MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
2, 32, 1994

ONE-DIMENSIONAL COUNTABLE MODEL OF STATIC EFFECT OF NON-LOCAL INTERACTION IN COMPOSITES

ROMAN NAGÓRSKI

SŁAWOMIR CZARNECKI

Faculty of Civil Engineering
Warsaw University of Technology

The paper is devoted to the static analysis of countable medium with an accumulation point in terms of a one-dimensional model of the non-local interactions in a two material interface contact area. The new approach is based on a concept of non-local interactions between the accumulation point defined as a hypothetical particle located on a theoretical boundary of two connected materials and the remaining particles of medium.
The problem of solvability of the equation of equilibrium (qualitative and quantitative analyses) is formulated, i.e. the existence and uniqueness theorems are demonstrated as well as the approximate method is formulated. Numerical results given in the form of diagrams are compared with the solutions which base on a classical model of contact forces.
The character of displacements of particles near the accumulation point (witin the joint region) is qualitatively consistent with the observed strain hardening (synergism) of material.

1. Introduction

At the interface of materials having various mechanical properties (e.g. in composites) there are often observed effects which consist in new mechanical properties, different from the properties of joined materials, which occure within the small contact zone.

The considerable increase in stiffness and strength of material in the interface contact area, quickly fading away from the contact area, as well as microscopic observation revealing a very complicated structure of the interface, caused by mixing of the various component often in the presence of additional reinforcing moderators, allow us to assume that the non-local interactions decreasing quickly with the distance from joint are of importance in the physical interpretation of this phenomenon.

The phenomenon of strength increase following the surface coating of material with a very thin layer of the hardening agent can be interpreted in a similar way.

The above phenomena are to be described within the framework of continuum mechanics, taking into account the non-local interactions in the contact zone of different materials. An adequate model with non-local interactions (cf Woźniak, 1969; Eringen and Edelen, 1972; Rogula, 1973; Sztyren, 1979; Gould, 1990) (e.g. with a density of these interactions given by two variable functions, decreasing quickly in an appropriate manner with the distance between the points in the medium and the contact surfaces, respectively) is available. However such a mathematical model of the phenomenon would lead to integral-differential equations. Thus, the effective solution to the problem becomes a difficult task.

The paper presents a concept of countable modelling in classical mechanics of deformable media (cf Nagórski, 1989; Czarnecki, 1990). On account of the preliminary stage of the study, only a one-dimensional problem is analysed. The countable chain of particles with the accumulation point lying on a straight line is proposed as the model of one-dimensional linear-elastic medium. The accumulation point defined as a particle located on the theoretical boundary of the connected materials (or material and reinforcing agent) describes such joint. The particles are subjected to displacements along the straight line of their initial configuration. The medium deformation is brought about by the external forces, internal elastic local interactions between the adjacent particles and internal elastic non-local interactions between the particle defined as an accumulation point and the remaining particles of the medium. The non-local interactions decrease with the distance from the accumulation point.

The main aim of this paper is:

- To formulate mathematically the problem of the medium of equilibrium
- To formulate the problem of solvability of the equilibrium equations i.e. the existence and uniqueness theorems, respectively
- To propose the effective method to determine the approximate solutions with an arbitrary accuracy
- To show the numerical results and their interpretation in order to confirm the hypothesis of the influence of non-local interactions on the observed increase in stiffness in the close proximity to the materials interface.

2. Basic definitions and equations

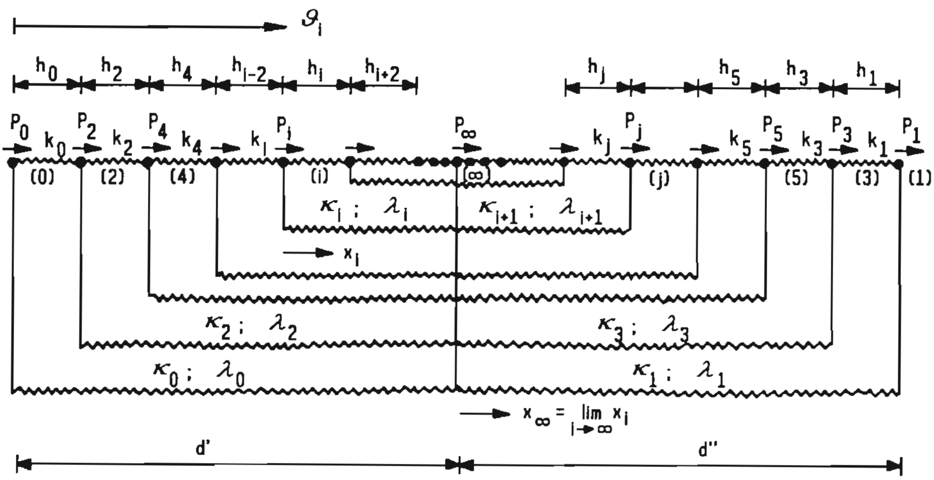

Fig. 1. One-dimensional countable model of the contact zone

Let (C) be a countable set of particles (i) lying on the euclidean straight line (see Fig.1)

$$(C) = \{(i): \ \ i \in \mathcal{N}^*\} \qquad\qquad (\mathcal{N}^* = \mathcal{N} \cup \{0\} = \{0, 1, 2, ...\}) \tag{2.1}$$

We assume the existance of a particle (∞) which is the accumulation point of the set (C).

By (D) we mean the set of particles

$$(D) = (C) \cup \{(\infty)\} \tag{2.2}$$

Under the external forces P_i $(i \in \mathcal{N}^*)$ and P_∞ as well as due to unknown internal interactions, let the particles (i) $(i \in \mathcal{N}^*)$ and the particle (∞) be in equilibrium in the locations determined by $\vartheta_i + x_i$ and $\vartheta_\infty + x_\infty$ respectively, where x_i, x_∞ denote the displacements of particles (i) and (∞) and ϑ_i, ϑ_∞ denote their coordinates in the natural configuration, $(P_i = P_\infty = 0)$.

We assume that (see Fig.1)

$$d' = \sum_{j=0}^{\infty} h_{2j} < \infty \qquad\qquad d'' = \sum_{j=0}^{\infty} h_{2j+1} < \infty \tag{2.3}$$

where $h_i > 0$ $(i \in \mathcal{N}^*)$ are given parameters determining the distribution of the set (D) particles in the state of natural equilibrium; i.e.

$$\vartheta_{2j} = \sum_{k=0}^{j} h_{2k} \qquad\qquad \vartheta_{2j+1} = \sum_{k=0}^{j} h_{2k+1} \tag{2.4}$$

We shall postulate that the interaction forces between particles $(i+2)-(i)$ are proportional to the relative difference in their displacements

$$F_i = k_i(x_{i+2} - x_i) \qquad\qquad (i \in \mathcal{N}^*) \tag{2.5}$$

where (see Fig.1)

$$k_i = \frac{e_i}{h_i} \qquad\qquad (i \in \mathcal{N}^*) \tag{2.6}$$

The coefficients e_i are given characteristics of this interactions refered to the relative strain and satisfying the condition

$$\exists e_*, e^* \in \mathcal{R} \quad \forall i \in \mathcal{N}^* \quad 0 < e_* \le e_i \le e^* < \infty \tag{2.7}$$

Furthermore we shall postulate that the particle (∞) interacts with all the remaining particles $(i) \in (C)$. The respective forces for this kind of interactions are assumed in the form

$$S_i = \kappa_i(x_\infty - x_i) \qquad\qquad (i \in \mathcal{N}^*) \tag{2.8}$$

where, like in Eq (2.6) (see Fig.1)

$$\kappa_i = \frac{\varepsilon_i}{\lambda_i} \qquad\qquad (i \in \mathcal{N}^*) \tag{2.9}$$

The coefficients ε_i are given characteristics of the interactions between particles $(i) - (\infty)$ refered to the relative strain.

We also assume that

$$\begin{aligned} \varepsilon_{2j} &= \varepsilon'\Big(1 - \frac{\lambda_{2j+2}}{d'}\Big)^{\alpha'} & \qquad \varepsilon_{2j+1} &= \varepsilon''\Big(1 - \frac{\lambda_{2j+3}}{d''}\Big)^{\alpha''} \\ \lambda_{2j} &= \sum_{k=j}^{\infty} h_{2k} & \qquad \lambda_{2j+1} &= \sum_{k=j}^{\infty} h_{2k+1} \end{aligned} \tag{2.10}$$

as well as

$$0 \le \varepsilon', \varepsilon'', \alpha', \alpha'' < \infty \tag{2.11}$$

The parameters ε', ε'' determine the magnitude of nonlocal interactions of the particle (∞), whereas the parameters α', α'' indicate their range, i.e. the decay rate depending on the increase in the distance λ_i from the particle (∞).

We emphasize here that the springs and the vertical segments shown in Fig.1 are of auxiliary significance only and merely visualize the scheme of internal interactions in the investigated system (D) which, in fact, is one-dimensional and rectilinear.

We define the countable medium (D) as a set of particles (2.2) together with the above postulated external and internal interactions. The particles of this medium are subjected to statical displacements x_i $(i \in \mathcal{N}^*)$ and x_∞, satisfying the support conditions as well as the following equation of equilibrium employing the principle of virtual work

$$\sum_{i=0}^{\infty} k_i(x_{i+2}-x_i)(y_{i+2}-y_i)+\sum_{i=0}^{\infty} \kappa_i(x_\infty-x_i)(y_\infty-y_i) = \sum_{i=0}^{\infty} P_i y_i + P_\infty y_\infty \qquad (2.12)$$

for the arbitrary allowable (virtual) displacements y_i $(i \in \mathcal{N}^*)$ and y_∞.

The medium (D) is our one-dimensional countable model of interactions two different materials connected at the point modelled by the particle (∞).

Let us notice that putting

$$x_1 = x_3 = x_5 = ... = x_\infty \qquad (2.13)$$

$$P_1 = P_3 = P_5 = ... = P_\infty = 0 \qquad (2.14)$$

(see Fig.1) we obtain – on the basis of Eq (2.12), (using the principle of virtual work) the equation of equilibrium for the countable medium modelling the interaction between given material and hardening agent (the right-hand side of the medium (D) undergoes rigid deformation i.e. translation).

3. The problem of equilibrium

Let

$$X = \{x = (x_i): \ x_i \in \mathcal{R}, \ i \in \mathcal{N}^*\} \qquad (3.1)$$

be a linear space of the series in which two operations are defined: addition and multiplication by scalars

$$(x_i) + (y_i) = (x_i + y_i) \qquad a(x_i) = (ax_i) \qquad (3.2)$$

Let next

$$V = \Big\{x = (x_i) \in X : \quad x_0 = 0, \quad \lim_{i \to \infty} x_i = x_\infty, \qquad\qquad (3.3)$$
$$\sum_{i=0}^{\infty} [k_i |x_{i+2} - x_i|^2 + \kappa_i |x_\infty - x_i|^2] < \infty \Big\}$$

Under the assumptions (2.3) ÷ (2.11) it can be proved that the set V is a Hilbert space with the inner product

$$< x, y > \equiv \mathcal{K}(x, y)$$

where

$$\mathcal{K}(x, y) \equiv \sum_{i=0}^{\infty} [k_i (x_{i+2} - x_i)(y_{i+2} - y_i) + \kappa_i (x_\infty - x_i)(y_\infty - y_i)] \qquad (3.4)$$

is a bilinear positive definite form over V.

Moreover, if the sequence (P_i) satisfies the condition

$$\sum_{i=1}^{\infty} \frac{P_i^2}{h_i} < \infty \qquad (3.5)$$

then

$$\mathcal{P}(y) \equiv \sum_{i=1}^{\infty} P_i y_i + P_\infty y_\infty \qquad (3.6)$$

is a continuous linear form defined over V.

Let us notice that for the medium (D) with fixed end at $i = 0$ $(x_0 = 0)$ according to Eqs (3.4) and (3.6) we can rewrite the Eq (2.12) in the form

$$\mathcal{K}(x, y) = \mathcal{P}(y) \qquad (3.7)$$

Hence by the Riesz representation theorem (Lax-Milgram lemma) there exists one and only one element (sequence) $x \in V$ which satisfies Eq (3.7) for all $y \in V$. The sequence $x = (x_i)$ determines the state of equilibrium of medium (D) (i.e. the statical displacement of its particles).

The above statement (under the same assumptions) holds also in the case when the end at $i = 1$ is fixed $(x_1 = 0)$.

If the displacements of the medium (D) ends are given, i.e.

$$x_0 = u_0 \qquad \text{or} \qquad x_1 = u_1 \qquad (3.8)$$

then the new unknown

$$v = x - u \tag{3.9}$$

can be introduced for arbitrary $u = (u_0, u_1, ...) \in X$ such that $\widetilde{u} = (0, u_1, u_2, u_3, ...) \in V$ (when $x_0 = u_0$, x_1 free) or relevantly $\widetilde{u} = (0, 0, u_2, u_3, ...) \in V$ (when $x_0 = u_0, x_1 = u_1$ and the problem (3.7) can be formulated as follows

$$\mathcal{K}(v, y) = \mathcal{P}(y) - \mathcal{K}(u, y) \qquad y \in V \tag{3.10}$$

It can be proved that the solution $x = v + u$ does not depend on the choice of u.

Similarly in the problem involving interaction between the medium and the hardening agent, the conclusions formulated in this paragraph remain true. It suffices to make use of Eq (2.13) and assume that the right end $(i = 1)$ is free.

4. Approximate solution

It is rather difficult or even impossible in most cases to find the exact solution to the problem presented in Section 3. Fortunately there exist a standard technique for obtaining approximate solutions with an arbitrary accuracy in a sense defined below. We shall now demonstrate this method, focusing our attention, for convenience, on the case when the medium (D) is fixed at the end $i = 0$ $(x_0 = 0)$.

Let

$$V^{(I)} = \{x = (x_i) \in X : \ x_i = x_I, \ i \geq I\} \qquad (I \in \mathcal{N}) \tag{4.1}$$

If condition

$$\exists \delta > 0 \quad \forall i \geq 2 \qquad \frac{1}{h_{i-2}} \sum_{j=i}^{\infty} h_j \leq \delta \tag{4.2}$$

is fulfilled, then it is not difficult to show that $V^{(I)}$ is the approximate subspace of V, i.e. $V^{(I)}$ is a finite-dimensional linear and closed subspace of V as well as

$$\forall y \in V \quad \lim_{i \to \infty} |y - y^{(I)}| = 0 \tag{4.3}$$

where $y^{(I)}$ is a projection of y onto $V^{(I)}$

$$y_i^{(I)} = \begin{cases} y_i^{(I)} & \text{for} \quad i < I \\ y_I & \text{for} \quad i \geq I \end{cases} \tag{4.4}$$

while $|\cdot|$ is the (energetic) norm in V i.e. (see (3.4))

$$|y| = \sqrt{\mathcal{K}(y, y)} \tag{4.5}$$

The Cea lemma (cf Ciarlet, 1978) implies that if x ($x \in V$) is a solution to Eq (3.7) and $x^{(I)}$ ($x^{(I)} \in V^{(I)}$ is a solution to the equation

$$\mathcal{K}(x^{(I)}, y) = \mathcal{P}(y) \qquad \forall y \in V^{(I)} \tag{4.6}$$

then

$$\lim_{I\to\infty} |x - x^{(I)}| = 0 \tag{4.7}$$

The problem (4.6) after taking into account Eqs (3.4), (3.6) and (4.1) can be easily rewritten in the form of the linear algebraic equations in I unknowns $x_1^{(I)}, ..., x_I^{(I)}$ which turns out to be suitable for algorithmization on a computer.

5. Examples

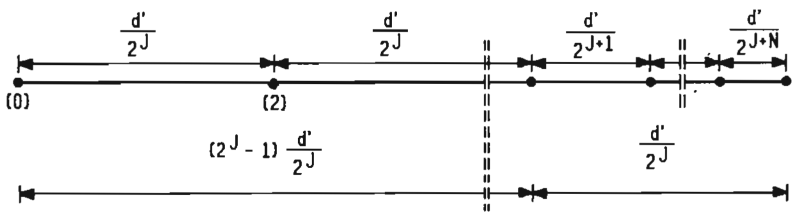

Fig. 2. Approximate analysis – distribution of the particles in the finite-dimensional model

Subsequently the method of solution described above is applied to some examples illustrating the medium (D) properties. The calculations have been done out for the following distribution of the particles (i) in the segments $[0, d']$, $[d', d' + d'']$ (see Fig.1 and Fig.2)

$$h_{2j} = \begin{cases} \dfrac{d'}{2^J} & 0 \le j \le 2^J - 2 \\ \dfrac{d'}{2^J}\Big(\dfrac{1}{2}\Big)^{2+j-2^J} & j \ge 2^J - 1 \end{cases}$$

$$h_{2j+1} = \begin{cases} \dfrac{d''}{2^J} & 0 \le j \le 2^J - 2 \\ \dfrac{d''}{2^J}\Big(\dfrac{1}{2}\Big)^{2+j-2^J} & j \ge 2^J - 1 \end{cases} \tag{5.1}$$

and on account of demonstrative (qualitative) character of the paper the following data have been assumed

$$
\begin{aligned}
& d' = d'' = 1 \\
& e_i = e = 1 && \forall i \in \mathcal{N}^* \\
& \varepsilon' = \varepsilon'' = \varepsilon = 0.1 && \text{or} \qquad \varepsilon' = \varepsilon'' = \varepsilon = 1.0 \\
& \alpha' = \alpha'' = \alpha = 3 && \text{or} \qquad \alpha' = \alpha'' = \alpha = 7
\end{aligned}
\tag{5.2}
$$

Moreover we put

$$P_1 = 1 \qquad\qquad P_\infty = 0 \tag{5.3}$$

for the medium fixed at the end $i = 0$ $(x_0 = 0)$ also

$$P_\infty = 1 \tag{5.4}$$

for the medium fixed at both ends $(x_0 = x_1 = 0)$ and $P_i = 0$ for the remaining indices i.

In order to find an approximate solution (sequence) $x_1^{(I)}, ..., x_I^{(I)}$ according to the procedure described in Section 4 we take

$$I = 2(2^J - 1 + N) \tag{5.5}$$

The parameter J determines the density of the distribution of the particles on the segment $[0, d' + d'']$ whereas the parameter N determines the range of the approximation zone of the sequence x.

The presentation of results is confined to the particles contained within the segment $[0, d']$ (left side of the medium) in view of the fact that the solution is roughly symmetrical with respect to the accumulation point.

For data given in Eqs (5.2), (5.4) and $J = 2$, $J = 3$ it was found that the numerical results show a good convergence $x_i^{(I)}$ to x_i with $N \to \infty$ (or $I \to \infty$, see Eq (5.5) and from $N = 10$ are practically undistinguishable.

In Fig.3 and Fig.4 the values of displacements of the particles are plotted in relation to the displacement diagram in the case when local interactions only are taken into account (full line). The increase in material stiffness in close proximity of the particle (∞) due to the existance of the non-local interactions is clearly visible. Naturally, the greater value of the parameter ε relative to the value of e the larger influence of the non-local interaction (see Eqs (5.2), (2.6), (2.9), (2.10)). On the other hand, the greater value of the parameter α the smaller zone of non-local interaction. The graphs of the internal forces F_i (see Eq (2.5)) and the forces

$$N_i = \sum_{j=0}^{i} S_j \tag{5.6}$$

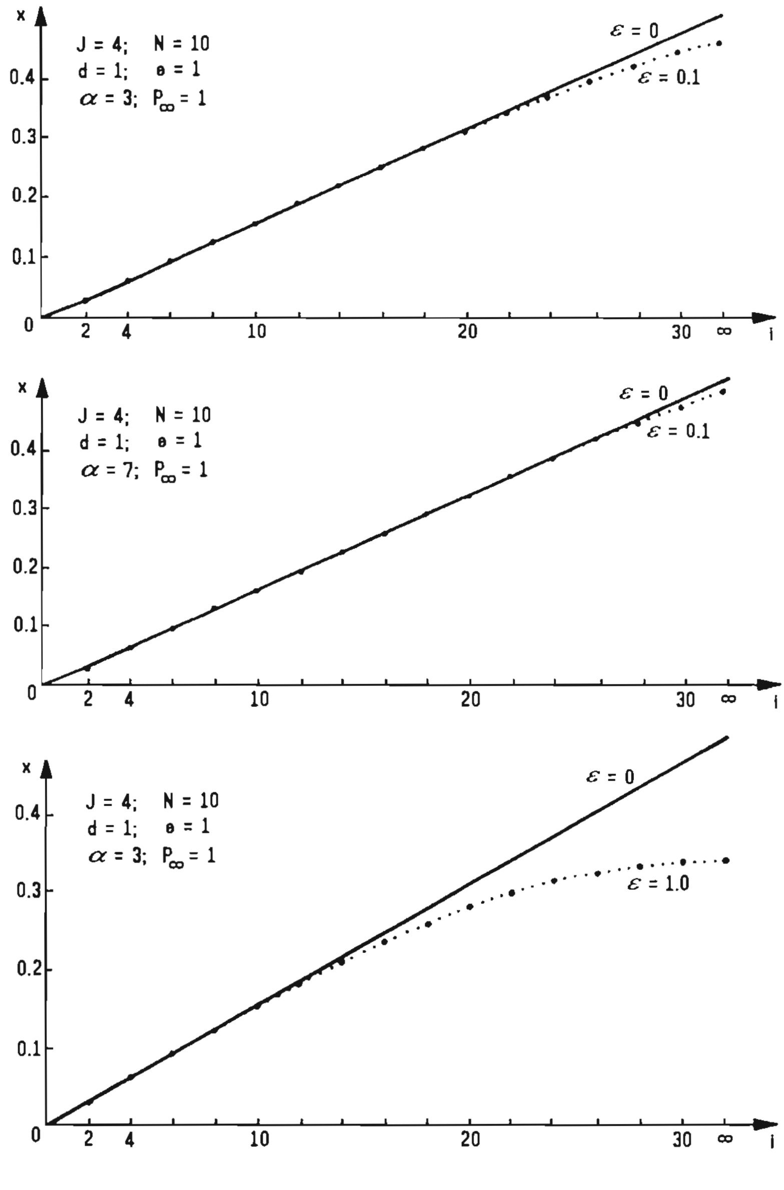

Fig. 3. Displacement x of the particle (i)

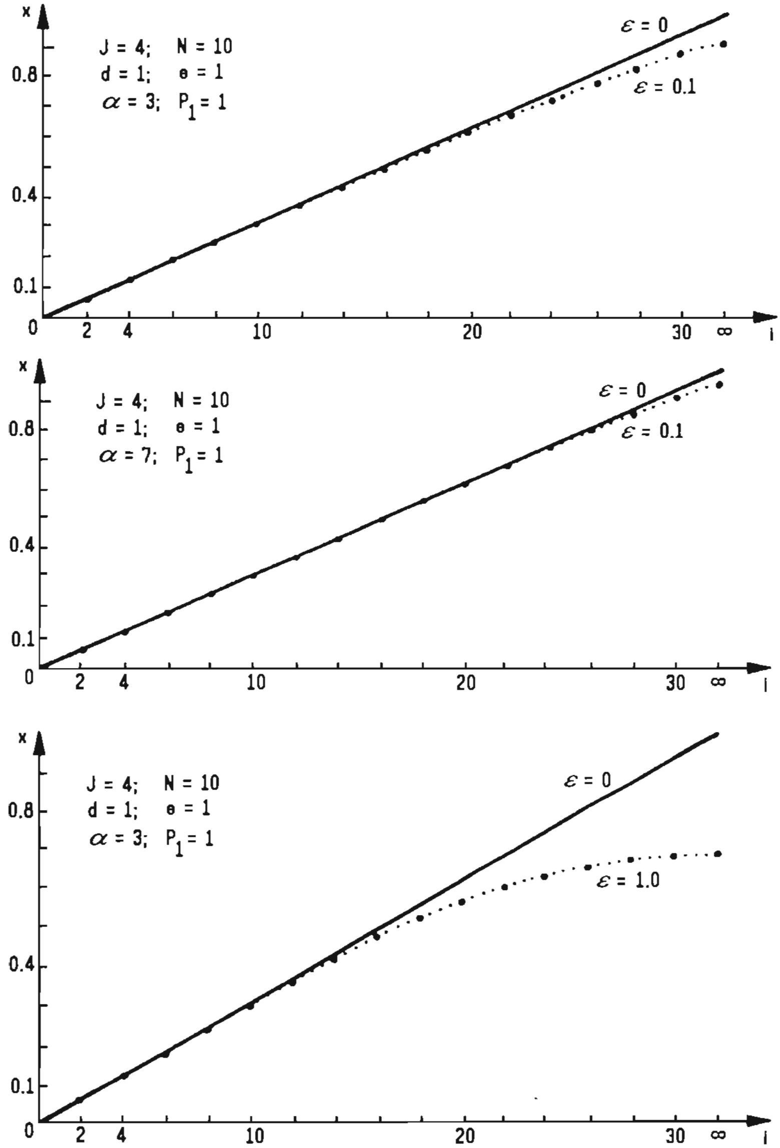

Fig. 4. Displacement x of the particle (i)

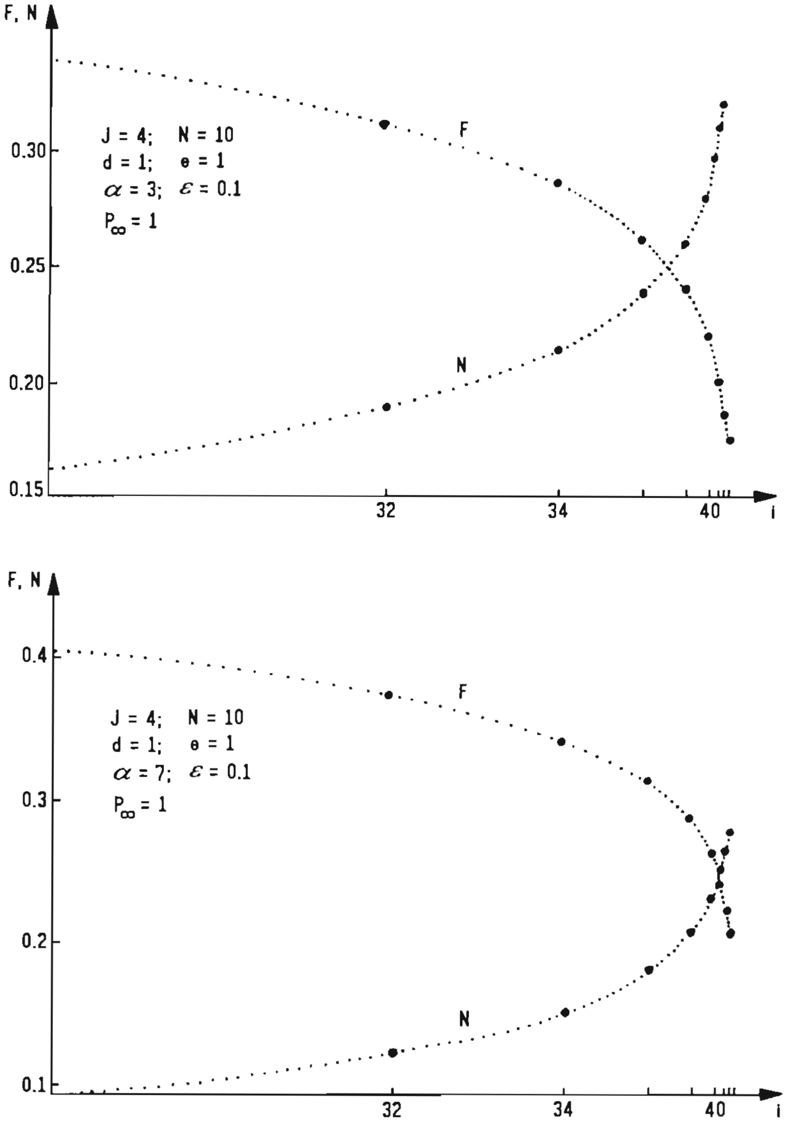

Fig. 5. Distribution of the internal forces F_i and N_i between particles $(i+2)-(i)$

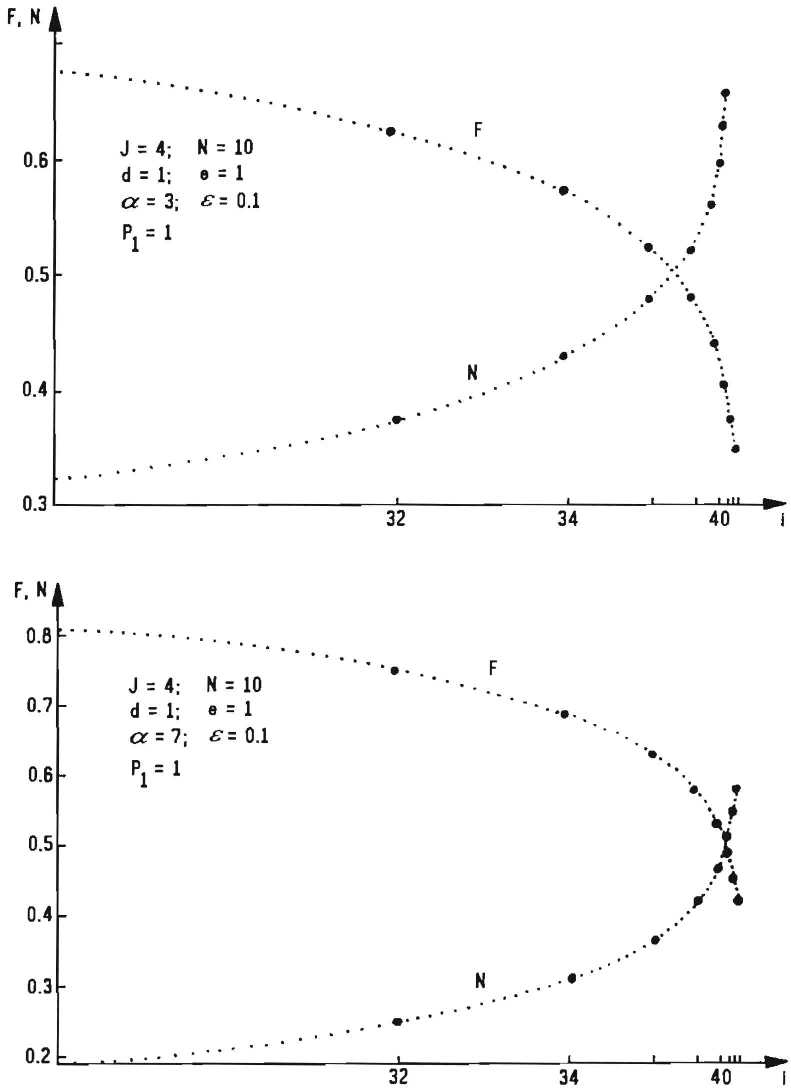

Fig. 6. Distribution of the internal forces F_i and N_i between particles $(i+2)-(i)$

(see Eq (2.8)) are presented in Fig.5 and Fig.6. Obviously we have

$$N_i + F_i = \text{const} \qquad \forall i \in \mathcal{N} \tag{5.7}$$

It is obvious that the local (classical) interactions quickly decrease and nonlocal interactions increase when the distance from the particle (∞) decreases.

6. Final remarks

The medium (D) presented in the Section $2 \div 5$ as a model of interactions "perpendicular to the boundary surface" of the connected materials has a demonstrative character. However its qualitative properties are in accordance with some experimental results in which the so called "synergic effect" is significant.

Further necessary investigations should focus at least on two-dimensional models of the phenomenon and on carrying out the experiments identifying the parameters used. However, they may require the unconventional experimental methods in comparison with those which are employed in the classical phenomenological mechanics.

References

1. CIARLET P., 1978, *The finite element method for elliptic problems,* Amsterdam-New York-Oxford, North Holland Pub. Co.
2. CZARNECKI S., 1990, *Countable modelling in the statics of some material media,* (in Polish), PhD thesis, Warsaw University of Technology
3. ERINGEN A.C., EDELEN D.G.B., 1972, *On nonlocal elasticity,* Int.J.Engng Sci., **10**, 233-248
4. GOULD L.I., 1990, *Balance laws associated with nonlocal equations of motion for theories containing one dependent variable,* Int.J.Engng Sci., **28**, 459-468
5. NAGÓRSKI R., 1989, *Sur une conception de la mecanique des milieux denombrables,* Mech.Teor.i Stos., **27**, 1, 23-32
6. ROGULA D., 1973, *On nonlocal continuum theories of elasticity,* Arch. Mech.Stos., **25**
7. SZTYREN M., 1979, *Boundary-value problems and surface forces for integral models of nonlocal elastic bodies,* Bull.Polish Acad.Sci., 8-9
8. WOŹNIAK CZ., 1969, *Introduction to the dynamics of deformable solids,* (in Polish), PWN Warsaw

Jednowymiarowy przeliczalny model efektu statycznego oddziaływań nielokalnych w kompozytach

Streszczenie

Celem pracy było studialne zbadanie możliwości zastosowania ośrodka przeliczalnego z punktem skupienia z uwzględnieniem jego oddziaływań nielokalnych z pozostałymi cząstkami ośrodka do modelowania efektu synergicznego w kompozytach w pobliżu miejsca (powierzchni) kontaktu dwóch materiałow. To oryginalne podejście do problemu zbadano wstępnie dla przypadku modelu jednowymiarowego. Zbadano równania tego modelu, dokonano jego jakościowej i ilościowej analizy statycznej przy zastosowaniu metod analizy funkcjonalnej, w tym przestrzeni ciągów oraz zaproponowanej i matematycznie uzasadnionej metody aproksymacyjnej, zezwalającej na efektywne otrzymywanie wyników liczbowych. Wyniki szeregu obliczeń wskazują na uzyskanie efektu wzmocnienia (usztywnienia) kompozytu w pobliżu miejsca kontaktu dwóch materiałow, jakościowo zgodnego z obserwacjami fizycznymi.

Manuscript received April 28, 1993; accepted for print August 18, 1993