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SIMPLE DETERMINATION OF STRETCH AND ROTATION TENSORS, MORE GENERAL ISOTROPIC TENSOR-VALUED FUNCTIONS OF DEFORMATION

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Using the representation theorem, interpolation methods for isotropic tensor-valued functions and the extended Cayley-Hamilton theorem, we present a way of determining $\mathbf{U}$, $\mathbf{U}^{-1}$, $\mathbf{V}$, $\mathbf{V}^{-1}$, $\mathbf{R}$ (without recourse to eigenvectors) in terms of $\mathbf{C}$, $\mathbf{B}$ and $\mathbf{F}$, respectively. We consider other more general isotropic tensor-valued functions of $\mathbf{C}$ and $\mathbf{B}$ ($\ln \mathbf{U}$, for example). We obtain some simple to calculate and new formulas for these functions in 2-dimensional and 3-dimensional cases of deformation.

1. Introduction

The polar decomposition

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R} \qquad\qquad \mathbf{R}\mathbf{R}^{\top} = \mathbf{R}^{\top}\mathbf{R} = \mathbf{I} \tag{1.1}$$

has played a major role in continuum mechanics (cf Truesdell and Noll (1965), Gurtin (1981), Marsden and Hughes (1983), Ciarlet (1988) and Bowen (1989), for example). Here, $\mathbf{I}$ is the identity tensor, the invertible second-order tensor $\mathbf{F}$ is the deformation gradient, the orthogonal tensor $\mathbf{R}$ is the rotation tensor and the positive definite symmetric tensors $\mathbf{U}$ and $\mathbf{V}$ are the right and left stretch tensors, respectively. The superscript $\top$ in Eq (1.1) means the transpose. Using the right and left Cauchy-Green tensors $\mathbf{C}$ and $\mathbf{B}$ defined by

$$\mathbf{U}^2 = \mathbf{C} = \mathbf{F}^{\top}\mathbf{F} \qquad\qquad \mathbf{V}^2 = \mathbf{B} = \mathbf{F}\mathbf{F}^{\top} \tag{1.2}$$

we have

$$\mathbf{U} = \sqrt{\mathbf{C}}$$

$$
\begin{aligned}
\mathbf{R} &= \mathbf{F}\mathbf{U}^{-1} \\
\mathbf{V} &= \sqrt{\mathbf{B}} = \mathbf{R}\mathbf{U}\mathbf{R}^{\top} = \mathbf{F}\mathbf{U}^{-1}\mathbf{F}^{\top}
\end{aligned}
\tag{1.3}
$$

Presuming that $\mathbf{F}$ is known, we see from Eq (1.2) that $\mathbf{C}$ and $\mathbf{B}$ are easy to calculate. Of course, once $\mathbf{U}$ and $\mathbf{U}^{-1}$ are known, $\mathbf{R}$ and $\mathbf{V}$ follow readily from Eqs (1.3). The problem of this paper can be reduced to determination of $\mathbf{U}$ and $\mathbf{U}^{-1}$ in terms of $\mathbf{C}$ (or $\mathbf{F}$).

There are several ways of determining $\mathbf{U}$ and $\mathbf{U}^{-1}$ in terms of $\mathbf{C}$ (cf Marsden and Hughes (1983), Hoger and Carlson (1984), Ting (1985), Curnier and Rakotomanana (1991), for example).

In this paper, we point out that by an simple application of the representation theorem to nonpolynomial isotropic tensor functions (see Appendix B), $\mathbf{U}$, $\mathbf{U}^{-1}$, $\mathbf{V}$ and $\mathbf{R}$ can be calculated directly from $\mathbf{C}$ (or $\mathbf{F}$) without recourse to tensor square roots, eigenvalues and eigenvectors (or without recourse to tensor square roots and eigenvectors). We consider the two-dimensional and three-dimensional cases in Section 2 and Section 3, respectively. The method for obtainig $\mathbf{U}$, $\mathbf{U}^{-1}$, $\mathbf{V}$, $\mathbf{R}$ and more general isotropic tensor-valued functions of $\mathbf{C}$ (or $\mathbf{B}$) is the same in both two and three dimensions, but the details of derivation are different. In Appendix A we briefly summarize some useful identities obtained from the Cayley-Hamilton theorem. We compare our results with those arrived at by Hoger and Carlson (1984) and by Ting (1985).

2. 2-dimensional case

The results for plane motion are contained, of course, as a special case in the results of the next section. However, it seems to us that their importance in continuum mechanics and the ease with they are obtained warrant their independent presentation.

2.1. Determination of the right stretch tensor

The right stretch tensor defined by

$$
\mathbf{U} = \sqrt{\mathbf{C}} = f(\mathbf{C}) \tag{2.1}
$$

is a smooth function of the right Cauchy-Green tensor (cf Gurtin (1981) and Stephenson (1980)) and is an isotropic function (see Ting (1985)).

In this case, $f(\mathbf{C})$ is a function, which is form-invariant under the full orthogonal group $O(2)$ (see Appendix B for more information about isotropic functions). This form-invariance requirement is specified by the following condition

$$\forall\,\mathbf{Q} \in O(2) \quad \mathbf{QUQ}^{\mathsf{T}} = f(\mathbf{QCQ}^{\mathsf{T}}) \qquad O(2) = \{\mathbf{Q}: \ \mathbf{QQ}^{\mathsf{T}} = \mathbf{Q}^{\mathsf{T}}\mathbf{Q} = \mathbf{I}\} \tag{2.2}$$

The representation theorem ((B.4) and Tables 2,4) states that Eq (2.2) is true if and only if the function (2.1) has the following representation

$$\mathbf{U} = \alpha\mathbf{I} + \beta\mathbf{C} \tag{2.3}$$

where α, β are functions of the invariants of $\mathbf{C}$

$$\alpha,\ \beta = f(\mathrm{tr}\mathbf{C}, \mathrm{tr}\mathbf{C}^2) = \tilde{f}(I_C, II_C) \tag{2.4}$$

and

$$I_C = \mathrm{tr}\mathbf{C} \qquad II_C = \det\mathbf{C} = \frac{1}{2}(\mathrm{tr}^2\mathbf{C} - \mathrm{tr}\mathbf{C}^2) \tag{2.5}$$

are the principal invariants of $\mathbf{C}$.

Using Eq (2.3) and the Cayley-Hamilton theorem (A.8), we have

$$\mathbf{U}^2 = \alpha^2\mathbf{I} + 2\alpha\beta\mathbf{C} + \beta^2\mathbf{C}^2 = (\alpha^2 - \beta^2 II_C)\mathbf{I} + (2\alpha\beta + \beta^2 I_C)\mathbf{C} \tag{2.6}$$

Combining Eqs (1.2) and (2.6), we get

$$\begin{aligned} \alpha^2 - \beta^2 II_C &= 0 \\ 2\alpha\beta + \beta^2 I_C &= 1 \end{aligned} \tag{2.7}$$

The solution of Eq (2.7) is

$$\alpha = \sqrt{\frac{II_C}{I_C + 2\sqrt{II_C}}} \qquad \beta = \frac{1}{\sqrt{I_C + 2\sqrt{II_C}}} \tag{2.8}$$

Substituting for α and β into Eq (2.3), we obtain

$$\mathbf{U} = \frac{1}{\sqrt{I_C + 2\sqrt{II_C}}}(\sqrt{II_C}\,\mathbf{I} + \mathbf{C}) \tag{2.9}$$

This agrees with the form of $\mathbf{U}$ determined by Hoger and Carlson (1984).

Solving

$$C^2 - I_C C + II_C = 0 \tag{2.10}$$

yields the eigenvalues of $\mathbf{C}$

$$C_{1,2} = \frac{1}{2}\Big(I_C \pm \sqrt{I_C^2 - 4II_C}\Big) \tag{2.11}$$

Since $U_i = \sqrt{C_i}$, $i = 1,2$ (U_i are the eigenvalues of $\mathbf{U}$), one derives immediately from Eq (2.9) that

$$\mathbf{U} = \frac{1}{\sqrt{C_1} + \sqrt{C_2}}(\sqrt{C_1 C_2}\mathbf{I} + \mathbf{C}) = \frac{1}{I_U}(II_U\mathbf{I} + \mathbf{C}) \tag{2.12}$$

where

$$I_U = \sqrt{I_C + 2\sqrt{II_C}} \qquad\qquad II_U = \sqrt{II_C} \tag{2.13}$$

are the principal invariants of $\mathbf{U}$. Eq (2.12) is identical to Eq (2.3) of Ting (1985).

If

$$I_C = 2\sqrt{II_C} \quad \Rightarrow \quad C_1 = C_2 = C \tag{2.14}$$

the representation in terms of Eqs (2.9) or (2.12) is not unique. In this case it is not difficult to show from the Wang's Lemma and Transfer Theorem (see Gurtin (1981)) that

$$\mathbf{U} = \sqrt{C}\mathbf{I} \tag{2.15}$$

2.2. Inverse of U

$\mathbf{U}$ is positive definite, the inverse of $\mathbf{U}$ is a smooth and isotropic function of the right Cauchy-Green tensor. The representation of this function is given by

$$\mathbf{U}^{-1} = \gamma\mathbf{I} + \delta\mathbf{C} \tag{2.16}$$

where γ and δ are invariants of $\mathbf{C}$.

Multiplying Eq (2.16) by $\mathbf{U}^2$

$$\mathbf{U} = \gamma\mathbf{U}^2 + \delta\mathbf{C}\mathbf{U}^2 = \gamma\mathbf{C} + \delta\mathbf{C}^2 = -\delta II_C\mathbf{I} + (\gamma + \delta I_C)\mathbf{C} \tag{2.17}$$

and comparing the result with Eq (2.3), yields

$$\begin{aligned} \alpha &= -\delta II_C \\ \beta &= \gamma + \delta I_C \end{aligned} \tag{2.18}$$

After solving Eq (2.18) we obtain the formulas for γ and δ in terms of the principal invariants of $\mathbf{C}$

$$\gamma = \Big(1 + \frac{I_C}{\sqrt{II_C}}\Big)\frac{1}{\sqrt{I_C + 2\sqrt{II_C}}} \qquad \delta = -\frac{1}{\sqrt{II_C}}\frac{1}{\sqrt{I_C + 2\sqrt{II_C}}} \tag{2.19}$$

Substituting for γ and δ into Eq (2.16), yields

$$\mathbf{U}^{-1} = \frac{1}{\sqrt{I_C + 2\sqrt{II_C}}}\Big[\Big(1 + \frac{I_C}{\sqrt{II_C}}\Big)\mathbf{I} - \frac{1}{\sqrt{II_C}}\mathbf{C}\Big] \tag{2.20}$$

It can be shown (after substituting Eq (2.13) into Eq (2.20)) that Eq (2.20) is identical to Eq (4.1) of Hoger and Carlson (1984).

2.3. Determination of the rotation tensor

Since $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$, we have from Eq (2.20)

$$\mathbf{R} = \frac{1}{\sqrt{I_C + 2\sqrt{II_C}}}\Big[\Big(1 + \frac{I_C}{\sqrt{II_C}}\Big)\mathbf{F} - \frac{1}{\sqrt{II_C}}\mathbf{F}\mathbf{F}^\top\mathbf{F}\Big] \tag{2.21}$$

Using Eq (A.10) for $\mathbf{F}\mathbf{F}^\top\mathbf{F}$ and the following identities

$$I_C = I_B = I_{FF^\top} \qquad II_C = II_B = II_F^2 \tag{2.22}$$

after some calculations we finally obtain the simplest formula for $\mathbf{R}$ in terms of $\mathbf{F}$ and its invariants

$$\mathbf{R} = \frac{1}{\sqrt{I_{FF^\top} + 2II_F}}(\mathbf{F} - \mathbf{F}^\top + I_F\mathbf{I}) \tag{2.23}$$

Of course, $\mathbf{R}$ is an isotropic nonpolynomial smooth function of $\mathbf{F}$ and $\mathbf{F}^\top$. It should noted that Eq (2.23) has not been in the literature to the best of author's knowlege. This formula is fundamental in showing the inner structure of a pure 2-dimensional rotation $\mathbf{R}$, namely its decomposition into the sum of skew-symmetric and isotropic tensors. Decomposition of the deformation gradient, defined by

$$\mathbf{F} = \frac{1}{2}(\mathbf{F} + \mathbf{F}^\top) + \frac{1}{2}(\mathbf{F} - \mathbf{F}^\top) = \widetilde{\mathbf{U}} + \widetilde{\mathbf{R}} \tag{2.24}$$

allows for the possibility to rewrite Eq (2.23) in the form

$$\mathbf{R} = \frac{1}{\sqrt{-2\mathrm{tr}\widetilde{\mathbf{R}}^2 + \mathrm{tr}^2\widetilde{\mathbf{U}}}}\Big[2\widetilde{\mathbf{R}} + (\mathrm{tr}\widetilde{\mathbf{U}})\mathbf{I}\Big] \tag{2.25}$$

The orthogonal tensor $\mathbf{R}$ is an isotropic nonpolynomial smooth function of $\widetilde{\mathbf{U}}$ and $\widetilde{\mathbf{R}}$. Since Eq (2.25) is true (see also Section 2.5), we can not agree with Curnier's and Rakotomanana's (1991) (p.475) opinion "*the additive decomposition*", Eq (2.24), "*becomes mathematicaly and physically meaningless*".

2.4. Determination of the left stretch tensor

We present this section to show the consistency of our approach. Of course, we can get a formula for $\mathbf{V}$ from Eq (2.9) with the replacement $\mathbf{C} - \mathbf{B}$, and *vice-versa*.

Since $\mathbf{V} = \mathbf{R}\mathbf{F}^{\mathsf{T}}$, $I_B = I_C$ and $II_B = II_C$, from Eq (2.21) it results

$$\mathbf{V} = \frac{1}{\sqrt{I_C + 2\sqrt{II_C}}}\Big[\Big(1 + \frac{I_C}{\sqrt{II_C}}\Big)\mathbf{B} - \frac{1}{\sqrt{II_C}}\mathbf{B}^2\Big] \tag{2.26}$$

Using the Cayley-Hamilton theorem (A.8), we obtain

$$\mathbf{V} = \frac{1}{\sqrt{I_B + 2\sqrt{II_B}}}(\sqrt{II_B}\,\mathbf{I} + \mathbf{B}) \tag{2.27}$$

2.5. Determination of the right and left stretch tensors in terms of the deformation gradients

Using Eqs (1.1) and (2.23), we derive new formulas for $\mathbf{U}$ and $\mathbf{V}$ in terms of $\mathbf{F}$

$$\mathbf{U} = \frac{1}{\sqrt{I_{FF^{\mathsf{T}}} + 2II_F}}(\mathbf{F}^{\mathsf{T}}\mathbf{F} + II_F\mathbf{I}) \tag{2.28}$$

$$\mathbf{V} = \frac{1}{\sqrt{I_{FF^{\mathsf{T}}} + 2II_F}}(\mathbf{F}\mathbf{F}^{\mathsf{T}} + II_F\mathbf{I}) \tag{2.29}$$

From a practical point of view they are most simple to calculate.

Since Eq (2.24), from Eqs (2.28) and (2.29), we have

$$\mathbf{U} = \frac{1}{\sqrt{-2\mathrm{tr}\tilde{\mathbf{R}}^2 + \mathrm{tr}^2\tilde{\mathbf{U}}}}\Big[\tilde{\mathbf{U}}^2 - \tilde{\mathbf{R}}^2 + (\tilde{\mathbf{U}}\tilde{\mathbf{R}} - \tilde{\mathbf{R}}\tilde{\mathbf{U}}) + \frac{1}{2}(\mathrm{tr}^2\tilde{\mathbf{U}} - \mathrm{tr}\tilde{\mathbf{U}}^2 - \mathrm{tr}\tilde{\mathbf{R}}^2)\mathbf{I}\Big] \tag{2.30}$$

$$\mathbf{V} = \frac{1}{\sqrt{-2\mathrm{tr}\tilde{\mathbf{R}}^2 + \mathrm{tr}^2\tilde{\mathbf{U}}}}\Big[\tilde{\mathbf{U}}^2 - \tilde{\mathbf{R}}^2 - (\tilde{\mathbf{U}}\tilde{\mathbf{R}} - \tilde{\mathbf{R}}\tilde{\mathbf{U}}) + \frac{1}{2}(\mathrm{tr}^2\tilde{\mathbf{U}} - \mathrm{tr}\tilde{\mathbf{U}}^2 - \mathrm{tr}\tilde{\mathbf{R}}^2)\mathbf{I}\Big] \tag{2.31}$$

and after using the Cayley-Hamilton theorem (A.8) we finally obtain the simplest formulas for $\mathbf{U}$ and $\mathbf{V}$ in terms of $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{R}}$

$$\mathbf{U} = \psi\Big[(\mathrm{tr}\tilde{\mathbf{U}})\tilde{\mathbf{U}} + (\tilde{\mathbf{U}}\tilde{\mathbf{R}} - \tilde{\mathbf{R}}\tilde{\mathbf{U}}) + \varphi\mathbf{I}\Big] \tag{2.32}$$

$$\mathbf{V} = \psi\Big[(\mathrm{tr}\tilde{\mathbf{U}})\tilde{\mathbf{U}} - (\tilde{\mathbf{U}}\tilde{\mathbf{R}} - \tilde{\mathbf{R}}\tilde{\mathbf{U}}) + \varphi\mathbf{I}\Big] \tag{2.33}$$

where

$$\psi = \frac{1}{\sqrt{-2\mathrm{tr}\tilde{\mathbf{R}}^2 + \mathrm{tr}^2\tilde{\mathbf{U}}}} \qquad \varphi = -\mathrm{tr}\tilde{\mathbf{R}}^2 \tag{2.34}$$

From these expressions we see the mathematical and physical meanings of the additive decomposition of $\mathbf{F}$, Eq (2.24), in the 2-dimensional case. Here a difference between $\mathbf{U}$ and $\mathbf{V}$ is the only one, namely the sign of the term $(\tilde{\mathbf{U}}\tilde{\mathbf{R}} - \tilde{\mathbf{R}}\tilde{\mathbf{U}})$.

2.6. Applications of interpolation methods to isotropic tensor functions

In this section we consider interpolation methods (see Boehler [ed.] (1987), Chapter 13) for isotropic smooth tensor-valued functions of $\mathbf{C}$ (or $\mathbf{B}$)

$$\mathbf{A} = \mathbf{F}^s(\mathbf{C}) = a_0\mathbf{I} + a_1\mathbf{C} \tag{2.35}$$

where

$$a_k = f_k(C_1, C_2) \qquad k = 0, 1 \tag{2.36}$$

and C_1, C_2 ($C_i = B_i > 0$, $i = 1, 2$) are the principal values of $\mathbf{C}$ (defined by Eq (2.11)).

Since, $C_i = B_i$ and $\mathbf{B} = \mathbf{R}\mathbf{C}\mathbf{R}^{\top}$, we obtain the relationship between Eq (2.35) and the isotropic functions of $\mathbf{B}$

$$\bar{\mathbf{A}} = \mathbf{F}^s(\mathbf{B}) = \mathbf{F}^s(\mathbf{R}\mathbf{C}\mathbf{R}^{\top}) = \mathbf{R}\mathbf{F}^s(\mathbf{C})\mathbf{R}^{\top} = \mathbf{R}\mathbf{A}\mathbf{R}^{\top} = a_0\mathbf{I} + a_1\mathbf{B} \tag{2.37}$$

From the Lagrange interpolation method for the functions (2.35) and the Cayley-Hamilton theorem Eq (A.8) it follows that the coefficients (2.36) can be found by solving the system of linear equations

$$\begin{aligned} A_1 &= a_0 + a_1 C_1 \\ A_2 &= a_0 + a_1 C_2 \end{aligned} \tag{2.38}$$

where A_i are the principal values of $\mathbf{A}$. Considering to above this seems to be a trivial result. On the other hand, Eqs (2.38) appear due to the requirement (2.2).

As an alternate approach, from the Newton interpolation method for the functions (2.35) and the Cayley-Hamilton theorem Eq (A.8), we have the formula

$$\mathbf{A} = b_0 \mathbf{I} + b_1(\mathbf{C} - C_1 \mathbf{I}) \tag{2.39}$$

where b_k are given by

$$b_0 = A_1 \qquad\qquad b_1 = \frac{A_1 - A_2}{C_1 - C_2} \tag{2.40}$$

Solving Eq (2.38) or comparing Eqs (2.39) and (2.40) with Eq (2.35), we obtain the invariants a_k. As we have easily proved, the Lagrange and the Newton interpolation formulas are identical for the smooth isotropic functions (2.35). The coefficients (2.36) are defined by

$$a_0 = \frac{C_1 A_2 - C_2 A_1}{C_1 - C_2} \qquad\qquad a_1 = \frac{A_1 - A_2}{C_1 - C_2} \tag{2.41}$$

If the principal values of $\mathbf{C}$ coincide, $C_1 = C_2 = C$, the interpolation formula for Eq (2.35) reduces to the results: $\mathbf{A} = A\mathbf{I}$, $A = A_1 = A_2$.

2.6.1. *Example 1*

As a simple example let us use the above method to find the function (2.1), $\mathbf{A} = \mathbf{U} = \sqrt{\mathbf{C}}$.

Since, $A_i = U_i = \sqrt{C_i}$, from Eq (2.41), we obtain

$$\begin{aligned} a_0 &= \frac{C_1\sqrt{C_2} - C_2\sqrt{C_1}}{C_1 - C_2} = \frac{\sqrt{C_1 C_2}}{\sqrt{C_1} - \sqrt{C_2}} \\ a_1 &= \frac{\sqrt{C_1} - \sqrt{C_2}}{C_1 - C_2} = \frac{1}{\sqrt{C_1} + \sqrt{C_2}} \end{aligned} \tag{2.42}$$

Eqs (2.35) and (2.42) are identical to Eq (2.12). Of course, substituting Eq (2.11) into Eq (2.42), we have Eq (2.9).

2.6.2. Example 2

In a similar way we can formulate the tensorial Hencky measure of strain, defined by (cf Curnier and Rokotomanana (1991), for example)

$$\mathbf{A} = \ln \mathbf{U} = \frac{1}{2} \ln \mathbf{C} \tag{2.43}$$

Since, $A_i = \ln U_i = \frac{1}{2} \ln C_i$, it results from Eq (2.41)

$$\begin{aligned} a_0 &= \frac{1}{a}(C_1 \ln C_2 - C_2 \ln C_1) \\ a_1 &= \frac{1}{a}(\ln C_1 - \ln C_2) \end{aligned} \tag{2.44}$$

where

$$a = 2(C_1 - C_2) \tag{2.45}$$

Substituting Eqs (2.11), (2.44) and (2.45) into Eq (2.35), we obtain the logarithmic strain measure in terms of $\mathbf{C}$ and the principal invariants of $\mathbf{C}$.

Other examples are $\mathbf{A} = \exp \mathbf{C}$, $\mathbf{A} = \sin \mathbf{C}$, $\mathbf{A} = \mathbf{U}^{-n}$ $(n = 1, 2, ..., N)$, $\mathbf{A} = \mathbf{F}_1^s(\mathbf{C}) + \mathbf{F}_2^s(\mathbf{C})$, etc. which can be treated in the same way.

2.7. Applications of the spectral theorem and comparision with interpolation methods

By the spectral theorem (Gurtin (1981)) and Eq (1.2), the right and the left Cauchy-Green tensors $\mathbf{C}$ and $\mathbf{B}$ admit the representations

$$\mathbf{C} = C_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + C_2 \mathbf{e}_2 \otimes \mathbf{e}_2 = C_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + C_2(\mathbf{I} - \mathbf{e}_1 \otimes \mathbf{e}_1) \tag{2.46}$$

and

$$\begin{aligned} \mathbf{B} &= \mathbf{R}\mathbf{C}\mathbf{R}^\top = C_1 \mathbf{R}(\mathbf{e}_1 \otimes \mathbf{e}_1)\mathbf{R}^\top + C_2[\mathbf{I} - \mathbf{R}(\mathbf{e}_1 \otimes \mathbf{e}_1)\mathbf{R}^\top] = \\ &= C_1 \mathbf{R}\mathbf{e}_1 \otimes \mathbf{R}\mathbf{e}_1 + C_2(\mathbf{I} - \mathbf{R}\mathbf{e}_1 \otimes \mathbf{R}\mathbf{e}_1) = C_1 \bar{\mathbf{e}}_1 \otimes \bar{\mathbf{e}}_1 + C_2(\mathbf{I} - \bar{\mathbf{e}}_1 \otimes \bar{\mathbf{e}}_1) \end{aligned} \tag{2.47}$$

where $\mathbf{e}_i$ and $\bar{\mathbf{e}}_i$ $(\|\mathbf{e}_i\| = 1, \|\bar{\mathbf{e}}_i\| = 1)$ are the eigenvectors of $\mathbf{C}$ an $\mathbf{B}$ (also $\mathbf{U}$ and $\mathbf{V}$), respectively.

The isotropic smooth functions Eq (2.35) have the representations

$$\mathbf{A} = \mathbf{F}^s(\mathbf{C}) = A_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + A_2(\mathbf{I} - \mathbf{e}_1 \otimes \mathbf{e}_1) \tag{2.48}$$

Similarly

$$\bar{\mathbf{A}} = \mathbf{F}^s(\mathbf{B}) = \mathbf{R}\mathbf{A}\mathbf{R}^\top = A_1\bar{\mathbf{e}}_1 \otimes \bar{\mathbf{e}}_1 + A_2(\mathbf{I} - \bar{\mathbf{e}}_1 \otimes \bar{\mathbf{e}}_1) \tag{2.49}$$

Suppose, that $C_1 \neq C_2$, then Eq (2.46) gives

$$\mathbf{e}_1 \otimes \mathbf{e}_1 = \frac{1}{C_1 - C_2}(\mathbf{C} - C_2\mathbf{I}) \tag{2.50}$$

which can be employed in Eq (2.48) with the result

$$\mathbf{A} = \frac{1}{C_1 - C_2}\Big[(C_1A_2 - C_2A_1)\mathbf{I} + (A_1 - A_2)\mathbf{C}\Big] \tag{2.51}$$

This agrees with Eqs (2.35) and (2.41).

Finally, we come to the conclusion that the methods presented here and in Section 2.6 are equivalent.

From a practical point of view we can not agree with Curnier and Rokotomanana (1991) conclusion (p.474): "*Tensorial forms of these tensors* (here – Eq (2.35)) *are also available but, as a rule, they are even more complicated to obtain than the spectral forms.*" From Sections 2.6 and 2.7 we can see that the above conclusion must have the reverse meaning.

Remarks

It is not easy to calculate the eigenvectors in a 3-dimensional case using the aforementioned way.

3. 3-dimensional case

3.1. Determination of the right and left stretch tensors

In this case, $\mathbf{f}(\mathbf{C})$ is a function, which is form-invariant under the full orthogonal group $O(3)$ (see Appendix B). This form-invariance requirement is specified by the following condition

$$\forall\, \mathbf{Q} \in O(3) \quad \mathbf{Q}\mathbf{U}\mathbf{Q}^\top = f(\mathbf{Q}\mathbf{C}\mathbf{Q}^\top) \qquad O(3) = \{\mathbf{Q} : \ \mathbf{Q}\mathbf{Q}^\top = \mathbf{Q}^\top\mathbf{Q} = \mathbf{I}\} \tag{3.1}$$

The representation theorem ((B.4) and Tables 1,3) states that Eq (3.1) is true if and only if the function (3.1) has the representation

$$\mathbf{U} = \sqrt{\mathbf{C}} = \alpha\mathbf{I} + \beta\mathbf{C} + \gamma\mathbf{C}^2 \tag{3.2}$$

where

$$\alpha, \beta, \gamma = f(\mathrm{tr}\mathbf{C}, \mathrm{tr}\mathbf{C}, \mathrm{tr}\mathbf{C}) = \tilde{f}(I_C, II_C, III_C) \tag{3.3}$$

are invariants.

Multiplying Eq (3.2) by $\mathbf{U}$, yields

$$\begin{aligned} \mathbf{U}^2 &= \mathbf{C} = \alpha\mathbf{U} + \beta\mathbf{U}^3 + \gamma\mathbf{U}^5 = [\beta III_U + \gamma(I_U^2 III_U - II_U III_U)]\mathbf{I} + \\ &+ [\alpha - \beta II_U + \gamma(I_U III_U - I_U^2 II_U + II_U^2)]\mathbf{U} + \\ &+ [\beta I_U + \gamma(I_U^3 - 2I_U II_U + III_U)]\mathbf{U}^2 \end{aligned} \tag{3.4}$$

From Eq (3.4) follows that

$$\begin{aligned} &\beta III_U + \gamma(I_U^2 III_U - II_U III_U) = 0 \\ &\alpha - \beta II_U + \gamma(I_U III_U - I_U^2 II_U + II_U^2) = 0 \\ &\beta I_U + \gamma(I_U^3 - 2I_U II_U + III_U) = 1 \end{aligned} \tag{3.5}$$

After solving Eq (3.5) we get α, β and γ in terms of the principal invariants of $\mathbf{U}$

$$\begin{aligned} \alpha &= \frac{-I_U III_U}{III_U - I_U II_U} \qquad\qquad \beta = \frac{II_U - I_U^2}{III_U - I_U II_U} \\ \gamma &= \frac{1}{III_U - I_U II_U} \end{aligned} \tag{3.6}$$

Since, $\mathbf{C}$ and $\mathbf{U}$ are the positive definite symmetric tensors and $U_i = \sqrt{C_i}$, $(i = 1, 2, 3)$, from Eq (3.6), we obtain

$$\begin{aligned} \alpha &= -\sqrt{C_1 C_2 C_3}(\sqrt{C_1} + \sqrt{C_2} + \sqrt{C_3})\gamma \\ \beta &= -(C_1 + C_2 + C_3 + \sqrt{C_1 C_2} + \sqrt{C_2 C_3} + \sqrt{C_1 C_3})\gamma \\ \gamma &= -\frac{1}{(\sqrt{C_1} + \sqrt{C_2})(\sqrt{C_2} + \sqrt{C_3})(\sqrt{C_1} + \sqrt{C_3})} \end{aligned} \tag{3.7}$$

This agrees with the form of $\mathbf{U}$ determined by Ting (1985).

Since, $\mathbf{C} = \mathbf{U}^2$, we have

$$\det\mathbf{C} = (\det\mathbf{U})^2 \qquad \text{or} \qquad III_U = \sqrt{III_C} \tag{3.8}$$

and

$$\mathrm{tr}\mathbf{C} = \mathrm{tr}\mathbf{U}^2 \qquad \text{or} \qquad II_U = \frac{1}{2}(I_U^2 - I_C) \tag{3.9}$$

Taking the trace of $\mathbf{U}$ given by Eq (3.2) yields

$$I_U = \mathrm{tr}\mathbf{U} = 3\alpha + \beta\mathrm{tr}\mathbf{C} + \gamma\mathrm{tr}\mathbf{C}^2 \tag{3.10}$$

and using Eqs (3.8) and (3.9), yields

$$I_U^4 - 2I_U^2\mathrm{tr}\mathbf{C} - 8I_U\sqrt{\det\mathbf{C}} + 2\mathrm{tr}^2\mathbf{C} - \mathrm{tr}\mathbf{C}^2 = 0 \tag{3.11}$$

Since, $2\mathrm{tr}^2\mathbf{C} - \mathrm{tr}\mathbf{C}^2 = I_C^2 - 4II_C$, Eq (3.11) coincides with that arrived at by Hoger and Carlson (1984), (p.116), with the aid of different methods.

The usual procedure for solving quartics (3.11) leads us to the following formula

$$\Big(\frac{u}{4}\Big)^3 - I_C\Big(\frac{u}{4}\Big)^2 + II_C\Big(\frac{u}{4}\Big) - III_C = 0 \tag{3.12}$$

From the characteristic equation of $\mathbf{C}$

$$C^3 - I_C C^2 + II_C C - III_C = 0 \tag{3.13}$$

it can be seen that the roots of Eq (3.12) and (3.13) are related as follows

$$u_i = 4C_i \qquad i = 1,2,3 \qquad C_i > 0 \tag{3.14}$$

where

$$C_i = \frac{1}{3}I_C + \frac{2}{3}\sqrt{I_C^2 - 3II_C}\cos\Big[\frac{2}{3}\pi(i-1) - \psi\Big] > 0 \tag{3.15}$$

and, in turn

$$\cos 3\psi = \frac{2I_C^3 - 9I_C II_C + 27III_C}{\sqrt{[2(I_C^2 - 3II_C)]^3}} \tag{3.16}$$

Since, $-8\sqrt{\det\mathbf{C}} < 0$ and Eq (3.14) is true, the solution of Eq (3.11) is given by

$$I_U = \frac{1}{2}(\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3}) = \sqrt{C_1} + \sqrt{C_2} + \sqrt{C_3} \tag{3.17}$$

when C_i are distinct. Of course, the other three roots of Eq (3.11) are of no use for us.

Formulas (3.2) and (3.7) become expressions for the left stretch tensor $\mathbf{V}$ with the replacement $\mathbf{C} - \mathbf{B}$ into Eq (3.2).

Remarks

If $d = 4II_C^3 - I_C^2II_C^2 + 4I_C^3III_C - 18I_CII_CIII_C + 27III_C^2 < 0$, we have $C_1 \neq C_2 \neq C_3$, $C_i > 0$, (*casus irreducibilis* of Eqs (3.12) and (3.13)) and the representation given by Eq (3.2) is unique. If $d = 0$, we have the case of two repeated eigenvalues and the representation of $\mathbf{U}$ is given by Eq (2.9) (this follows from the Wang's Lemma, see Gurtin (1981)). If $I_C^2 = 3II_C$, we have the case of three repeated eigenvalues and $\mathbf{U}$ is given by Eq (2.15).

Using the method presented in Section 2.1 (the knowlege of the invariants of $\mathbf{U}$ is not required), we get

$$\begin{aligned} \mathbf{U}^2 &= (\alpha^2 + 2\beta\gamma III_C + \gamma^2 I_C III_C)\mathbf{I} + \\ &+ [2\alpha\beta - 2\beta\gamma II_C + \gamma^2(III_C - I_C II_C)]\mathbf{C} + \\ &+ [2\alpha\gamma + \beta^2 + 2\beta\gamma I_C + \gamma^2(I_C^2 - II_C)]\mathbf{C}^2 \end{aligned} \tag{3.18}$$

Combining Eqs (1.2) and (3.18), we have

$$\begin{aligned} &\alpha^2 + 2\beta\gamma III_C + \gamma^2 I_C III_C = 0 \\ &2\alpha\beta - 2\beta\gamma II_C + \gamma^2(III_C - I_C II_C) = 1 \\ &2\alpha\gamma + \beta^2 + 2\beta\gamma I_C + \gamma^2(I_C^2 - II_C) = 0 \end{aligned} \tag{3.19}$$

Rewriting Eq $(3.19)_{1,3}$ in the form

$$\begin{aligned} &v^2 + 2III_C w + I_C III_C = 0 \\ &2v + w^2 + 2I_C w + I_C^2 - II_C = 0 \end{aligned} \tag{3.20}$$

where

$$v = \frac{\alpha}{\gamma} \qquad\qquad w = \frac{\beta}{\gamma} \tag{3.21}$$

yields

$$\begin{aligned} &v^4 - 2I_C III_C v^2 + 8III_C^2 v + III_C^2(I_C^2 - 4II_C) = 0 \\ &w = -\frac{2III_C}{v^2 + I_C III_C} \end{aligned} \tag{3.22}$$

From Eqs (3.22) and (3.13) it follows

$$\Big(\frac{x}{4III_C}\Big)^3 - I_C\Big(\frac{x}{4III_C}\Big)^2 + II_C\Big(\frac{x}{4III_C}\Big) - III_C = 0 \tag{3.23}$$

and

$$x_i = 4III_C C_i \qquad\qquad i = 1,2,3 \tag{3.24}$$

Since, $8III_C^2 > 0$, the solution of Eq (3.22) is given by

$$v = -\frac{1}{2}(\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}) = -\sqrt{III_C}(\sqrt{C_1} + \sqrt{C_2} + \sqrt{C_3}) \tag{3.25}$$

Using Eqs (3.21) and $(3.22)_2$ we obtain

$$
\begin{aligned}
w &= -\frac{1}{2}\Big[(\sqrt{C_1}+\sqrt{C_2}+\sqrt{C_3})^2+I_C\Big] \\
\gamma^2 &= \frac{1}{2w(v-II_C)+III_C-I_C II_C} \\
\alpha &= v\gamma \qquad\qquad \beta = w\gamma
\end{aligned}
\tag{3.26}
$$

It is not difficult to show that the formulae agree with those for α, β and γ obtained above.

We went out of the way to obtain Eqs (3.12) and (3.23) because we can not agree with the final results given (without a proof) by Hoger and Carlson (1984) (pp. 116-117).

By applying the representation theorem (B.4) (Tables 1 and 3, Appendix B), a formula for U in terms of $\tilde{\mathsf{U}}$ and $\tilde{\mathsf{R}}$ (or F and $\tilde{\mathsf{F}}^{\mathsf{T}}$) of the form

$$
\begin{aligned}
\mathsf{U} &= \alpha_1\mathsf{I}+\alpha_2\tilde{\mathsf{U}}+\alpha_3\tilde{\mathsf{U}}^2+\alpha_4\tilde{\mathsf{R}}^2+\alpha_5(\tilde{\mathsf{U}}\tilde{\mathsf{R}}-\tilde{\mathsf{R}}\tilde{\mathsf{U}})+\alpha_6\tilde{\mathsf{R}}\tilde{\mathsf{U}}\tilde{\mathsf{R}}+ \\
&+ \alpha_7(\tilde{\mathsf{U}}^2\tilde{\mathsf{R}}-\tilde{\mathsf{R}}\tilde{\mathsf{U}}^2)+\alpha_8\tilde{\mathsf{R}}(\tilde{\mathsf{U}}\tilde{\mathsf{R}}-\tilde{\mathsf{R}}\tilde{\mathsf{U}})\tilde{\mathsf{R}} \\
\alpha_k &= f_k(\mathrm{tr}\tilde{\mathsf{U}},\mathrm{tr}\tilde{\mathsf{U}}^2,\mathrm{tr}\tilde{\mathsf{U}}^3,\mathrm{tr}\tilde{\mathsf{R}}^2,\mathrm{tr}\tilde{\mathsf{U}}\tilde{\mathsf{R}}^2,\mathrm{tr}\tilde{\mathsf{U}}^2\tilde{\mathsf{R}}^2,\mathrm{tr}\tilde{\mathsf{U}}^2\tilde{\mathsf{R}}^2\tilde{\mathsf{U}}\tilde{\mathsf{R}}) \qquad k=1,...,8
\end{aligned}
\tag{3.27}
$$

can be obtained theoretically. However, the formulae for the coefficient α_k in terms of the invariants of $\tilde{\mathsf{U}}$ and $\tilde{\mathsf{R}}$ are extremely complicated.

3.2. Inverse of U

The representation of U^{-1} is given by

$$
\mathsf{U}^{-1} = \delta\mathsf{I}+\varepsilon\mathsf{C}+\varphi\mathsf{C}^2 \tag{3.28}
$$

where δ, ε and φ are invariants of C.

Multiplying Eq (3.28) by C and using the identity (A.1), yields

$$
\mathsf{U} = \delta\mathsf{C}+\varepsilon\mathsf{C}^2+\varphi\mathsf{C}^3 = \varphi III_C\mathsf{I}+(\delta-II_C\varphi)\mathsf{C}+(\varepsilon+I_C\varphi)\mathsf{C}^2 \tag{3.29}
$$

Comparing this with Eq (3.2), we have

$$
\alpha = \varphi III_C \qquad\qquad \beta = \delta - II_C\varphi \qquad\qquad \gamma = \varepsilon+\varphi I_C \tag{3.30}
$$

After solving Eq (3.30) we obtain the formulae for δ, ε and φ in terms of the invariants of $\mathbf{C}$

$$\delta = \beta - \frac{\alpha II_C}{III_C} \qquad \varepsilon = \gamma - \frac{\alpha I_C}{III_C} \qquad \varphi = \frac{\alpha}{III_C} \tag{3.31}$$

where α, β and γ are given by Eq (3.7).

Substituting Eq (3.6) and the identity

$$II_C = II_U^2 - 2I_U III_U \tag{3.32}$$

into Eq (3.31), we finally obtain δ, ε and φ in terms of the principal invariants of $\mathbf{U}$,

$$\begin{aligned} \delta &= \psi\Big[(I_U II_U^2 - III_U(I_U^2 + II_U)\Big] \\ \varepsilon &= -\psi\Big[III_U + I_U(I_U^2 - 2II_U)\Big] \\ \varphi &= \psi I_U \end{aligned} \tag{3.33}$$

where

$$\psi = \frac{1}{III_U(I_U II_U - III_U)} \tag{3.34}$$

Eqs (3.28) and (3.33) are identical to Eq (3.2) of Ting (1985).

3.3. Determination of R

Since $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$, we obtain from Eq (3.28) that

$$\mathbf{R} = \mathbf{F}(\delta\mathbf{I} + \varepsilon\mathbf{C} + \varphi\mathbf{C}^2) = \delta\mathbf{F} + \varepsilon\mathbf{F}\mathbf{F}^\top\mathbf{F} + \varphi\mathbf{F}\mathbf{F}^\top\mathbf{F}\mathbf{F}^\top\mathbf{F} = (\delta\mathbf{I} + \varepsilon\mathbf{B} + \varphi\mathbf{B}^2)\mathbf{F} \tag{3.35}$$

where δ, ε and φ are given by Eq (3.31) or by Eq (3.33).

3.4. More general isotropic tensor-valued functions of C or B

In this case, isotropic smooth tensor-valued functions of $\mathbf{C}$ and $\mathbf{B}$ have the representations

$$\mathbf{A} = \mathbf{F}^s(\mathbf{C}) = a_0\mathbf{I} + a_1\mathbf{C} + a_2\mathbf{C}^2 \tag{3.36}$$

$$\bar{\mathbf{A}} = \mathbf{F}^s(\mathbf{B}) = a_0\mathbf{I} + a_1\mathbf{B} + a_2\mathbf{B}^2 \tag{3.37}$$

where

$$a_k = f_k(C_1, C_2, C_3) \qquad k = 0, 1, 2 \tag{3.38}$$

and C_i $(C_i = B_i > 0,\ i = 1,2,3)$ are the principal values of $\mathbf{C}$ (defined by Eq (3.15)).

In the way similar to that described in Section 2.6 we can find the coefficients (3.38). If, $C_1 \neq C_2 \neq C_3$, we have

$$\begin{aligned} a_0 &= A_1C_2C_3b_1 + A_2C_3C_1b_2 + A_3C_1C_2b_3 \\ a_1 &= -A_1(C_2 + C_3)b_1 - A_2(C_3 + C_1)b_2 - A_3(C_1 + C_2)b_3 \\ a_2 &= A_1b_1 + A_2b_2 + A_3b_3 \end{aligned} \tag{3.39}$$

where

$$b_1 = \frac{1}{(C_1 - C_2)(C_1 - C_3)} \qquad b_2 = \frac{1}{(C_2 - C_3)(C_2 - C_1)}$$
$$b_3 = \frac{1}{(C_3 - C_1)(C_3 - C_2)} \tag{3.40}$$

If, for example: $C_1 \neq C_2 = C_3$, the representations of functions (3.36) are given by Eqs (2.35) and (2.41).

3.4.1. Example 3

In a similar way to that described in Section 2.6 we can form the tensorial Hencky measure of strain, defined by Eq (2.43). Since, $A_i = \ln U_i = \frac{1}{2}\ln C_i$, $(i = 1,2,3)$ from Eq (3.39), we get

$$\begin{aligned} a_0 &= \frac{1}{2}[\ln(C_1)C_2C_3b_1 + \ln(C_2)C_3C_1b_2 + \ln(C_3)C_1C_2b_3] \\ a_1 &= -\frac{1}{2}[\ln(C_1)(C_2 + C_3)b_1 + \ln(C_2)(C_3 + C_1)b_2 + \ln(C_3)(C_1 + C_2)b_3] \\ a_2 &= \frac{1}{2}[\ln(C_1)b_1 + \ln(C_2)b_2 + \ln(C_3)b_3] \end{aligned} \tag{3.41}$$

where b_i are given by Eq (3.40). Substituting Eqs (3.41), (3.40) and (3.15) into Eq (3.36), yields the logarithmic strain measure in terms of $\mathbf{C}$ and the principal invariants of $\mathbf{C}$.

Other examples of the functions (3.36) and (3.37) can be treated in the same way.

Appendix A

Identities obtained from the Cayley-Hamilton theorem

The reader is referred to the following papers: Rivlin (1955), Spencer and Rivlin (1958/59), Wesołowski (1964), Spencer (1971) and Boehler (1987) [ed.].

Three-dimensional case

The Cayley-Hamilton theorem for a second order tensor $\mathbf{J}$ reads

$$\mathbf{J}^3 - I_J\mathbf{J}^2 + II_J\mathbf{J} - III_J\mathbf{I} = \mathbf{0} \tag{A.1}$$

where

$$\begin{gathered} I_J = \mathrm{tr}\mathbf{J} \qquad\qquad II_J = \frac{1}{2}(\mathrm{tr}^2\mathbf{J} - \mathrm{tr}\mathbf{J}^2) \\ III_J = \det\mathbf{J} = \frac{1}{6}\mathrm{tr}^3\mathbf{J} - \frac{1}{2}\mathrm{tr}\mathbf{J}\mathrm{tr}\mathbf{J}^2 + \mathrm{tr}\mathbf{J}^3 \end{gathered} \tag{A.2}$$

Multiplying (A.1) by $\mathbf{J}$ and using (A.1), yields

$$\mathbf{J}^4 = (I_J^2 - II_J)\mathbf{J}^2 + (III_J - I_JII_J)\mathbf{J} + I_JIII_J\mathbf{I} \tag{A.3}$$

Similarly, from (A.3), we get

$$\mathbf{J}^5 = (I_J^3 - 2I_JII_J + III_J)\mathbf{J}^2 + (I_JIII_J - I_J^2II_J + II_J^2)\mathbf{J} + (I_J^2III_J - II_JIII_J)\mathbf{I} \tag{A.4}$$

If in (A.1) we replace $\mathbf{J}$ by $a\mathbf{J} + b\mathbf{K} + c\mathbf{L}$, where a, b, c are arbitrary scalars, $\mathbf{K}$, $\mathbf{L}$ are second order tensors, and equate to zero the terms with the same coefficients we obtain important identities, for example

$$\begin{aligned} &\mathbf{J}^2\mathbf{K} + \mathbf{JKJ} + \mathbf{KJ}^2 = (\mathrm{tr}\mathbf{K})\mathbf{J}^2 + \mathrm{tr}\mathbf{J}(\mathbf{JK} + \mathbf{KJ}) + (\mathrm{tr}\mathbf{JK} - \mathrm{tr}\mathbf{J}\mathrm{tr}\mathbf{K})\mathbf{J} + \\ &-\frac{1}{2}(\mathrm{tr}^2\mathbf{J} - \mathrm{tr}\mathbf{J}^2)\mathbf{K} + \left[\mathrm{tr}\mathbf{J}^2\mathbf{K} - \mathrm{tr}\mathbf{JK}\mathrm{tr}\mathbf{J} + \frac{1}{2}\mathrm{tr}\mathbf{K}(\mathrm{tr}^2\mathbf{J} - \mathrm{tr}\mathbf{J}^2)\right]\mathbf{I} = \\ &= I_K\mathbf{J}^2 + I_J(\mathbf{JK} + \mathbf{KJ}) + (I_{JK} - I_JI_K)\mathbf{J} - II_J\mathbf{K} + (I_{J^2K} - I_{JK}I_J + I_KII_J)\mathbf{I} \end{aligned} \tag{A.5}$$

$$\begin{aligned} &\mathbf{JKL} + \mathbf{JLK} + \mathbf{KJL} + \mathbf{KLJ} + \mathbf{LJK} + \mathbf{LKJ} = \mathrm{tr}\mathbf{J}(\mathbf{KL} + \mathbf{LK}) + \mathrm{tr}\mathbf{K}(\mathbf{JL} + \mathbf{LJ}) + \\ &+\mathrm{tr}\mathbf{L}(\mathbf{JK} + \mathbf{KJ}) + (\mathrm{tr}\mathbf{KL} - \mathrm{tr}\mathbf{K}\mathrm{tr}\mathbf{L})\mathbf{J} + (\mathrm{tr}\mathbf{JL} - \mathrm{tr}\mathbf{J}\mathrm{tr}\mathbf{L})\mathbf{K} + (\mathrm{tr}\mathbf{JK} - \mathrm{tr}\mathbf{J}\mathrm{tr}\mathbf{K})\mathbf{L} + \\ &+(\mathrm{tr}\mathbf{JKL} + \mathrm{tr}\mathbf{LKJ} - \mathrm{tr}\mathbf{J}\mathrm{tr}\mathbf{KL} - \mathrm{tr}\mathbf{K}\mathrm{tr}\mathbf{JL} - \mathrm{tr}\mathbf{L}\mathrm{tr}\mathbf{JK} + \mathrm{tr}\mathbf{J}\mathrm{tr}\mathbf{K}\mathrm{tr}\mathbf{L})\mathbf{I} = \\ &= I_J(\mathbf{KL} + \mathbf{LK}) + I_K(\mathbf{JL} + \mathbf{LJ}) + I_L(\mathbf{JK} + \mathbf{KJ}) + (I_{KL} - I_KI_L)\mathbf{J} + \\ &+(I_{JL} - I_JI_L)\mathbf{K} + (I_{JK} - I_JI_K)\mathbf{L} + \\ &+(I_{JKL} + I_{LKJ} - I_JI_{KL} - I_KI_{JL} - I_LI_{JK} + I_JI_KI_L)\mathbf{I} \end{aligned} \tag{A.6}$$

The identities (A.1) and (A.5) are recovered from (A.6) by setting $\mathbf{J}=\mathbf{K}=\mathbf{L}$ and $\mathbf{J}=\mathbf{L}$, respectively. A derivation of Eq (A.6) which is independent of Eq (A.1) was given by Rivlin (1955).

From Eqs (A.1) and (A.5), yields

$$\begin{aligned}\mathbf{J}^2\mathbf{K}\mathbf{J}+\mathbf{J}\mathbf{K}\mathbf{J}^2 &= -I_J(\mathbf{J}^2\mathbf{K}+\mathbf{K}\mathbf{J}^2)+I_J^2(\mathbf{J}\mathbf{K}+\mathbf{K}\mathbf{J})+(I_{JK}+I_JI_K)\mathbf{J}^2+ \\ &+(I_{J^2K}-I_J^2I_K)\mathbf{J}-II_J\mathbf{K}+\Big[\frac{1}{3}I_K(I_{J^3}+2I_J^3)+I_J(I_{J^2K}-I_JI_{JK}+I_J^2I_K)\Big]\mathbf{I}\end{aligned} \tag{A.7}$$

Two-dimensional case

In this case the counterparts of (A.1) and (A.6) are

$$\mathbf{J}^2-I_J\mathbf{J}+II_J\mathbf{I}=\mathbf{0} \tag{A.8}$$

and

$$\mathbf{J}\mathbf{K}+\mathbf{K}\mathbf{J}=(\mathrm{tr}\mathbf{K})\mathbf{J}+(\mathrm{tr}\mathbf{J})\mathbf{K}+(\mathrm{tr}\mathbf{K}\mathbf{J}-\mathrm{tr}\mathbf{K}\,\mathrm{tr}\mathbf{J})\mathbf{I}=I_K\mathbf{J}+I_J\mathbf{K}+(I_{KJ}-I_KI_J)\mathbf{I} \tag{A.9}$$

respectively.

From Eqs (A.8) and (A.9), we get

$$\mathbf{J}\mathbf{K}\mathbf{J}=I_{JK}\mathbf{J}+II_J\mathbf{K}-II_JI_K\mathbf{I} \tag{A.10}$$

Appendix B

Nonpolynomial representations for isotropic scalar-valued and symmetric tensor-valued functions

Let $\mathbf{A}_i$ $(i=1,...,N)$ and $\mathbf{W}_p$ $(p=1,...,M)$ denote N symmetric second-order tensors and M skew-symmetric second-order tensors respectively (for 3-dimensional or 2-dimensional cases). The scalar-valued and symmetric tensor-valued functions $f(\mathbf{A}_i,\mathbf{W}_p)$, $\mathbf{F}^s(\mathbf{A}_i,\mathbf{W}_p)$ are said to be isotropic if following equations

$$f(\mathbf{A}_i,\mathbf{W}_p)=f(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^\top,\mathbf{W}_p\mathbf{Q}^\top) \tag{B.1}$$

$$\mathbf{Q}\mathbf{F}^s(\mathbf{A}_i,\mathbf{W}_p)\mathbf{Q}^\top=\mathbf{F}^s(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^\top,\mathbf{Q}\mathbf{W}_p\mathbf{Q}^\top) \tag{B.2}$$

hold for any orthogonal tensor $\mathbf{Q}$ ($\mathbf{Q}\in O(2)$ and $\mathbf{Q}\in O(3)$ for 2-dimensional and 3-dimensional cases, respectively).

Determination of the nonpolynomial representations for isotropic functions (B.1) and (B.2) for $O(3)$ has undergone repeated corrections by Wang (1969), (1970), (1971), by Smith (1970), (1971) and by Boehler (1977). The irreducibility of the sharpened Smith's and Boehler's functions has been proved by Pennisi and Trovato (1987). In Korsgaard's (1990b) and Zheng (1993) papers the representations of (B.1) and (B.2) functions have been obtained by different methods than in the aforementioned contributions. The representations of 2-dimensional functions (B.1) and (B.2) by direct methods independently of the 3-dimensional case have been obtained by Korsgaard (1990a).

The general representation theorem for scalar-valued functions (B.1) states, that f can be expressed as a function of the invariants of the functional basis of the argument tensors

$$f(\mathbf{A}_i,\mathbf{W}_p) = f(I_s) \tag{B.3}$$

where I_s $(s = 1, ..., S)$ are the invariants of the functional basis (Table 1. and Table 2., for 3-dimensional and 2-dimensional cases, respectively).

Table 1. Functional basis for the full orthogonal group $O(3)$

$\mathrm{tr}\mathbf{A}_i$, $\mathrm{tr}\mathbf{A}_i^2$, $\mathrm{tr}\mathbf{A}_i^3$, $\mathrm{tr}\mathbf{A}_i\mathbf{A}_j$, $\mathrm{tr}\mathbf{A}_i^2\mathbf{A}_j$, $\mathrm{tr}\mathbf{A}_i\mathbf{A}_j^2$, $\mathrm{tr}\mathbf{A}_i^2\mathbf{A}_j^2$, $\mathrm{tr}\mathbf{A}_i\mathbf{A}_j\mathbf{A}_k$,	
$\mathrm{tr}\mathbf{W}_p^2$, $\mathrm{tr}\mathbf{W}_p\mathbf{W}_q$, $\mathrm{tr}\mathbf{W}_p\mathbf{W}_q\mathbf{W}_r$,	$i, j, k = 1, ..., N;\ i < j < k$
$\mathrm{tr}\mathbf{A}_i\mathbf{W}_p^2$, $\mathrm{tr}\mathbf{A}_i^2\mathbf{W}_p^2$, $\mathrm{tr}\mathbf{A}_i^2\mathbf{W}_p^2\mathbf{A}_i\mathbf{W}_p$, $\mathrm{tr}\mathbf{A}_i\mathbf{W}_p\mathbf{W}_q$, $\mathrm{tr}\mathbf{A}_i\mathbf{W}_p^2\mathbf{W}_q$, $\mathrm{tr}\mathbf{A}_i\mathbf{W}_p\mathbf{W}_q^2$, $\mathrm{tr}\mathbf{A}_i\mathbf{A}_j\mathbf{W}_p$	
$\mathrm{tr}\mathbf{A}_i^2\mathbf{A}_j\mathbf{W}_p$, $\mathrm{tr}\mathbf{A}_i\mathbf{A}_j^2\mathbf{W}_p$, $\mathrm{tr}\mathbf{A}_i\mathbf{W}_p^2\mathbf{A}_j\mathbf{W}_p$,	$p, q, r = 1, ..., M;\ p < q < r$

Table 2. Functional basis for the full orthogonal group $O(2)$

$\mathrm{tr}\mathbf{A}_i$, $\mathrm{tr}\mathbf{A}_i^2$, $\mathrm{tr}\mathbf{A}_i\mathbf{A}_j$,	$i, j = 1, ..., N;\ i < j$
$\mathrm{tr}\mathbf{W}_p^2$, $\mathrm{tr}\mathbf{W}_p\mathbf{W}_q$,	$p, q = 1, ..., M;\ p < q$
$\mathrm{tr}\mathbf{A}_i\mathbf{A}_j\mathbf{W}_p$	

The representation for the tensor-valued function $\mathbf{F}^s$ Eq (B.2) with a generating set of tensors $\mathbf{G}^s$ can be expressed as the linear combinations

$$\mathbf{F}^s(\mathbf{A}_i,\mathbf{W}_p) = \sum_{l=1}^{L} f_l(\mathbf{A}_i,\mathbf{W}_p)\mathbf{G}_l^s \tag{B.4}$$

where the scalar-valued function $f_l(\mathbf{A}_i,\mathbf{W}_p)$ are general functions of the invariants of the functional basis obtained from Eq (B.3). The general representation theorem for tensor-valued function (B.2) state, that $\mathbf{G}^s$ are the generators given in Table 3 and Table 4.

Table 3. Generators $\mathbf{G}^s$ of a symmetric tensor-valued function for the full orthogonal group $O(3)$

$\mathbf{I}, \mathbf{A}_i, \mathbf{A}_i^2, \mathbf{A}_i\mathbf{A}_j + \mathbf{A}_j\mathbf{A}_i, \mathbf{A}_i^2\mathbf{A}_j + \mathbf{A}_j\mathbf{A}_i^2, \mathbf{A}_i\mathbf{A}_j^2 + \mathbf{A}_j^2\mathbf{A}_i$	$i,j = 1,...,N;\ i < j$
$\mathbf{W}_p^2, \mathbf{W}_p\mathbf{W}_q + \mathbf{W}_q\mathbf{W}_p, \mathbf{W}_p^2\mathbf{W}_q - \mathbf{W}_q\mathbf{W}_p^2, \mathbf{W}_p\mathbf{W}_q^2 - \mathbf{W}_q^2\mathbf{W}_p$	$p,q = 1,...,M;\ p < q$
$\mathbf{A}_i\mathbf{W}_p - \mathbf{W}_p\mathbf{A}_i, \mathbf{W}_p\mathbf{A}_i\mathbf{W}_p, \mathbf{A}_i^2\mathbf{W}_p - \mathbf{W}_p\mathbf{A}_i^2, \mathbf{W}_p(\mathbf{A}_i\mathbf{W}_p - \mathbf{W}_p\mathbf{A}_i)\mathbf{W}_p$	

Table 4. Generators $\mathbf{G}^s$ of a symmetric tensor-valued function for the full orthogonal group $O(2)$

$\mathbf{I},\ \mathbf{A}_i,\ \mathbf{A}_i\mathbf{W}_p - \mathbf{W}_p\mathbf{A}_i$	$i = 1,...,N;\ \ p = 1,...,M$

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Proste wyznaczenie tensorów wydłużenia i rotacji oraz bardziej ogólnych izotropowych tensorowych funkcji deformacji

Streszczenie

Stosując twierdzenia o reprezentacjach, metody interpolacji izotropowych funkcji tensorowych i rozszerzone twierdzenie Cayleya-Hamiltona wyznaczymy $\mathbf{U}$, $\mathbf{U}^{-1}$, $\mathbf{V}$, $\mathbf{V}^{-1}$, $\mathbf{R}$ (bez obliczania wektorów własnych) jako funkcje odpowiednio $\mathbf{C}$, $\mathbf{B}$ i $\mathbf{F}$. Rozpatrujemy także bardziej ogólne funkcje izotropowe $\mathbf{C}$ lub $\mathbf{B}$ np. $\ln \mathbf{U}$. Otrzymujemy pewne nowe i latwe do stosowania wzory na powyższe funkcje zarówno dla dwuwymiarowej jak i trójwymiarowej deformacji.

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