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COUNTABLE MODEL OF STATIC EFFECT OF NON-LOCAL INTERACTION IN ELASTIC COMPOSITE CIRCULAR DISC

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The paper deals with the static analysis of circular disc in a rotationally-symmetric state of deformation. The effect of elastic non-local interaction between the material of disc and the thin layer of hardening agent on the edge is taken into consideration. A countable model of the problem is formulated and a solution method of the govering equation is proposed. Numerical results show that the synergetic effect is observed in the boundary zone of the disc.

1. Introduction

At the interface of the materials having various mechanical properties effects are often observed which consist in new mechanical properties, different from those of the joined materials, occuring in the small contact zone. These so called synergetic effects, are situated for example near the surface of body hardening by an agent in special technology process or in the contact zone of two materials of a composite, if an additional moderator in the joint is used. The considerable increase in stiffness and strength of the material in the contact zone, quickly fading away from this zone as well as microsopic observation revealing very complicated structure of the joint, allows us to suppose that non-local interactions decreasing quickly with the distance, from this joint can be of importance in the physical interpretation of this phenomenon.

An adequate model of the phenomenon within the framework of continuum mechanics taking into account non-local interactions in the contact zone of different materials can be applied (cf Woźniak, 1969; Eringen and Edelen, 1972; Gould, 1990; Vukobrat and Kuzmanović, 1992). For example the density of these interactions represented by the function of two variables decreasing quickly in an appriopriate manner with the distance between the points in the medium and the contact surface, or a medium with the higher derivatives of the deformation function could be used. Such mathematical models of the phenomenon would lead to complicated integral-differential equations or differential equations of high order. Then the effective solution of the problem would be a difficult task. Only a method of discretization or a finite dimensional approximation method could be considered.

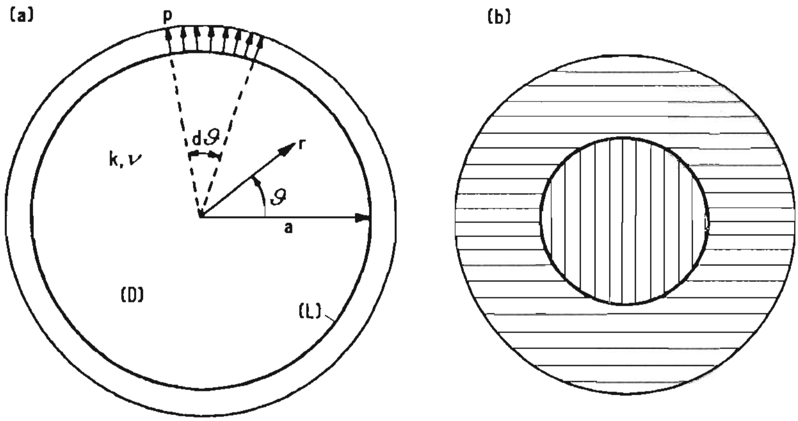


Fig. 1.

In this paper an original concept of countable modelling in the classical mechanics of material media (cf Nagórski, 1989; Czarnecki, 1990) is applied to the static analysis of elastic circular disc with the thin layer of a hardening agent on the edge under assumption of plane and rotationally symmetric state of stress (cf Fig.1a). The countable model of the disc is formulated following closely the idea of the FEM in analogical manner as for a thin elastic rotationally symmetric plate with concentrated force acting in the centre being the accumulation point of nodes (cf Nagórski, 1990) and utilising the concept of one-dimensional countable chain of particles with accumulation point being the particle interacting non-locally with the remaining particles (cf Nagórski and Czarnecki, 1994).

In the proposed model the layer of hardening agent is described by the boundary circle being the accumulation line of circles determined by radial discretization mesh and "interacting" non-locally with remaining circles of the mesh (cf Fig.2). The intensity of these interactions is simulated by a function quickly decreasing with the distance from the boundary of disc.

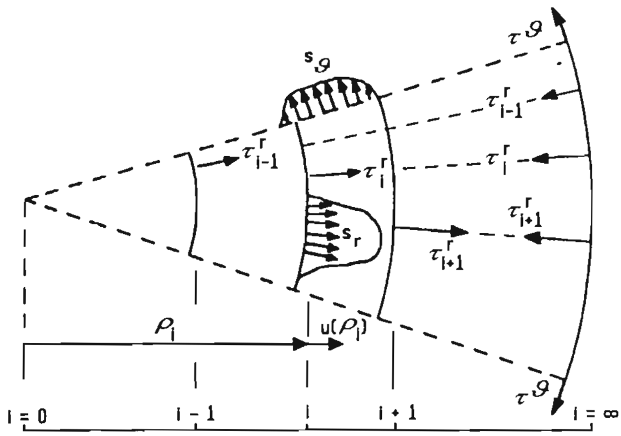


Fig. 2.

Next, an effective method of solution of the governing equation of static problem is presented. It allows us to determine radial displacements and internal stresses with an arbitrary accuracy.

Numerical examples confirm expected qualitative properties of the presented model corresponding with the effects mentionned in the beginnig.

Note that in the analogical way it can be described and analyzed the static problem of two or more discs in the rotationally symmetric state of stress connected on the edges in the presence additional reinforcing moderator (cf Fig.1b).

2. Basic notions and equations

Let us consider a homogeneous and isotropic linearly elastic circular disc (D) in the plane and rotationally symmetric state of stress.

Additionally we assume that the edge of disc is covered by a thin layer of hardening agent (L) in such technology process that the particles of agent interact with the particles of disc in a boundary zone and the interactions decrease quickly with the distance from the edge (cf Fig.1a).

Suppose that the total energy E can be expressed as a sum of the elastic energy E_c of disc, the energy E_n of linearly elastic interactions of edge layer with disc and the potential energy E_p of external uniform load of intensity p on the edge.

Introducing polar coordinates in the plane of (D): undimensional radial $\rho = r/a$ ($\rho \in [0,1]$) and angular ϑ ($\vartheta \in [0,2\pi]$) we discretize the segment $[0,1]$ by nodes ρ_i ($i \in \mathcal{N}$) such that

$$0 = \rho_0 < \rho_1 < \ldots < \rho_i < \ldots \qquad \lim_{i\to\infty} \rho_i = \rho_\infty = 1 \tag{2.1}$$

According to the FEM algorithm we represent the energy

$$E_c = \frac{1}{2} 2\pi a^2 \int\limits_0^1 (s_r\varepsilon_r + s_\vartheta\varepsilon_\vartheta)\rho\, d\rho \ , \tag{2.2}$$

where (cf Fig.2)

$$s_r = k(\varepsilon_r + \nu\varepsilon_\vartheta) \qquad s_\vartheta = k(\varepsilon_\vartheta + \nu\varepsilon_r) \tag{2.3}$$

denote the internal forces and

$$\varepsilon_r = \frac{1}{a}\frac{du}{d\rho} \qquad \varepsilon_\vartheta = \frac{1}{a}\frac{u}{\rho} \tag{2.4}$$

the strains of (D), whereas k, ϑ and u denote stiffness coefficient ($[k] = \mathrm{N/m}$), the Poisson ratio and radial displacement in the form

$$E_c = \pi k \sum_{i=0}^{\infty} \int\limits_{\rho_i}^{\rho_{i+1}} \Big[\Big(\frac{du}{d\rho}\Big)^2 + 2\nu\frac{du}{d\rho}\frac{u}{\rho} + \Big(\frac{u}{\rho}\Big)^2\Big]\rho\, d\rho = \pi k \sum_{i=0}^{\infty} \mathbf{u}_i^\top \mathbf{k}_i \mathbf{u}_i \tag{2.5}$$

where

$$\begin{aligned}
&\mathbf{u}_i = [u_i, u_{i+1}]^\top \qquad (u_i \approx u(\rho_i))\\
&\mathbf{k}_i = \int\limits_{\rho_i}^{\rho_{i+1}} (\mathbf{dN}_i)^\top \mathbf{e}(\mathbf{dN}_i)\rho\, d\rho\\
&\mathbf{d} = \begin{bmatrix} d/d\rho \\ 1/\rho \end{bmatrix} \qquad \mathbf{e} = \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}\\
&\mathbf{N}_i = \begin{bmatrix} \dfrac{\rho_{i+1} - \rho}{\rho_{i+1} - \rho} & \dfrac{\rho - \rho_i}{\rho_{i+1} - \rho_i} \end{bmatrix}
\end{aligned} \tag{2.6}$$

It can be proved that there exists the limit

$$u_\infty = \lim_{i\to\infty} u_i \tag{2.7}$$

if only $E_c < \infty$.

We postulate the energy E_n in the form

$$E_n = \frac{1}{2}\sum_{i=0}^{\infty} 2\pi\kappa_i(u_\infty - u_i) + \frac{1}{2}2\pi\kappa_\vartheta(u_\infty)^2 \tag{2.8}$$

where (cf Fig.2)

$$r_i = \kappa_i \frac{u_\infty - u_i}{\rho_\infty - \rho_i} \tag{2.9}$$

denote the intensity (in N/rad) of non-local interaction between layer (L) treated as the material circle of radius a ($\rho = \rho_\infty = 1$) and the disc (D) represented by the circle of radius $r_i = a\rho_i$, whereas κ_i is a stiffness coefficient which we propose, for instance, to be

$$\kappa_i = \kappa_r \rho_i^\alpha \qquad\qquad i \in \mathcal{N} \tag{2.10}$$

with κ_r (in N/m) and α (nondimensional) being parameters of these interactions. The second term in E_n (2.8) denotes the tensile elastic energy of the layer (L) in which u_∞ is the radial displacement (i.e. $\varepsilon_\vartheta = u_\infty/a$ is the unit elongation of (L)) and κ_ϑ is a coefficient of tensile stiffness.

The potential energy E_p of radial uniform load of intensity p (in N/m) on the edge can be written in the form

$$E_p = -2\pi a p u_\infty \tag{2.11}$$

Let

$$x = (u_0, u_1, \ldots) = (u_i) \qquad\qquad (u_0 = 0) \tag{2.12}$$

and (cf Eqs (2.5), (2.7) ÷ (2.10))

$$\mathcal{V} = \left\{ x = (u_i);\ k\sum_{i=0}^{\infty} \mathbf{u}_i^\top \mathbf{k}_i \mathbf{u}_i + \sum_{i=0}^{\infty} \kappa_r \rho_i^\alpha \frac{(u_\infty - u_i)^2}{1-\rho_i} + \kappa_\vartheta (u_\infty)^2 < \infty \right\} \tag{2.13}$$

and

$$< x, y >= k\sum_{i=0}^{\infty} \mathbf{u}_i^\top \mathbf{k}_i \mathbf{v}_i + \sum_{i=0}^{\infty} \kappa_r \frac{\rho_i^\alpha}{1-\rho_i}(u_\infty - u_i)(v_\infty - v_i) + \kappa_\vartheta u_\infty v_\infty \tag{2.14}$$

for $x, y \in \mathcal{V}$ such that

$$\begin{aligned} &x = (u_i) \qquad && u_0 = 0 \qquad && u_\infty = \lim_{i\to\infty} u_i \qquad && \mathbf{u}_i = [u_i, u_{i+1}]^\top \\ &y = (v_i) \qquad && v_0 = 0 \qquad && v_\infty = \lim_{i\to\infty} v_i \qquad && \mathbf{v}_i = [v_i, v_{i+1}]^\top \end{aligned} \tag{2.15}$$

One can prove that the set $\mathcal{V}$ of sequences (2.12) is a Hilbert space with the inner product (2.14) and

$$Px = pau_\infty \qquad x \in \mathcal{V} \tag{2.16}$$

is the linear continuous form defined on $\mathcal{V}$.

Then the functional

$$F(y) = \frac{1}{2} < y, y > -Py \qquad y \in \mathcal{V} \tag{2.17}$$

is derived (in the Fréchet's sense) on $\mathcal{V}$ and the equation

$$dF(x)y = 0 \qquad \forall y \in \mathcal{V} \tag{2.18}$$

has the unique solution $x \in \mathcal{V}$ being the stationnary point of total energy of the disc (D) with the layer (L)

$$E = E_c + E_n + E_p = 2\pi F \tag{2.19}$$

We postulate that Eq (2.18) is the equilibrium condition of the considered disc whose countable model is described above.

3. Solution method

It is rather impossible to find the exact solution of Eq (2.18). Fortunately there exists a standard technique for obtaining the approximate solution with an arbitrary accuracy in the sense defined below.

Let

$$\mathcal{V}^{(I)} = \{x = (u_i) \in \mathcal{V};\ u_i = u_I,\ i \geq I\} \qquad (I \in \mathcal{N}) \tag{3.1}$$

If the condition

$$\exists \delta > 0 \qquad \frac{1 - \rho_i}{\rho_{i+1} - \rho_i} < \delta \qquad \forall i \geq 1 \tag{3.2}$$

is fulfilled, then it is not too difficult to show that $\mathcal{V}^{(I)}$ is the approximate subspace of $\mathcal{V}$, i.e. $\mathcal{V}^{(I)}$ is finite dimensional linear subspace of $\mathcal{V}$ and

$$\forall y \in \mathcal{V} \qquad \lim_{i\to\infty} |y - y^{(I)}| = 0 \tag{3.3}$$

where $y^{(I)} = (v_i^{(I)})$ is the projection of $y = (v_i)$ onto $\mathcal{V}^{(I)}$

$$v_i^{(I)} = \begin{cases} v_i & i < I \\ v_I & i \geq I \end{cases} \tag{3.4}$$

while $|\cdot|$ is the (energetic) norm in $\mathcal{V}$ (cf (2.14))

$$|y| =< y, y >^{\frac{1}{2}} \tag{3.5}$$

The Cea lemma implies that if $x = (u_i) \in \mathcal{V}$ is the solution of Eq (2.18), i.e.

$$< x, y >= Py \qquad \forall y \in \mathcal{V} \tag{3.6}$$

and $x^{(I)} = (u_i^{(I)}) \in \mathcal{V}^{(I)}$ is the solution of the equation

$$< x^{(I)}, y^{(I)} >= Py^{(I)} \qquad \forall y^{(I)} \in \mathcal{V}^{(I)} \tag{3.7}$$

then

$$\lim_{I \to \infty} |x - x^{(I)}| = 0 \tag{3.8}$$

It should be noted that for the decreasing factor $h = \sup(\rho_{i+1} - \rho_i)$ of mesh nodes the ratio ρ_1/h must be sufficiently positive because the point $\rho = 0$ is singular (cf (2.5)).

The problem (3.7) after taking into account Eqs (2.6), (2.14) and (3.1) can be easily written in the form of the linear algebraic system of equations

$$\mathbf{K}^{(I)}\mathbf{q}^{(I)} = \mathbf{Q}^{(I)} \tag{3.9}$$

in which

$$\begin{aligned} \mathbf{q}^{(I)} &= [u_i^{(I)}, \ldots, u_{I-1}^{(I)}, u_I^{(I)}]^\top \\ \mathbf{Q}^{(I)} &= [0, \ldots, 0, pa]^\top \end{aligned} \tag{3.10}$$

whereas $\mathbf{K}_{I\times I}^{(I)}$ is the stiffness matrix being the sum of stiffness matrix of disc (D) (as in FEM) and a matrix describing non-local interactions between (L) and (D) having the form

$$\begin{bmatrix} \times & & & \times \\ & \cdot & & \cdot \\ & & \cdot & \cdot \\ & & & \cdot \quad \cdot \\ \times & \cdot & \cdot \quad \cdot & \times \end{bmatrix}$$

4. Examples

We shall now apply the method of solution presented in the Section 3 to some examples illustrating the properties of system (D) and (L) whose model was presented in Section 2.

The calculations have been carried out for the following distribution of the nodes over the segment $\rho \in [0,1]$

$$\rho_i = \frac{1}{2^J} \qquad (i = 0, 2, \dots, 2^J - 1)$$
$$\rho_i = 1 - \frac{1}{2^{i'}} \qquad i' = 1 - 2^J + J + 1 \qquad (i = 2^J, 2^J + 1, \dots) \tag{4.1}$$

The tests made for fixed data

$$pa = 1 \qquad k = 1 \qquad \nu = 0.1, 0.3 \qquad \alpha = 3, 5$$
$$\kappa_r = 0.1,\ 1.0 \qquad \kappa_\vartheta = 0, 0.1 \qquad J = 2, 3, 4$$

and for different I $(I > 2^J)$ confirm the good convergence of $u_i^{(I)}$ with increasing I. This convergence is quicker for $J = 3$ and $J = 4$ than for $J = 2$ as well as for $\alpha = 5$ than for $a = 3$.

We can refer the results to the well known exact analytical solution for the disc (without the layer (L))

$$u = \frac{pa}{(1+\nu)k}\rho \qquad s_r = s_\vartheta = p \tag{4.2}$$

which is in a homogeneous state of stress.

For $\kappa_r = \kappa_\vartheta = 0$, i.e. in the absence of non-local interactions, the approximate results obtained for $J = 2$, $I = 6$ are very near the exact results and practically not distinguishable for increasing J and I, respectively.

The values of displacement u are presented in Fig.3 ÷ Fig.6. They are referred to the values determined from Eq $(4.2)_1$. The decrease in displacements of the disc in close proximity to the edge is clearly visible form the plots of u. Naturally, the greater value of the parameter κ_r relative to the value of k, the greater influence of non-local interactions. On the other hand, the greater value of the parameter α, the smaller the zone of influence of these interactions. The significant influence of the tensile stiffness of layer (L) is also worth to underline.

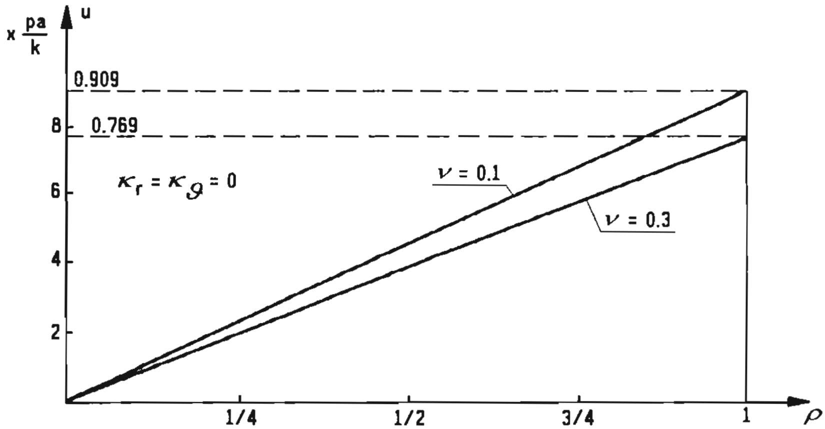


Fig. 3.

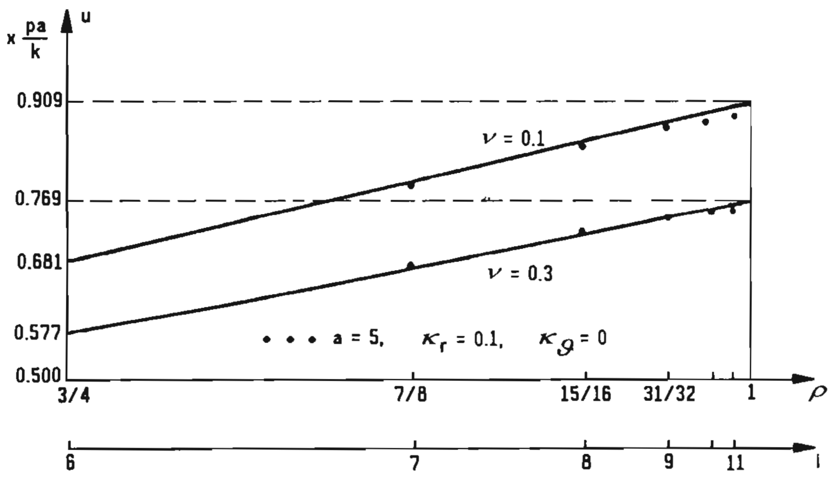


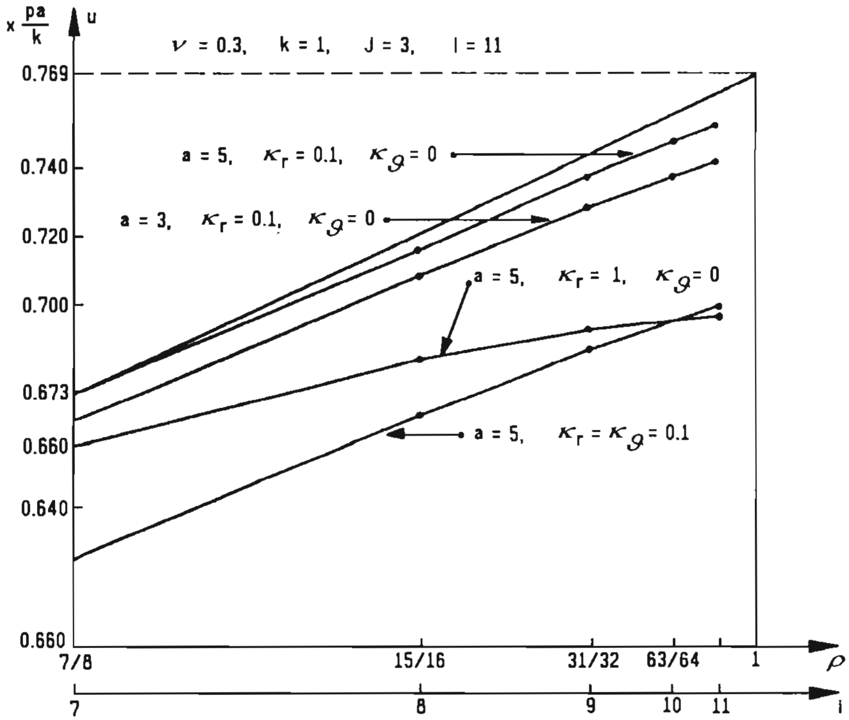
Fig. 4.

The forces s_r, s_ϑ (cf (2.3)) and the ,,total" radial forces (cf (2.9), (2.10)) were also calculated at the points $\rho = \rho_i$ $(i = 1, \ldots, I-1)$

$$s_t = s_r + \frac{1}{a\rho_i} \sum_{j=0}^{i} r_j \tag{4.3}$$

The plots of these forces are presented in Fig.7 ÷ Fig.12.

Note that the values of s_ϑ decrease in the boundary zone in relation to the values for the case when local interactions in the disc only are taken into


x pa/k
u
ν = 0.3, k = 1, J = 3, I = 11
0.769
0.740
0.720
0.700
0.673
0.660
0.640
0.660
a = 5, κr = 0.1, κϑ = 0
a = 3, κr = 0.1, κϑ = 0
a = 5, κr = 1, κϑ = 0
a = 5, κr = κϑ = 0.1
7/8
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31/32
63/64
1
ρ
7
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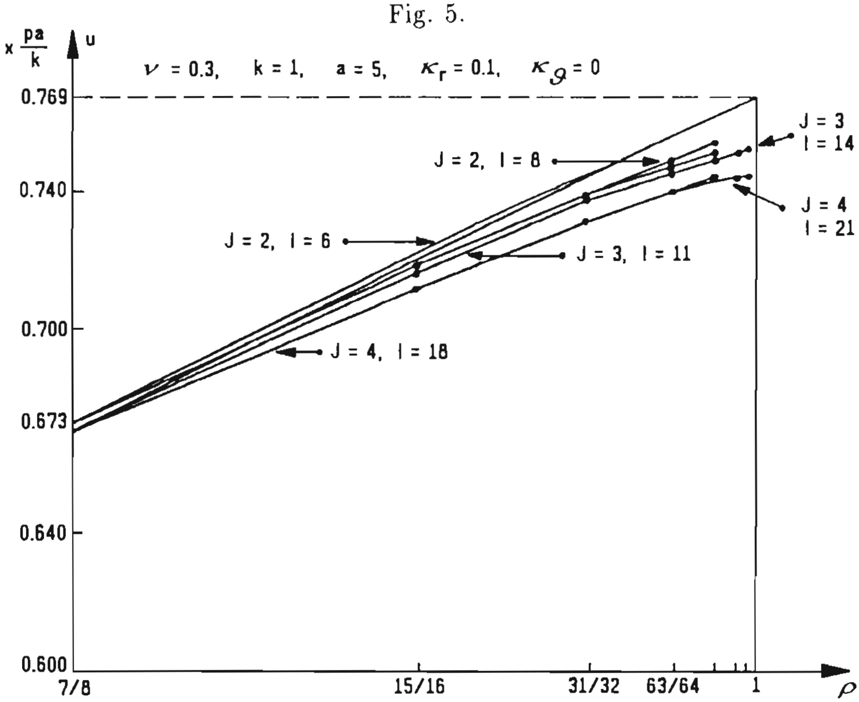
11
i

Fig. 5.


x pa/k
u
ν = 0.3, k = 1, a = 5, κr = 0.1, κϑ = 0
0.769
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0.673
0.640
0.600
J = 3
I = 14
J = 2, I = 8
J = 4
I = 21
J = 2, I = 6
J = 3, I = 11
J = 4, I = 18
7/8
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63/64
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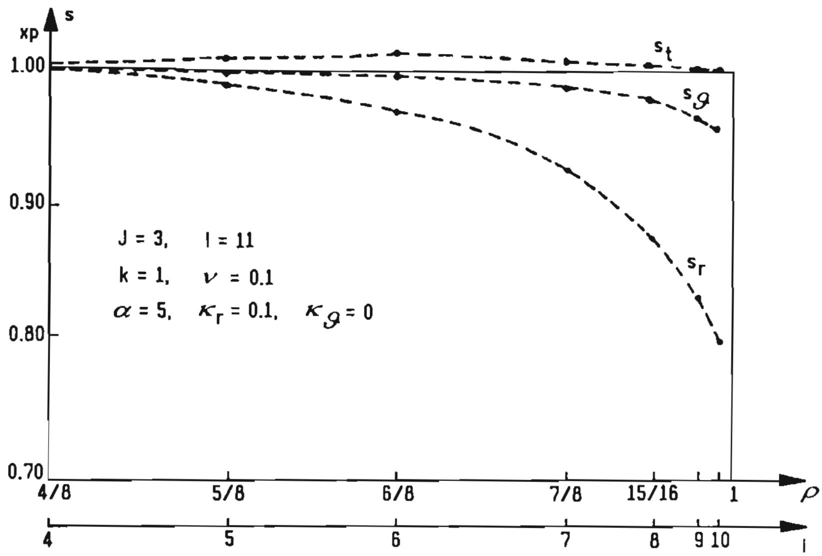
Fig. 6.



s
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s_ϑ
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J = 3, I = 11
k = 1, ν = 0.1
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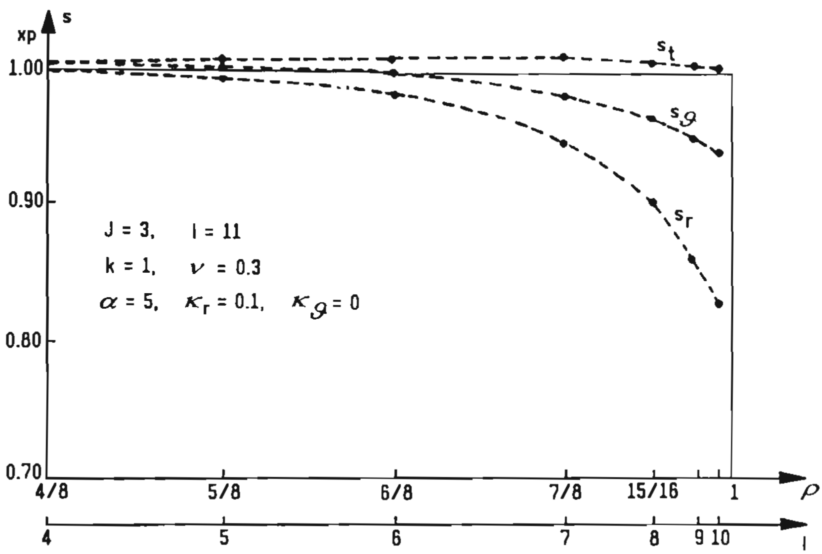
Fig. 7.



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J = 3, I = 11
k = 1, ν = 0.3
α = 5, κ_r = 0.1, κ_ϑ = 0
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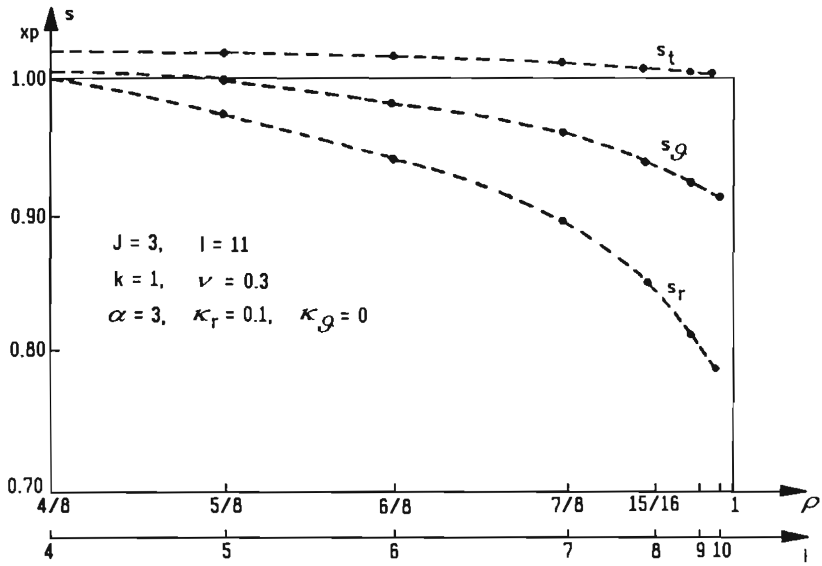
Fig. 8.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0.80
0.70
s_t
s_ϑ
s_r
J = 3, I = 11
k = 1, ν = 0.3
α = 3, κ_r = 0.1, κ_ϑ = 0
4/8
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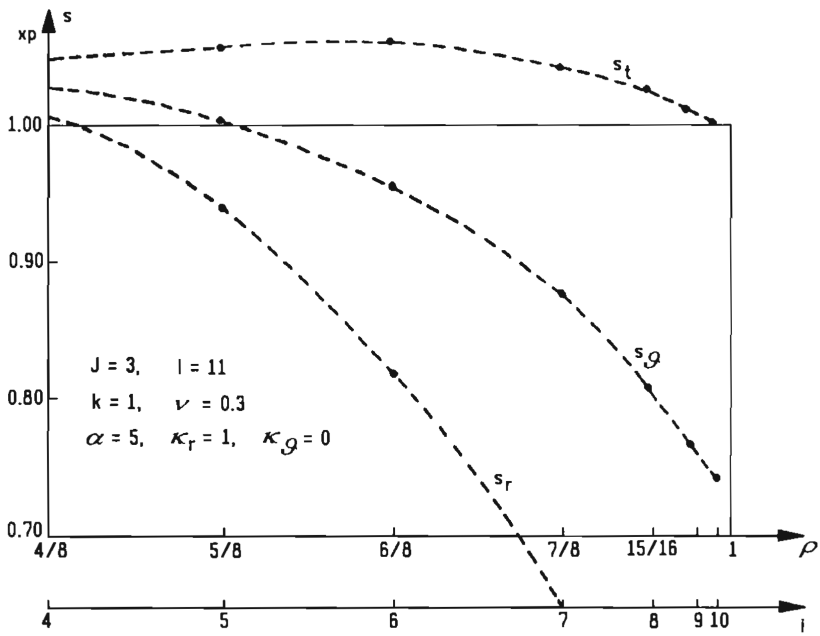
Fig. 9.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J = 3, I = 11
k = 1, ν = 0.3
α = 5, κ_r = 1, κ_ϑ = 0
4/8
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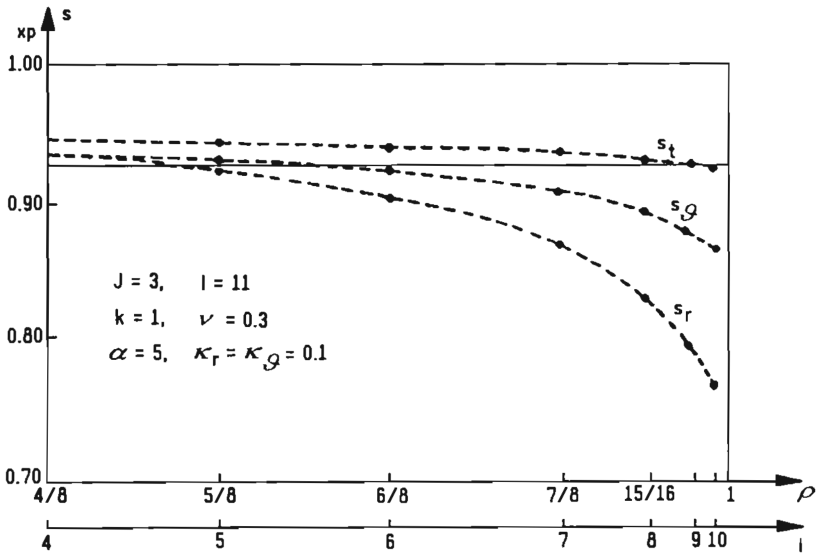
Fig. 10.


xp
s
1.00
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s_t
s_ϑ
s_r
J = 3, I = 11
k = 1, ν = 0.3
α = 5, $\kappa_r = \kappa_\vartheta = 0.1$
4/8
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ρ
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Fig. 11.

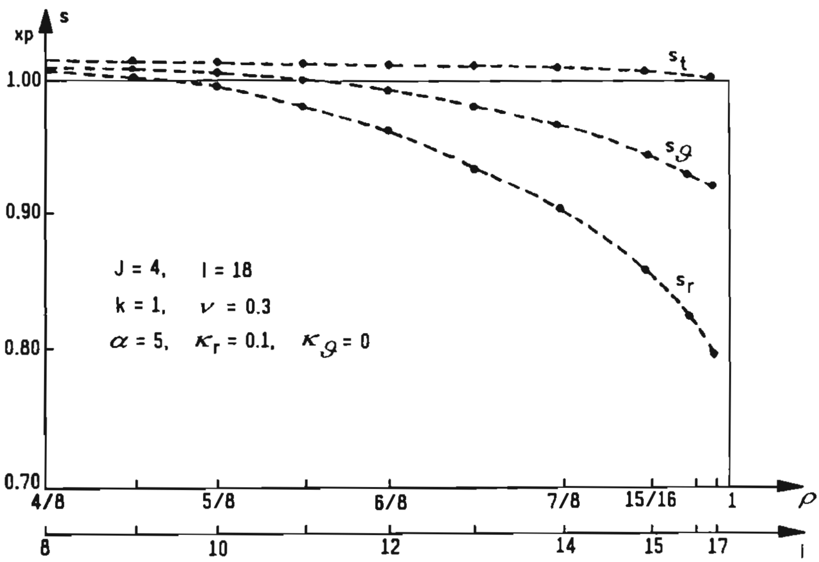
xp
s
1.00
0.90
0.80
0.70
s_t
s_ϑ
s_r
J = 4, I = 18
k = 1, ν = 0.3
α = 5, $\kappa_r = 0.1$, $\kappa_\vartheta = 0$
4/8
5/8
6/8
7/8
15/16
1
ρ
8
10
12
14
15
17
i

Fig. 12.

account, but the values of forces s_t exceed insignificantly only the values of forces s_r, determined in the case of absence of non-local interactions (also for $\kappa_\vartheta > 0$, cf Fig.11). It is also interesting that the greater values of s_t than the values of "classical" s_r appear behind the boundary zone of the disc, i.e. at a distance from the edge.

5. Final remarks

The discrete and countable model of non-local interactions between the disc and the thin layer of hardening agent on the edge has some qualitative properties conforming to the experimental results in which the so called "synergetic effect" is significant. In the model, only the non-local interactions "perpendicular to the edge" have been taken into consideration what seems to be substanciated. The proposed function describing intensity, range and speed of decreasing of non-local interactions with the distance from the "connection surface" of two materials is only one of the possible but it is simple and dependent on three parameters only (cf Eq (2.10)). It seems that an exponential function could be also taken into account. Experiments identifying the parameters used and the correctness of the form of the function as well as the correctness of the approach proposed for the description of the considered phenomena should be undertaken. However they may require unconventional experimental methods in comparison with those which are employed in the classical phenomenological mechanics.

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Przeliczalny model statycznego efektu oddziaływań nielokalnych w sprężystej kompozytowej tarczy kołowej

Streszczenie

Przedmiotem pracy jest analiza statyczna tarczy kołowej w obrotowo-symetrycznym stanie deformacji liniowo-sprężystej. W rozważaniach uwzględnionio efekt oddzialywań nielokalnych między materiałem tarczy a cienką warstwą środka utwardzającego na jej brzegu. Sformułowano przeliczalny model powyższego zagadnienia a następnie zaproponowano metodę rozwiązania otrzymanego równania problemu. Przedstawione przykłady liczbowe wykazały, że oczekiwany efekt wzmocnienia (efekt synergiczny) jest wyraźnie obserwowany w strefie brzegowej tarczy.

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