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VIBRATIONS OF LAYERED VISCOELASTIC BEAM WITH INTERLAYER SLIPS

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Two methods for calculating both eigenfrequencies and the logarithmic decrement for layered beams with interlayer slips and consisting of viscoelastic stiffness-comparable layers have been presented in this paper. The hybrid method described in Section 4 is a new one since formulation of the boundary value problem considered has been derived in the new way i.e., by linking different kinematical patterns within adjacent layers. The second method presented in Section 2 has been derived within the linear theory of (visco)elasticity as a result of modification of the formulation (that is interlayer continuity conditions) given by Karczmarzyk (1993). Both the methods have been applied to investigation of influence of the interlayer slips on eigenfrequencies and vibration damping of two- and three-layer beams.

1. Introduction

A few types of composite structures with interlayer slips (delaminations) have been investigated recently. Armanios (1991) developed a two-dimensional theory enabling calculation of stresses distribution within a lap-strap specimen consisting of orthotropic layers subjected to a static force tension. Between the lap and the strap a local delamination has been assumed thus stress concentrations in the vincinity of the crack tip has also been studied. Frostig (1992) presented a theory enabling the study of behaviour of a sandwich beam consisting of isotropic layers subjected to a static bending force when a local delamination appears between an outer layer (face) and the middle layer (core). An influence of delamination length and location on the peeling stress at the face-core interface has been studied. The same problem was investigated both numerically and experimentally by Zenkert (1991). The peeling stresses, increasing in the vincinity of delamination, have also been calculated in the paper. The reader may also find earlier papers on static behaviour of some composite structures with interlayer slips (Goodman and Popov (1968)).

Dynamic behaviour of layered beams with both local and large delaminations has rather rarely been investigated. Mujumdar and Suryanarayan (1988) developed a theory of flexural vibrations of homogeneous, isotropic beam with longitudinal displacement discontinuities. The theory is based on the Bernoulli-Euler equation of motion which has been employed along with the adequate continuity conditions for each part of the beam. The researchers have confirmed their theory experimentally all numerical and experimental results, however, have been presented for slender beams. Since the thickness over length ratio has not been given in the aforementioned paper the author calculated that the parameter for beams from Table 1 on page 458 was between 0.04 and 0.026. On the other hand the researchers have stated on page 457: "There is a slight increase in the error as the beam span reduces, as expected". Taking into account both the statement and well established knowledge concerning inaccuracy of the Bernoulli-Euler theory (cf Huang (1961), Karczmarzyk (1993)) one may conclude that application of the Mujumdar and Suryanrayan approach is limited. Wang et al. (1982) presented the approach based on the Bernoulli-Euler theory, more general however than the one presented by Mujumdar and Suryanrayan (1985). They assumed a number of discrete delaminations within a homogeneous, isotropic beam and derived equations for calculating eigenfrequencies. Unfortunately one can notice a mistake in their paper – see Table 3 on page 498. The eigenfrequencies of beams in the case of off-midplane delaminations are lower than in the case of midplane delaminations!

In this paper the author has presented two methods for calculating both the eigenfrequencies and the logarithmic decrement of layered, simply supported beam with interlayer slips occuring along whole the beam length. Both the methods can easily be extended for clamped-clamped beam (cf Karczmarzyk (1992)). The method described in Section 4 is a new one, since formulation of the boundary value problem considered has been derived by linking different patterns of mechanical behaviour (that is two different kinematical models) of adjacent layers. Because of this it is called here a hybrid method. The method outlined in Section 2 is a modification of the one given by Karczmarzyk (1993). It is more general (and a little more complicated) than the hybrid method, since it has been completely developed within the linear theory of (visco)elasticity. To verify the methods presented in the present paper several numerical results have been given in Tables 2 $\div$ 5.

The beams considered here consist of any number of the stiffness-comparable, both isotropic and anisotropic (i.e., unidirectionally fibrous) layers. The number of interfaces with slips can be less or equal to the number of all the interfaces within the beam – see Fig.1. Both rectangular and non-rectangular cross-sections have been taken into account.

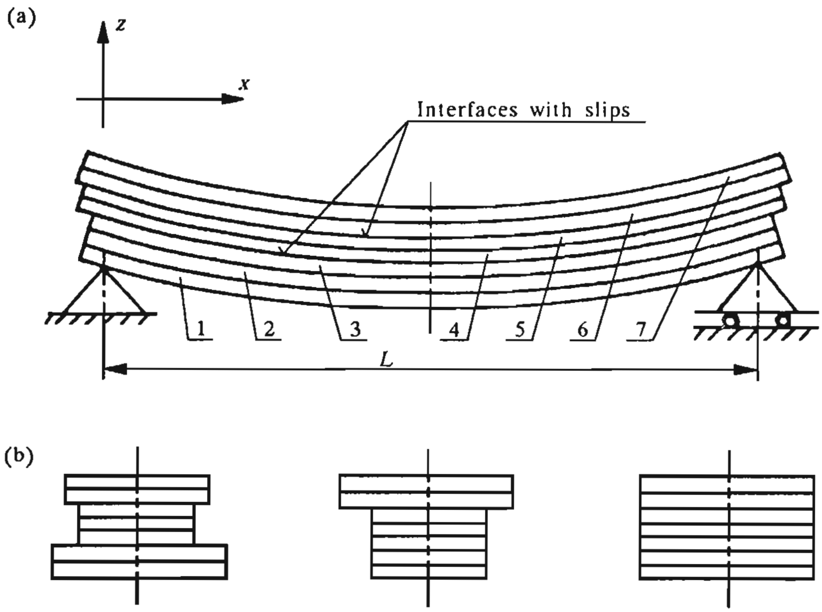

Fig. 1. Layered beam considered in the paper: (a) – deformation of the beam with slips at two interfaces, (b) – exemplary cross-section; numbers 1,2,3,... denote layers

2. Stresses within isotropic layer – an exact solution

We present in this section the exact formulas for stresses within the jth rectangular, isotropic, viscoelastic layer of a vibrating, simply supported beam. Presentation is directed on the free vibration case however, after replacing the eigenfrequency ω_m with the frequency of a sinusoidal force, it is also valid in the case of forced vibration. Within the jth layer of the beam we have a plane stress state so on the basis of Hooke's law we obtain

$$(\sigma_{yy})_j \equiv (\sigma_{22})_j = (\sigma_{yz})_j \equiv (\sigma_{23})_j = (\sigma_{xy})_j \equiv (\sigma_{12})_j = 0 \tag{2.1}$$

By using displacement field functions obtained by Karczmarzyk (1993) one can derive the following formulas for the stresses σ_{zz}, σ_{zx}

$$
\begin{aligned}
(\sigma_{zz})_j &= 2\mu_j\beta_{1j}[\sinh(\beta_{1j}z)X_{1j} + \cosh(\beta_{1j}z)X_{2j}]W(x)\exp(\mathrm{i}\omega_m t) + \\
&+ \frac{\mu_j(\alpha_m^2+\beta_{1j}^2)}{\beta_{2j}}[\sinh(\beta_{2j}z)X_{3j} + \cosh(\beta_{2j}z)X_{4j}]W(x)\exp(\mathrm{i}\omega_m t)
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
(\sigma_{zx})_j &= \frac{\mu_j(\alpha_m^2+\beta_{1j}^2)}{\alpha_m^2}[\cosh(\beta_{1j}z)X_{1j} + \sinh(\beta_{1j}z)X_{2j}]\frac{dW}{dx}\exp(\mathrm{i}\omega_m t) + \\
&+ 2\mu_j[\cosh(\beta_{2j}z)X_{3j} + \sinh(\beta_{2j}z)X_{4j}]\frac{dW}{dx}\exp(\mathrm{i}\omega_m t)
\end{aligned}
\tag{2.3}
$$

where $\alpha_m = m\pi/L$, $m = 1,2,3,...$, L is the beam length, X_{ij} $(i = 1,2,3,4)$ are unknown constants. Function $W(x)$, β_{1j}, β_{2j} appearing above are defined in Section 3.

We notice that formulas for stresses given above are of the same form when the plane strain state within the layered structure is assumed. In such a case the parameter λ' should be replaced with the Lame constant λ (cf Karczmarzyk (1993)). It is noted that the approach presented by Karczmarzyk (1993) can be applied, after some modifications, to formulate boundary problems of the layered beams consisting of isotropic or fibrous layers, with interlayer slips. In this case we have to equate to zero the shear stresses on surfaces of layers at the interfaces with slips and exclude the continuity conditions of longitudinal displacements between layers at the interfaces. The numerical results with subscripts AS (i.e., ω_{AS}, δ_{AS}) presented in Tables 2 ÷ 5 were calculated according to the method.

3. New formulas for displacements and stresses within the Bernoulli-Euler beam theory

The formulas derived in this section are new and necessary to formulate the boundary value problem considered in Section 4. We assume that the displacements within a layer are given by the following equations

$$
\begin{aligned}
u_{x_j} &= -g_j(z)\frac{dW(x)}{dx}\exp(\mathrm{i}\omega_m t) \qquad & u_{z_j} &= f_j(z)W(x)\exp(\mathrm{i}\omega_m t) \\
g_j(z) &= \widetilde{\alpha}_j z + \widetilde{\beta}_j \qquad & W(x) &= W_m\sin\Big(m\pi\frac{x}{L}\Big) \qquad m = 1,2,3,...
\end{aligned}
\tag{3.1}
$$

where $f_j(z)$ is an unknown function whereas $\widetilde{\alpha}_j$, $\widetilde{\beta}_j$ are unknown constants. Assuming that the displacements satisfy the first Navier's equation of motion one obtains, after integrating the equation (in order to obtain $f_j(z)$) and applying the right-hand side formula of Eqs set (3.1), the following expression for displacement $(u_z)_j \equiv u_{zj}$

$$(u_z)_j = \Big[-\Xi_j\Big(\widetilde{\alpha}_j\frac{z^2}{2} + \widetilde{\beta}_j z\Big) + \widetilde{\gamma}_j\frac{1}{\lambda'_j + \mu_j}\Big)\Big]W(x)\exp(\mathrm{i}\omega_m t) \tag{3.2}$$

where

$$\Xi_j = \frac{(\lambda'_j + 2\mu_j)\beta_{2j}^2}{\lambda'_j + \mu_j} \qquad\qquad \beta_{2j}^2 = \alpha_m^2 - \frac{\rho_j\omega_m^2}{\lambda'_j + 2\mu_j} \tag{3.3}$$

and λ'_j defined by Karczmarzyk (1993). The symbol $\widetilde{\gamma}_j$ denotes unknown constant appearing after integrating first Navier's equation of motion. By using Eqs (3.1) $\div$ (3.3) and the constitutive Hooke's law one obtains

$$(\sigma_{zz})_j = [-(\lambda_j + 2\mu_j)\Xi_j + \lambda_j\alpha_m^2](\widetilde{\alpha}_j z + \widetilde{\beta}_j)W(x)\exp(\mathrm{i}\omega_m t) \tag{3.4}$$

$$(\sigma_{zx})_j = \Big[-\mu_j\Big(\frac{\Xi_j z^2}{2} + 1\Big)\widetilde{\alpha}_j - \mu_j\Xi_j z\widetilde{\beta}_j + \frac{\mu_j}{\lambda'_j + \mu_j}\widetilde{\gamma}_j\Big]\frac{dW}{dx}\exp(\mathrm{i}\omega_m t) \tag{3.5}$$

A reader can notice that we have derived functions of displacements and stresses without taking into account the third Navier's equation of the transverse motion. The foregoing functions depend on unknown constants $\widetilde{\alpha}_j$, $\widetilde{\beta}_j$, $\widetilde{\gamma}_j$ i.e.

$$\begin{aligned} (u_x)_j &= u_x(\widetilde{\alpha}_j,\widetilde{\beta}_j) & \qquad (u_z)_j &= u_z(\widetilde{\alpha}_j,\widetilde{\beta}_j,\widetilde{\gamma}_j)\\ (\sigma_{zz})_j &= \sigma_{zz}(\widetilde{\alpha}_j,\widetilde{\beta}_j) & \qquad (\sigma_{zx})_j &= \sigma_{zx}(\widetilde{\alpha}_j,\widetilde{\beta}_j,\widetilde{\gamma}_j) \end{aligned}$$

In further considerations we assume that the equation of transverse motion of the beam is of the following form

$$\int\limits_{z_1}^{z_2}\Big[\frac{\partial(\sigma_{zx})_j}{\partial x} + \frac{\partial(\sigma_{zz})_j}{\partial z}\Big]\,dz = \int\limits_{z_1}^{z_2}\rho_j\frac{\partial^2(u_z)_j}{\partial t^2}\,dz \tag{3.6}$$

Basing on Eqs (3.4) and (3.6) one can write

$$(\sigma_{zz}(x,z_2,t))_j - (\sigma_{zz}(x,z_1,t))_j = \int\limits_{z_1}^{z_2}\Big[\rho_j\frac{\partial^2(u_z)_j}{\partial t^2} - \frac{\partial(\sigma_{zx})_j}{\partial x}\Big]\,dz \tag{3.7}$$

In order to include the equation of motion (3.6) into formulation of any boundary problem we have to calculate $\sigma_{zz}(x,z_1,t)$ (or $\sigma_{zz}(x,z_2,t)$) according to Eq (3.4) and $\sigma_{zz}(x,z_2,t)$ (or $\sigma_{zz}(x,z_1,t)$) according to Eq (3.7). The approach is simplified however quite exact for the purpose of the present paper. Eq (3.7) is structurally equivalent to the Bernoulli-Euler equation of motion of the classical beam theory based on the Kirchhoff's assumption of plane cross-sections. Let us note that within the theory we have the following relationships

$$\int\limits_{z_1}^{z_2} \frac{\partial(\sigma_{zx})_j}{\partial x}\, dz = \frac{dQ(x)_j}{dx} = \frac{d^2(M_g(x))_j}{dx^2} = -E_j I_j \frac{\partial^4(u_z)_j}{\partial x^4} \tag{3.8}$$

where E_j, I_j are the Young modulus and the moment of inertia of a cross-section of the jth layer, respectively. Since the deflection in the classical beam theory is independent of the space variable z then one can write Eq (3.7) in the well known form

$$q_j = \rho_j F_j \frac{\partial^2(u_z)_j}{\partial t^2} + E_j I_j \frac{\partial^4(u_z)_j}{\partial x^4} \tag{3.9}$$

where $q_j = (\sigma_{zz}(x,z_2,t))_j - (\sigma_{zz}(x,z_1,t))_j$ and F_j cross-sectional area of the jth layer. By using Eqs (3.2), (3.3) and (3.5) one can write the difference of normal stresses q_j (thus Eq (3.7)) in the form

$$\begin{aligned} q_j = & -\mu_j \Big[\varXi_j \beta_{1j}^2 \frac{z_2^3 - z_1^3}{6} + \alpha_m^2 (z_2 - z_1)\Big] \widetilde{\alpha}_j W(x) \exp(\mathrm{i}\omega_m t) + \\ & -\mu_j \varXi_j \beta_{1j}^2 \frac{z_2^2 - z_1^2}{2} \widetilde{\beta}_j W(x) \exp(\mathrm{i}\omega_m t) + \frac{\mu_j \beta_{1j}^2}{\lambda_j' + \mu_j} (z_2 - z_1) \widetilde{\gamma}_j W(x) \exp(\mathrm{i}\omega_m t) \end{aligned} \tag{3.10}$$

where

$$\beta_{1j}^2 = \alpha_m^2 - \frac{\rho_j \omega_m^2}{\mu_j}$$

A reader can notice that the factor $\varXi_j$ occurs in Eqs (3.2), (3.4), (3.5) and (3.10). Taking into account the assumptions introduced at the begining of this section we simplify the aforementioned formulas by omitting the factor $\varXi_j$. In further considerations we utilize the following formulas

$$(u_z)_j = \widetilde{\gamma}_j \frac{1}{\lambda_j' + \mu_j} W(x) \exp(\mathrm{i}\omega_m t) \tag{3.11}$$

$$(\sigma_{zz})_j = \lambda_j \alpha_m^2 (\widetilde{\alpha}_j z + \widetilde{\beta}_j) W(x) \exp(\mathrm{i}\omega_m t) \tag{3.12}$$

$$(\sigma_{zx})_j = \Big[-\mu_j \widetilde{\alpha}_j + \frac{\mu_j}{\lambda_j' + \mu_j} \widetilde{\gamma}_j\Big] \frac{dW}{dx} \exp(\mathrm{i}\omega_m t) \tag{3.13}$$

$$q_j = \Big[-\mu_j\alpha_m^2\widetilde{\alpha}_j + \frac{\mu_j\beta_{1j}^2}{\lambda_j' + \mu_j}\widetilde{\gamma}_j\Big](z_2 - z_1)W(x)\exp(\mathrm{i}\omega_m t) \tag{3.14}$$

Let us notice that after the simplification made in Eqs (3.2) ÷ (3.10) the normal stress $\sigma_{zz}(\widetilde{\alpha}_j,\widetilde{\beta}_j,z)$ is a linear function of the space variable z, the shear stress $\sigma_{zx}(\widetilde{\alpha}_j,\widetilde{\gamma}_j)$ does not depend on the space variable z and $q_j(\widetilde{\alpha}_j,\widetilde{\gamma}_j)$ is linearly dependent on the layer thickness. Shear stress σ_{zx} will be equivalent to zero within the layer when homogeneous boundary conditions on free surface of the layer are fulfiled. Thus the formula (3.13) corresponds very well to equation of motion (3.9).

4. A new hybrid formulation of the eigenvalue problem of a split two-layer beam

In this section it will be shown how to connect the simplified solution from Section 3, satisfying the Euler-Bernoulli equation of motion, with the exact solution outlined in Section 2 in order to formulate the eigenvalue problem of a split two-layer beam. It is assumed that one layer of the beam is thin in comparison with the other. Besides we assume that mechanical behaviour of material within the thiner layer is governed by formulas given in Section 3 while behaviour of the thicker layer is determined by formulas mentioned in Section 2. Thus within the thin layer we have plane cross-sections while within the thick layer we have wrapped cross-sections – see Karczmarzyk (1993). Due to the feature the formulation derived in this section is a new one. Let us note that similar approach has been successfully employed by the author to formulate the eigenvalue problem of unsplit two-layer beam composed of stiffness-comparable layers (cf Karczmarzyk (1989)).

Let us introduce a set of boundary and continuity conditions for the two-layer beam with delaminations between layers. The homogeneous stress boundary conditions on the free surface of the thin layer $(z = z_1)$ are as follows

$$\begin{aligned}(\sigma_{zz})_1 &= \lambda_1\alpha_m^2(\widetilde{\alpha}_1 z_1 + \widetilde{\beta}_1)W(x)\exp(\mathrm{i}\omega_m t) = 0\\ (\sigma_{zx})_1 &= \Big[-\mu_1\widetilde{\alpha}_1 + \frac{\mu_1}{\lambda_1' + \mu_1}\widetilde{\gamma}_1\Big]\frac{dW}{dx}\exp(\mathrm{i}\omega_m t) = 0\end{aligned} \tag{4.1}$$

The continuity equations (for stresses and deflection) between layers $(z = z_2)$

are as follows

$$q_1 = 2\mu_2\beta_{12}[\sinh(\beta_{12}z_2)X_{12} + \cosh(\beta_{12}z_2)X_{22}]W(x)\exp(\mathrm{i}\omega_m t) + + \frac{\mu_2(\alpha_m^2+\beta_{12}^2)}{\beta_{22}}[\sinh(\beta_{22}z_2)X_{32} + \cosh(\beta_{22}z_2)X_{42}]W(x)\exp(\mathrm{i}\omega_m t) \qquad (4.2)$$

$$\tau_1 = \frac{\mu_2(\alpha_m^2+\beta_{12}^2)}{\alpha_m^2}[\cosh(\beta_{12}z_2)X_{12} + \sinh(\beta_{12}z_2)X_{22}]\frac{dW}{dx}\exp(\mathrm{i}\omega_m t) + + 2\mu_2[\cosh(\beta_{22}z_2)X_{32} + \sinh(\beta_{22}z_2)X_{42}]\frac{dW}{dx}\exp(\mathrm{i}\omega_m t) = 0 \qquad (4.3)$$

$$u_1 = [X_{12}\cosh(z_2\beta_{12}) + X_{22}\sinh(z_2\beta_{12})]W(x)\exp(\mathrm{i}\omega_m t) + + [X_{32}\cosh(z_2\beta_{22}) + X_{42}\sinh(z_2\beta_{22})]W(x)\exp(\mathrm{i}\omega_m t) \qquad (4.4)$$

Due to first equation of set (4.1) we have the following relationship $q_1 \equiv (\sigma_{zz}(x,z_2,t))_1$ where the quantity q_1 is given in Eq (3.14). The symbols τ_1, u_1 denote shear stress and deflection of the thin layer (i.e. layer 1) for $z = z_2$, $x \in <0, L>$, respectively. The functions are given in Eqs (3.13) and (3.11), respectively. Right-hand sides of Eqs (4.2) ÷ (4.4) refer to the thick layer (i.e. layer 2) and their form results from the solution outlined in Section 2 and derived by Karczmarzyk (1993). Since we consider in this paper layered beams with delaminations appearing between layers thus we omit the continuity equation of longitudinal displacements i.e., the equation $(u_x)_1 = (u_x)_2$ is not taken into account. It is noted that the shear stresses at the interface with slips have been neglected. Such an assumption is consistent with the considerations on formula (3.13) in Section 3.

The homogeneous stress boundary conditions on the free surface of the thick layer $(z = z_3)$ are as follows

$$(\sigma_{zz})_2 = 2\mu_2\beta_{12}[\sinh(\beta_{12}z_3)X_{12} + \cosh(\beta_{12}z_3)X_{22}]W(x)\exp(\mathrm{i}\omega_m t) + + \frac{\mu_2(\alpha_m^2+\beta_{12}^2)}{\beta_{22}}[\sinh(\beta_{22}z_3)X_{32} + \cosh(\beta_{22}z_3)X_{42}]W(x)\exp(\mathrm{i}\omega_m t) = 0 \qquad (4.5)$$

$$(\sigma_{zx})_2 = \frac{\mu_2(\alpha_m^2+\beta_{12}^2)}{\alpha_m^2}[\cosh(\beta_{12}z_3)X_{12} + \sinh(\beta_{12}z_3)X_{22}]\frac{dW}{dx}\exp(\mathrm{i}\omega_m t) + +2\mu_2[\cosh(\beta_{22}z_3)X_{32} + \sinh(\beta_{22}z_3)X_{42}]\frac{dW}{dx}\exp(\mathrm{i}\omega_m t) = 0 \qquad (4.6)$$

In the case of non-rectangular cross-section one has to replace the continuity conditions for stresses with the continuity conditions for forces (cf

Karczmarzyk (1993)). Eigenfrequencies ω_{SS} presented in Tables 2 $\div$ 4 have been calculated according to the method described in this section. The formulation developed can be extended to calculate dynamic parameters of a beam composed of any number of layers provided that each pair of adjacent layers satisfies the fundamental assumption introduced i.e., one of the layers is much more thick than the other one and the formulation outlined in Section 2 is obligatory within the thick layer.

5. Final forms of boundary value problems

The equations (4.1) $\div$ (4.6) can be presented in a short form of algebraic, homogeneous, matrix equation

$$\mathsf{A}Y = 0 \tag{5.1}$$

The column vector Y contains seven unknown constants i.e., $\widetilde{\alpha}_j, \widetilde{\beta}_j, \widetilde{\gamma}_j$ as well as the vector $\boldsymbol{X}_{j+1}$ refering to the solution from Section 2 which is presented in details by Karczmarzyk (1993). In the case considered in Section 3 we have $j = 1$, however the approach presented there can easily be extended for more layers within the beam with interlayer delaminations. If a beam is composed of p_1 thick layers and p_2 thin layers and between thick and thin layers appear slips, the dimension of the square matrix A will be $(4p_1 + 3p_2) \times (4p_1 + 3p_2)$.

Final form of the eigenvalue problem formulated according to the method outlined in Section 2 is also described by Eq (5.1), however in this case the vector $\boldsymbol{Y}$ is composed of vectors $\boldsymbol{X}_l$ for $l = 1, 2, ..., p$ whereas the dimension of the matrix A is $4p \times 4p$ where p denotes number of layers.

In the case of forced sinusoidal vibrations we have to expand into the Fourier series an external load of the beam and then to impose nonhomogeneous boundary conditions. Instead of Eq (5.1) one obtains

$$\mathsf{A}Y = B \tag{5.2}$$

where $\boldsymbol{B}$ is a given vector of external forces acting on the surface of vibrating beam.

6. A method of verifying the solutions presented in the paper

The formulations and solutions of the eigenvalue problem given in previous sections of the present paper can be verified by using a simple formula defining eigenfrequency and resulting from the Rayleigh's approach. When the interlayer delaminations are assumed in each interface of the beam the eigenfrequency of the structure will be defined as follows

$$\omega_m^2 = \frac{\sum\limits_{j=1}^{p} (V_j)_{max}}{\sum\limits_{j=1}^{p} (T_j)_{max}} \tag{6.1}$$

where V_j, T_j denote the potential energy and kinetic, calculated for frequency equal 1, energy of jth layer respectively and p is a number of layers. Assuming that displacements of layers are defined by the Kirchhoff formulas one obtains in the case of a two-layer simply supported beam of non-rectangular cross-section the following formula

$$\omega_m^2 = \frac{1}{12} \frac{\sum\limits_{j=1}^{2} E_j \widetilde{b}_j h_j^3 \alpha_m^2}{\sum\limits_{j=1}^{2} \rho_j \widetilde{b}_j h_j} \tag{6.2}$$

where E_j, ρ_j, $\widetilde{b}_j$, h_j are Young modulus, mass density, width and thickness of the jth layer, respectively.

Validity of the formulas (6.1) and (6.2) is restricted to the first mode of vibration of slender beams, however due to their simplicity they are useful for the purpose of assessing of the theory presented in the previous sections. Results with subscripts BS (i.e., ω_{BS} and δ_{TBS}) given in Tables 2 ÷ 5 have been calculated according to the formula (6.2).

7. Numerical results and discussion

The numerical results have been shown in Tables 2 ÷ 5. The second subscript S denotes parameters calculated for the beams with interlayer slips. In the case of two-layer beams the slips appear between adjacent layers, while in the case of three-layer beam the slips have been assumed between the middle layer and one of the outer (steel) layers. The numerical results with subscripts AS (i.e., ω_{AS}, δ_{TAS}) have been calculated according to the method outlined in Section 2. Results with subscripts BS (i.e., ω_{BS} and δ_{TBS}) in Tables 2 ÷ 4 have been calculated according to formula (6.2). Eigenfrequencies ω_{SS} presented in Tables 2 ÷ 4 have been obtained following the simplified method described in Section 4. Results with the subscript A only are predicted by the method developed in Section 2, Karczmarzyk (1993) for the case without interlayer slips. Values with the subscript B have been calculated for unsplit beams by using the simplified method given in Section 4, Karczmarzyk (1993). Subscript 2 displays the fact that the second layer (according to the sequence given in Table 1) is assumed to be viscoelastic.

Table 1. Material and geometrical (cross-sectional) parameters of beams investigated in the paper. Symbols $2b_j$, h_j denote width and thickness of layers however symbols E_{j1}, ν_j, ρ_j denote the Young modulus, the Poisson ratio and the mass density, respectively, for $j = 1,2$. The beams 1,2 are of non-rectangular (T-) cross-section

	$2b_1$	$2b_2$	h_1	h_2	E_{11}	E_{21}	ν_1	ν_2	ρ_1	ρ_2
	[mm]				[Pa]				[kg/m^3]	
beam 1	105	35	35	105	$0.3\cdot10^{11}$	$0.3\cdot10^{11}$	0.05	0.05	600	600
beam 2	105	35	10	140	$207\cdot10^{9}$	$1.6\cdot10^{10}$	0.25	0.30	7860	1750
beam 3	80	80	10	10	$200\cdot10^{9}$	$1.6\cdot10^{10}$	0.25	0.30	7860	1750
The beam 3 is three-layer. Geometrical and material parameters of layer 3 of the beam are the same as for the layer 1										

Table 2. Eigenfrequencies and logarithmic decrements for the 1st mode of vibration of the beam 1. Values with subscript 2 are obtained for $\eta_{E1} = 0$, $\eta_{E2} = 0.1$, η_{Ej} is the loss factor of jth layer

	L [mm]	1000	1500	2000	2500	3000	3650
x_1	ω_A [rad/s]	2806.86	1271.57	720.23	462.45	321.72	217.62
x_2	ω_B [rad/s]	2907.28	1292.12	726.82	465.16	323.03	218.22
x_3	ω_{AS} [rad/s]	1557.11	696.84	392.93	251.76	174.94	118.24
x_4	ω_{BS} [rad/s]	1576.69	700.75	394.17	252.27	175.19	118.35
x_5	ω_{SS} [rad/s]	1476.17	660.88	372.70	238.81	165.95	112.16
$\xi_{13}(x_1, x_3)$		80.26	82.48	83.30	83.69	83.90	84.05
$\xi_{35}(x_3, x_5)$		5.483	5.441	5.428	5.423	5.417	5.421
$\xi_{43}(x_4, x_3)$		1.257	0.561	0.316	0.203	0.143	0.093
x_6	$(\delta_{TA})_2$	0.19775	0.19535	0.19445	0.19402	0.19378	0.19361
x_7	$(\delta_{TB})_2$	0.19404	0.19404	0.19404	0.19404	0.19404	0.19404
x_8	$(\delta_{TAS})_2$	0.28162	0.28193	0.28204	0.28209	0.28211	0.28213
x_9	$(\delta_{TBS})_2$	0.28274	0.28274	0.28274	0.28274	0.28274	0.28274
$\xi_{86}(x_8, x_6)$		42.36	44.32	45.045	45.39	45.58	45.72
$\xi_{98}(x_9, x_8)$		0.398	0.287	0.248	0.230	0.223	0.216

Table 3. Eigenfrequencies and logarithmic decrements for the 1st mode of vibration of the beam 2. Values with subscript 2 are obtained for $\eta_{E1} = 0$, $\eta_{E2} = 0.1$, η_{Ej} is the loss factor of jth layer

	L [mm]	1000	1500	2000	2500	3000	3650
x_1	ω_A [rad/s]	1492.1	693.0	396.3	255.7	178.3	120.8
x_2	ω_B [rad/s]	1621.0	720.5	405.3	259.4	180.1	121.7
x_3	ω_{AS} [rad/s]	845.63	381.02	215.38	138.16	96.062	64.954
x_4	ω_{BS} [rad/s]	867.02	385.34	216.75	138.72	96.336	65.079
x_5	ω_{SS} [rad/s]	839.51	378.32	213.86	137.18	95.387	64.499
$\xi_{13}(x_1, x_3)$		76.45	81.88	84.00	85.08	85.61	85.98
$\xi_{35}(x_3, x_5)$		0.729	0.714	0.711	0.714	0.708	0.705
$\xi_{43}(x_4, x_3)$		2.529	1.134	0.636	0.405	0.285	0.192
x_6	$(\delta_{TA})_2$	0.26212	0.25730	0.25534	0.25438	0.25384	0.25344
x_7	$(\delta_{TB})_2$	0.25345	0.25345	0.25345	0.25345	0.25345	0.25345
x_8	$(\delta_{TAS})_2$	0.30886	0.30895	0.30899	0.30900	0.30901	0.30902
x_9	$(\delta_{TBS})_2$	0.30978	0.30978	0.30978	0.30978	0.30978	0.30978
$\xi_{86}(x_8, x_6)$		17.83	20.07	21.01	21.47	21.73	21.93

Table 4. Eigenfrequencies and logarithmic decrements for the 3rd mode of vibration of the beam 2. Values with subscript 2 are obtained for $\eta_{E1} = 0$, $\eta_{E2} = 0.1$, η_{Ej} is the loss factor of the jth layer

	L [mm]	1000	1500	2000	2500	3000	3650
x_1	ω_A [rad/s]	9155.7	4974.2	3083.2	2081.1	1492.1	1033.6
x_2	ω_B [rad/s]	14589.3	6484.1	3647.3	2334.3	1621.0	1095.0
x_3	ω_{AS} [rad/s]	6459.7	3160.60	1847.46	1204.90	845.62	575.83
x_4	ω_{BS} [rad/s]	7803.2	3468.08	1950.79	1248.51	867.02	585.71
x_5	ω_{SS} [rad/s]	6397.8	3135.21	1833.49	1196.04	839.51	571.70
$\xi_{13}(x_1, x_3)$		41.74	57.38	66.89	72.72	76.45	79.50
$\xi_{35}(x_3, x_5)$		0.968	0.810	0.762	0.741	0.729	0.722
$\xi_{43}(x_4, x_3)$		20.80	9.729	5.593	3.619	2.531	1.716
x_6	$(\delta_{TA})_2$	0.28813	0.27746	0.27014	0.26530	0.26212	0.25943
x_7	$(\delta_{TB})_2$	0.25345	0.25345	0.25345	0.25345	0.25345	0.25345
x_8	$(\delta_{TAS})_2$	0.30726	0.30831	0.30864	0.30878	0.30886	0.30892
x_9	$(\delta_{TBS})_2$	0.30978	0.30978	0.30978	0.30978	0.30978	0.30978
$\xi_{86}(x_8, x_6)$		6.639	11.12	14.25	16.40	17.83	19.08

Table 5. Eigenfrequencies and logarithmic decrements for the 1st mode of vibration of the beam 3. Values with subscript 2 are obtained for $\eta_{E1} = 0$, $\eta_{E2} = 0.1$, η_{Ej} is the loss factor of the jth layer

	L [mm]	800	900	1000	1100	1200	1300
x_1	ω_A [rad/s]	759.35	601.65	488.24	404.08	339.91	289.88
x_2	ω_{AS} [rad/s]	259.40	204.98	166.05	137.24	115.33	98.27
x_3	$(\delta_{TA})_2$	0.00794	0.00651	0.00548	0.00471	0.00412	0.00366
x_4	$(\delta_{TAS})_2$	0.09544	0.09545	0.09546	0.09547	0.09548	0.09549
$\xi_{12}(x_1, x_2)$		192.73	193.52	194.03	194.43	194.73	194.98
$\xi_{43}(x_4, x_3)$		1102.02	1366.21	1641.97	1926.96	2217.48	2509.02

In order to compare both eigenfrequencies and the logarithmic decrements obtained by different methods and to show influence of the interlayer slips on the values of parameters the following comparative parameter ξ_{ij} has been introduced

$$\xi_{ij} \equiv \xi_{ij}(x_i, x_j) = \frac{x_i - x_j}{x_j} 100 \tag{7.1}$$

On the grounds of ξ_{ij} values given in Tables 2 ÷ 5 we can see that eigenfrequencies of layered beams with the interlayer slips are lower than eigenfrequencies of the structures without slips. In contrast the logarithmic decrement values for split beams are higher than in the case of absence of the interlayer slips. For instance in the case of three-layer beam the logarithmic decrement increase resulting from slips between the middle layer and one of the outer layers varies from 11 times for $L = 1000$ [mm] to 25 times for $L = 3650$ [mm] (see Table 5).

The method outlined in Section 2 predicts accurate and lower values of eigenfrequencies and slightly lower values of the periodic logarithmic decrement, for split beams, than the simple method given in Section 6. With regard to eigenfrequencies the latter conclusion is consistent with the results of Huang (1961). Considering the higher modes of vibrations of thickset beams we can see that the formula (6.2) is useless for calculating eigenfrequencies.

The hybrid method described in Section 4 predicts lower values of eigenfrequencies than the linear elasticity method. The hybrid method enables us to calculate quite accurately eigenfrequencies of layered beam with the interlayer slips provided that each pair of adjacent layers satisfies the fundamental assumption introduced i.e., one of the layers is much more thin than the other one and the simplified formulation from Section 3 is obligatory within the thin layer. For instance when h_2/h_1 ratio (i.e., thick layer thickness over thin layer thickness ratio) is equal to 3 the values ω_{SS} are 5.4% lower than the values ω_{AS} (see Table 2) however for $h_2/h_1 = 14$ the inaccuracy of the eigenfrequencies calculated according to the method from Section 4 varies from 0.7% to 1% (see Tables 3 ÷ 4).

8. Final conclusions

The new hybrid method presented in Section 4 is quite accurate provided that each pair of adjacent layers of a layered beam with the interlayer slips satisfies the fundamental assumption introduced i.e., one of the layers is much more thin than the other one and the simplified formulation from Section 3 is obligatory within the thin layer. The restriction does not concern the second method outlined in Section 2 which is accurate despise the adjacent layers thicknesses ratio.

The logarithmic decrement for split beams is much higher than in the case without the interlayer slips. Increase of vibration damping is very considerable for three-layer beam consisting of layers of comparative thickness when slips appear between one of the outer layers and the middle layer.

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Drgania warstwowej belki lepkosprężystej z międzywarstwowymi poślizgami

Streszczenie

W pracy przedstawiono dwie metody obliczania częstości wlasnych i logarytmicznego dekrementu tlumienia belek warstwowych z międzywarstwowymi poślizgami, skladających się z lepkosprężystych warstw o porównywalnej sztywności. Metoda hybrydowa opisana w rozdziale 4 jest nowa ponieważ sformulowanie rozważanego problemu brzegowego otrzymano w nowy sposób tzn. przez polączenie dwóch różnych modeli kinematycznych w sąsiednich warstwach. Druga metoda prezentowana w rozdziale 2 jest otrzymana w ramach liniowej teorii (lepko)sprężystości w wyniku modyfikacji sformulowania (tzn. międzywarstwowych warunków ciągłości) podanych w pracy Karczmarzyk (1993). Obie metody zastosowano do zbadania wplywu poślizgów między warstwami na częstości wlasne i tlumienie drgań belek dwu- i trójwarstwowych.

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