi}\sum_{k=1}^{n}w_k\varphi(r_k)K(\rho_m - r_k) = 0 \qquad m = 1,\dots,n$$
$$\sum_{k=1}^{n}w_k\varphi(r_k) = -1$$

where

$$P_n^{(\alpha,\beta)}(r_k) = 0 \qquad w_k = -\frac{\pi}{2^k \sin\pi\alpha}\,\frac{P_{n+1}^{(-\alpha,-\beta)}(r_k)}{P_n^{(\alpha,\beta)\prime}(r_k)} \qquad k = 1,\dots,n$$
$$P_{n+1}^{(-\alpha,-\beta)}(s_m) = 0 \qquad m = 1,\dots,n+1$$
$$\rho_k = -1 + \frac{2k-1}{n} \qquad k = 1,\dots,n$$
$$\Theta_{km} = \frac{\pi}{2\mathrm{Bi}}(\mathrm{sgn}X_1 - \mathrm{sgn}X_2) - \frac{1}{\mathrm{Bi}}[R(X_1) - R(X_2)]$$
$$X_1 = \frac{2}{n}\Big(m - k + \frac{1}{2}\Big) \qquad X_2 = \frac{2}{n}\Big(m - k - \frac{1}{2}\Big) \qquad k, m = 1,\dots n$$

$P_n^{(\alpha,\beta)}(\cdot)$ denotes the Jacobi polynomial of degree n with constants α and β, w_k are the corresponding wieghts.

The constant γ_{0n} was introduced into system (4.1). It is known (cf Belocerkovskiĭ and Lifanov (1985)) that

$$\lim_{n\to\infty}\gamma_{0n} = 0$$

Thus the condition $|\gamma_{0n}| < \varepsilon$ permits the choice of number n.

5. Results

Following Hills and Barber (1986), we assume that the contact zone is the same as in the corresponding isothermal case, i.e. $a_0/a_H = b_0/b_H = 1$. To determination of the value P_H/P Eq (3.9) is applied. Note that another way of solution is possible, i.e. assuming that the ratio P_H/P is an independent parameter and the contact area limits are found by using the iteration procedures.

The parameters f, B, Bi and $f\mathrm{Pe}H$ are given. Hills and Barber (1985) gave the values of parameters B and H for some metals. In our analysis we take $f = 0.4$, B $= 0.3$.

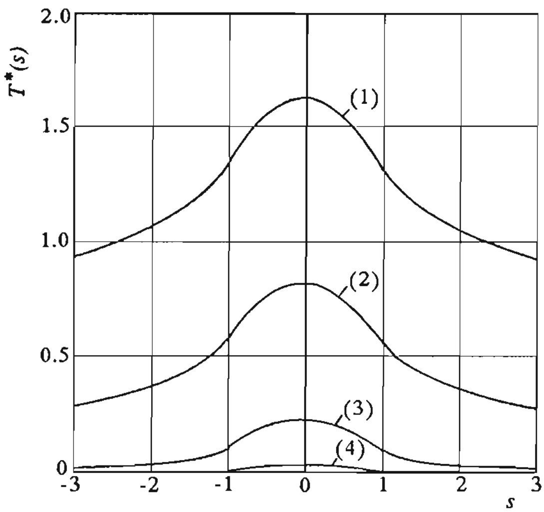

Fig. 2. Surface temperature for various values of Biot's number: (1) – Bi = 0.01; (2) – Bi = 0.1; (3) – Bi = 1; (4) – Bi = 100 and $f\mathrm{Pe}H = 0.5$

The temperature of punch surface inside and outside the contact region is shown in Fig.2 for four values of Biot's number. The great influence of the heat radiation is observed. When increasing the radiation the decrease of temperature appears. The analogous effect is observsd in Fig.3, where the diagrams of temperature in the punch are shown for two values of the parameter Bi. Note here that in the case of Bi $= 0$ the stationary temperature can not be evaluated because in Eq (3.2) the constant c is undefinite.

In the case of free surface heat insulation for contact problem Barber (1976) has shown that the contact zone is smaller than that obtained employing the Hertz theory. It was also shown that the critical values of the complex

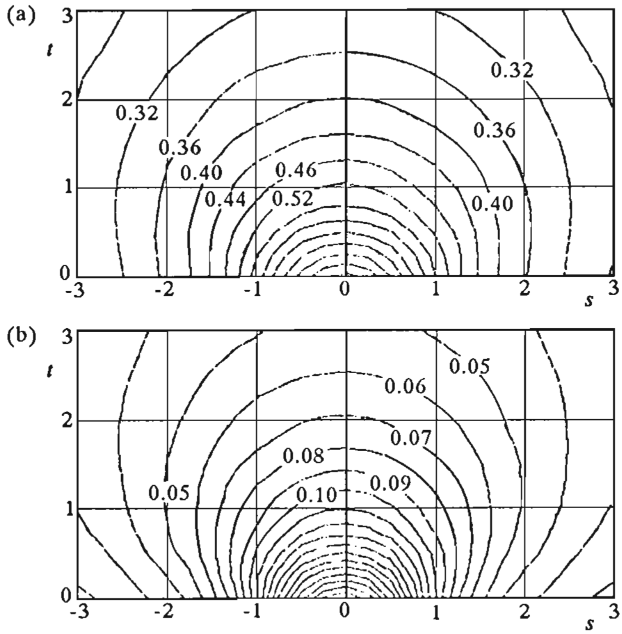

Fig. 3. Temperature contours: (a) – Bi = 0.1; (b) – Bi = 1 for $f\text{Pe}H = 0.5$

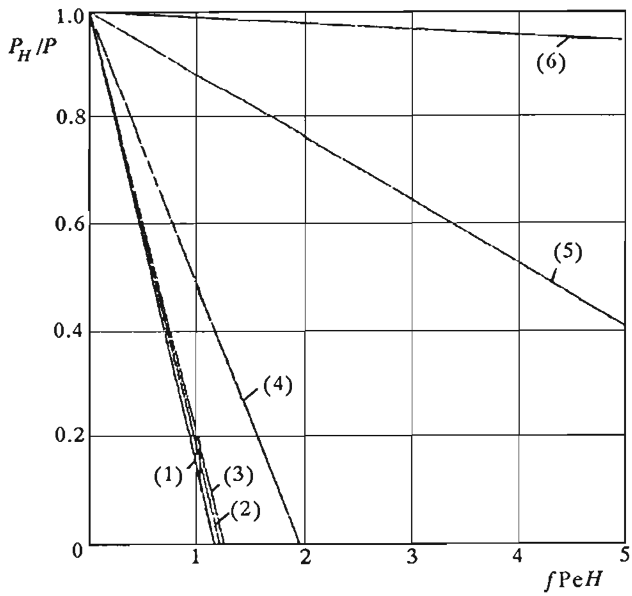

Fig. 4. Effect of Biot's number on the ratio P_H/P: (1) – Bi = 0 (cf Barber (1976)); (2) – Bi = 0.01; (3) – Bi = 0.1; (4) – Bi = 1; (5) – Bi = 10; (6) – Bi = 100

parameter $f\mathrm{Pe}H$, for which $P_H/P = 0$ exist. In our case the value of P_H/P decreases linearly also with the parameter $f\mathrm{Pe}H$ (see Fig.4). However, due to radiation this falling is less than that observed in Barber problem. It is clear that P_H/P depends on $f\mathrm{Pe}H$ under the formula

$$\frac{P_H}{P} = 1 - \gamma f\mathrm{Pe}H$$

where γ is the function of Bi. Numerical analysis permits the critical values of complex parameter $f\mathrm{Pe}H$ and the values of γ for different values of Bi to be found. These results are shown in Table 1.

Table 1. Change of the critical values of parameter $f\mathrm{Pe}H$ and parameter γ with Bi

Bi	0 ([3])	0.01	0.1	0.5	1	2	10	100
cr.val. $f\mathrm{Pe}H$	1.16	1.18	1.22	1.54	1.92	2.70	8.33	–
γ	0.86	0.85	0.82	0.65	0.52	0.37	0.12	0.02

6. Conclusions

More realistic heat radiation condition has been used for the statement and solution of the contact problem involving frictional heating. It leads to the following results:

- The temperature level falls with the radiation growth on the surface and inside the body

- The contact zone is wider than that observed in the insulated case and the contact zone width increases for higher heat radiation levels

- The complex parameter $f\mathrm{Pe}H$ critical values are increasing with the radiation growth.

These results can be applied to the investigation of grinding proccess, the work of brakes, electrical contacts and other frictional pairs, in which heat radiation is present.

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Płaskie zagadnienie kontaktowe uwzględniające generację ciepła i promieniowanie

Streszczenie

W pracy rozpatrzono zagadnienie dla sprężystego stempla, ślizgającego się po sztywnej pólprzestrzeni. Założono, że stempel jest przewodnikiem ciepla, pólprzestrzeń jest termicznie izolowana, oraz że w paśmie kontaku jest generowane cieplo, a poza nim zachodzi promieniowanie. Zagadnienie zredukowano do ukladu dwóch osobliwych równań całkowych, który rozwiązano numerycznie. Zbadano wplyw promieniowania na obszar kontaktu i na temperaturę w stemplu.

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