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A CONTRIBUTION TO THE MODELLING OF MACRO-HETEROGENEOUS COMPOSITES

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An asymptotic method for the modelling of macro-heterogeneous linear-elastic composites has been established by Woźniak (1992). In this note we propose an alternative approach to the aforementioned modelling problem which has rather a simple analytical form and a clear physical interpretation. The obtained equations constitute a convenient basis for both theoretical and computational analysis of macro-heterogeneous composite structures.

1. Preliminaries

As it is known every composite constitutes a certain micro-heterogeneous medium with macro-properties different from these of the individual constituents. If these macro-properties are constant throughout the whole body then the composite is said to be macro-homogeneous; if they depend on a position in the body then we deal with a certain macro-heterogeneous composite structure; an example of macro-heterogeneous composite is shown in Fig.1. For a general description and a motivation of researches for such composites the reader is referred to Woźniak (1992). In this note we propose a new approach to the formulation of engineering models of macro-heterogeneous composites which seems to be rather simple compared to that of Woźniak (1992), has a clear physical interpretation and can be easily applied to engineering problems.

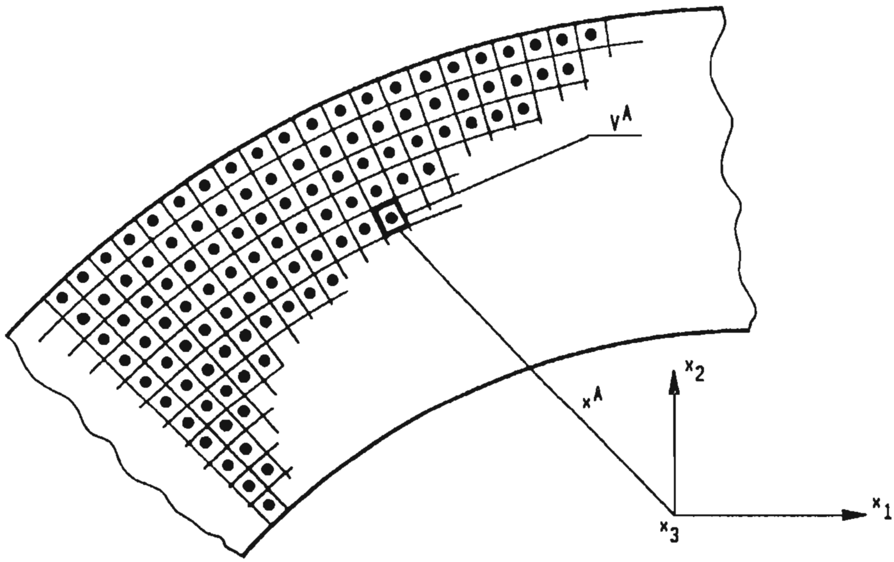

Fig. 1. A fragment of a macro-heterogeneous composite structure

Throughout the note subscripts i, j, k, l run over $1, 2, 3$ being related to Cartesian orthogonal coordinates x_1, x_2, x_3 in the referential space. Points of this space are denoted by $\boldsymbol{x} \equiv (x_1, x_2, x_3)$, $\boldsymbol{z} \equiv (z_1, z_2, z_3)$ etc. and τ stands for a time coordinate. Indices a, b run over $1, 2, 3, ..., n$ and summation convention holds for both i, j, k, l and a, b.

Let Ω be a regular region in the referential space occupied by a linear-elastic composite medium in its natural state. Material properties of this medium are determined by the elasticity tensor field $a_{ijkl}(\cdot)$ and the mass density scalar field $\rho(\cdot)$, which are supposed to be highly oscillating functions having jump discontinuities across the interfaces between material constituents of the composite. We shall restrict ourselves to composites for which there exist a decomposition of Ω into a very large number R of mutually disjoined small volume elements V^A, $A = 1, ..., R$, such that (cf Fig.1):

- Every V^A is a small piece of a body in which all typical material micro-heterogeneities can be detected but large enough to represent local macro-properties (apparent properties) of a composite structure

- Every two adjacent volume elements have very similar distribution of material constituents but the remote volume elements can be distinctly different.

Volume elements V^A, $A = 1,...,R$, will be referred to as macro-representative elements (m.v.e.). By $\boldsymbol{x}^A$ we shall denote a geometric center and by δ_A the maximum characteristic length dimension of V^A. Setting $\delta \equiv \max \delta_A$, $A = 1,...,R$, we refer to δ as a microstructure parameter, whis is assumed to be sufficiently small compared to the minimum characteristic length dimension of the region Ω.

In order to formulate an engineering model of a composite under consideration we introduce two fundamental concepts: the concept of a macro-function and that of a shape function. To this end let δ be a microstructure parameter and λ, $\lambda > 0$, be a maximum value of an admissible approximation error related to the computations of a continuous function $F(\cdot)$ within the framework of the modelling procedure.

A continuous (real valued) function $F(\cdot)$ defined on Ω will be called a *macro function* (of an order (δ, λ)) if for every $\boldsymbol{x}, \boldsymbol{z} \in \Omega$ and $|\boldsymbol{x} - \boldsymbol{z}| < \delta$ it satisfies condition $|F(\boldsymbol{x}) - F(\boldsymbol{z})| < \lambda$. It means that from a computational viewpoint a value of $F(\boldsymbol{x})$ can be replaced by $F(\boldsymbol{z})$ and inversely, provided that $|\boldsymbol{x} - \boldsymbol{z}| < \delta$. Similarly, function $F(\cdot)$ which is continuous and has continuous derivatives up to kth order will be called a macro function (of an order (δ, λ)) if $F(\cdot)$ together with all its derivatives are macro functions (of an order (δ, λ)). In the sequel we shall tacitly assume that (δ, λ) is known and we shall deal with a class of macro-functions related to the composite structure under consideration and to a certain approximation error.

Let $f(\cdot)$ be an arbitrary integrable function defined almost everywhere on Ω. For every m.v.e. V^A we shall introduce the averaging operator $< f >_A$ setting

$$< f >_A \equiv \frac{1}{\mathrm{vol}(V^A)} \int_{V^A} f(x)\, dv \qquad dv \equiv dx_1 dx_2 dx_3 \qquad A = 1,...,R$$

Now assume that there exist a macro function $F(\cdot)$ defined on Ω such that condition $|F(x) - < f >_A | < \lambda$ hold for every $\boldsymbol{x} \in V^A$ and $A = 1,...,R$. Then, for every continuous macro function $G(\cdot)$ defined on Ω, we obtain

$$\int_\Omega f(\boldsymbol{x}) G(\boldsymbol{x})\, dv = \sum_{A=1}^{R} < f >_A G(\boldsymbol{x}^A) \mathrm{vol}(V^A) + O(\lambda) = \int_\Omega F(\boldsymbol{x}) G(\boldsymbol{x})\, dv + O(\lambda) \tag{1.1}$$

where $O(\lambda) \to 0$ together with $\lambda \to 0$. Eq (1.1), after neglecting terms $O(\lambda)$, will be used in the sequel as a certain asymptotic approximation formula.

The second fundamental concept is that of a *micro-shape function* which is defined on Ω and restricted to an arbitrary m.v.e. V^A has an interpretation similar to the known shape function of the finite element method.

By micro-shape functions we shall understand a system of independent real-valued functions $h_a(\cdot)$, $a = 1, ..., n$, defined and continuous on Ω, having piecewise continuous derivatives $h_{a,i}(\cdot)$ in Ω and such that: (i) $< h_{a,i} >_A = 0$ for $A = 1, ..., R$, (ii) $h_a(\boldsymbol{x}) \in O(\delta)$ for every $\boldsymbol{x} \in \Omega$ but values $h_{a,i}(\boldsymbol{x})$ are independent of the microstructure parameter δ, (iii) for every two adjacent m.v.e. V^A, V^B functions $h_a(\boldsymbol{x}^A + \boldsymbol{y})$, $h_{a,i}(\boldsymbol{x}^A + \boldsymbol{y})$, $\boldsymbol{y} \in V^A - \boldsymbol{x}^A$ have a form similar to the corresponding functions $h_a(\boldsymbol{x}^B + \boldsymbol{y})$, $h_{a,i}(\boldsymbol{x}^B + \boldsymbol{y})$, $\boldsymbol{y} \in V^B - \boldsymbol{x}^B$, respectively. Micro-shape functions can be related to a discretization of every m.v.e. and hence they have to be postulated independently in every problem under consideration. From condition (iii) it follows that the form of these functions can be found by means of a certain interpolation formula on a basis of a few typical m.v.e. (cf Section 3).

2. Analysis

The procedure leading from the equations of linear elasticity theory for a micro-heterogeneous composite body under consideration to an engineering "averaged" model of this body will be based on two preliminary concepts introduced in Sect.1, on certain smoothness conditions and on two fundamental assumptions.

The first assumption has a heuristic character and will be called the *Micro-Macro Kinematic Hypothesis* and states that a displacement field $u_i(\cdot, \tau)$ at every time instant τ can be assumed in the form

$$u_i(\boldsymbol{x}, \tau) = U_i(\boldsymbol{x}, \tau) + h_a(\boldsymbol{x}) Q_i^a(\boldsymbol{x}, \tau) \qquad \boldsymbol{x} \in \Omega \tag{2.1}$$

where $U_i(\cdot, \tau)$, $Q_i^a(\cdot, \tau)$, together with their first and second time derivatives are arbitrary independent differentiable macro functions (which are independent of a microstructure parameter δ) and $h_a(\cdot)$, $a = 1, ..., n$, are micro-shape functions postulated a priori for every composite structure under consideration.

Fields $U_i(\cdot, \tau)$, $Q_i^a(\cdot, \tau)$ will be called macro-displacements and correctors, respectively. The term $h_a(\boldsymbol{x}) Q_i^a(\boldsymbol{x}, \tau)$ describes small disturbances of the displacement field $u_i(\cdot, \tau)$; it will be shown below that these disturbances are due to the micro-heterogeneous structure of a body. By means of $h_a\, Q_i^a \in O(\delta)$, we also obtain

$$\begin{aligned} u_i(\boldsymbol{x}, \tau) &= U_i(\boldsymbol{x}, \tau) + O(\delta) \\ u_{i,j}(\boldsymbol{x}, \tau) &= U_{i,j}(\boldsymbol{x}, \tau) + h_{a,j}(\boldsymbol{x}) Q_i^a(\boldsymbol{x}, \tau) + O(\delta) \end{aligned} \tag{2.2}$$

Let us denote by σ_{ij}, b_i, p_i stresses, body forces (supposed to be constant in Ω) and boundary tractions, respectively. Let us also assume, on a basis of considerations in Sect.1, that there exist sufficiently regular macro fields $S_{ij}(\cdot,\tau)$, $H_{ai}(\cdot,\tau)$, $M(\cdot)$ defined on Ω, such that conditions

$$\begin{aligned} &S_{ij}(x^A,\tau) = <\sigma_{ij}>_A \\ &H_{ai}(x^A,\tau) = <\sigma_{ij}h_{a,j}>_A \\ &M(x^A) = <\rho>_A \end{aligned} \tag{2.3}$$

hold for $A = 1, ..., R$ and at every time instant τ. Then, introducing the principle of virtual work

$$\int_\Omega \sigma_{ij}\delta u_{i,j}\, dv = \oint_\Omega p_i\delta u_i\, da + \int_\Omega \rho(b_i - \ddot{u}_i)\delta u_i\, dv \tag{2.4}$$

which will be assumed to hold for every $\delta u_i = \delta U_i$, $\delta u_{i,j} = \delta U_{i,j} + h_{a,j}\delta Q^a_i$, (cf Eqs (2.2); δU_i, δQ^a_i are arbitrary independent macro functions), by means of Eq (1.1) we obtain the following averaged form of both sides of (2.4)

$$\begin{aligned} &\int_\Omega \sigma_{ij}\delta u_{i,j}\, dv = \int_\Omega (S_{ij}\delta U_{i,j} + H_{ai}\delta Q^a_i)\, dv + O(\lambda) \\ &\oint_{\partial\Omega} p_i\delta U_i\, da + \int_\Omega \rho(b_i - \ddot{u}_i)\delta u_i\, dv = \oint_{\partial\Omega} p_i\delta U_i\, da + \\ &\qquad + \int_\Omega M(b_i - \ddot{U}_i)\delta U_i\, dv + O(\lambda) + O(\delta) \end{aligned} \tag{2.5}$$

Terms $O(\lambda)$ in Eq (2.5) are implied by the formula (1.1) and terms $O(\delta)$ are responsible for $u_i = U_i + O(\delta)$.

The second fundamental assumption, which be called *Asymptotic Approximation Hypothesis*, states that terms $O(\lambda)$ and $O(\delta)$ in all resulting equations (obtained by an averaging procedure) can be neglected. Hence from Eqs (2.4), (2.5) we obtain the following condition

$$\int_\Omega (S_{ij}\delta U_{i,j} + H_{ai}\delta Q^a_i)\, dv = \oint_{\partial\Omega} p_i\delta U_i\, da + \int_\Omega M(b_i - \ddot{U}_i)\delta U_i\, dv \tag{2.6}$$

which holds for every independent (macro) fields δU_i and δQ^a_i. It can be shown that restriction of δU_i, δQ^a_i to macro fields in Eq (2.6) is irrelevant and the condition (2.6) implies the following field equations

$$
\begin{aligned}
&S_{ij,j}(\boldsymbol{x},\tau) + M(\boldsymbol{x})b_i = M(\boldsymbol{x})\ddot{U}_i(\boldsymbol{x},\tau) \\
&H_{ai}(\boldsymbol{x},\tau) = 0 \qquad\qquad \boldsymbol{x} \in \Omega
\end{aligned}
\tag{2.7}
$$

and natural boundary conditions

$$
S_{ij}(\boldsymbol{x},\tau)n_j(\boldsymbol{x}) = p_i(\boldsymbol{x},\tau) \qquad\qquad \boldsymbol{x} \in \partial\Omega \tag{2.8}
$$

where $n_j(\boldsymbol{x})$ is an outward normal to $\partial\Omega$ at $\boldsymbol{x}$. Eqs (2.7) and (2.8) will be called *macro-equations of motion* and *macro-natural boundary conditions*, respectively; S_{ij} are said to be *averaged stresses* and H_{ai} will be referred to as *structural forces.*

Combining (2.3), with the known stress-strain relations $\sigma_{ij}(\boldsymbol{x},\tau) = a_{ijkl}(\boldsymbol{x})u_{(k,l)}(\boldsymbol{x},\tau)$ and using Eq $(2.2)_2$ we obtain

$$
\begin{aligned}
&S_{ij}(\boldsymbol{x}^A,\tau) = <a_{ijkl}>_A U_{(k,l)}(\boldsymbol{x}^A,\tau) + <a_{ijkl}h_{a,(k}>_A Q^a_{l)}(\boldsymbol{x}^A,\tau) + O(\delta) \\
&H_{ai}(\boldsymbol{x}^A,\tau) = <a_{ijkl}h_{a,l}>_A U_{(j,k)}(\boldsymbol{x}^A,\tau) + <a_{ijkl}h_{a,j}h_{b,l}>_A Q^b_k(\boldsymbol{x}^A,\tau) + O(\delta)
\end{aligned}
\tag{2.9}
$$

Let $A_{ijkl}(\cdot)$, $A_{aijk}(\cdot)$, $A_{abik}(\cdot)$ be certain macro fields defined on Ω which satisfy conditions

$$
\begin{aligned}
&A_{ijkl}(\boldsymbol{x}^A) = <a_{ijkl}>_A \\
&A_{aijk}(\boldsymbol{x}^A) = <a_{ijkl}h_{a,l}>_A \\
&A_{abik}(\boldsymbol{x}^A) = <a_{ijkl}h_{a,j}h_{b,l}>_A
\end{aligned}
\tag{2.10}
$$

for $A = 1,...,R$. Using the Asymptotic Approximation Hypothesis, from Eqs (2.9) and (2.10) we arrive at the following interrelations between S_{ij}, H_{ai} and $U_{(k,l)}$, Q^a_k

$$
\begin{aligned}
&S_{ij}(\boldsymbol{x},\tau) = A_{ijkl}(\boldsymbol{x})U_{(k,l)}(\boldsymbol{x},\tau) + A_{aijk}(\boldsymbol{x})Q^a_k(\boldsymbol{x},\tau) \\
&H_{ai}(\boldsymbol{x},\tau) = A_{aikl}(\boldsymbol{x})U_{(l,k)}(\boldsymbol{x},\tau) + A_{abij}(\boldsymbol{x})Q^b_j(\boldsymbol{x},\tau) \qquad \boldsymbol{x} \in \Omega
\end{aligned}
\tag{2.11}
$$

Eqs (2.11) will be referred to as *macro-constitutive relations* of linear elastic (macro-heterogeneous) composites. For micro-periodic structures the fields $A_{ijkl}(\cdot)$, $A_{aijk}(\cdot)$, $A_{abij}(\cdot)$, $M(\cdot)$ are constant and a composite is said to be macro-homogeneous.

Eqs (2.7), (2.11) constitute the system of governing relations for macro-displacements U_i and correctors Q_i^a. These equations involve exclusively macro functions and hence can be used as a convenient basis for analysis and computations of macro-heterogeneous composites under consideration. Let us observe that correctors Q_i^a can be eliminated from Eqs (2.7), (2.11) since they are governed by a system of linear algebraic equations $A_{abij}Q_j^b = -A_{aijk}U_{(j,k)}$; it can be shown that A_{abij} represent a certain invertible linear transformation. In a special case of micro-homogeneous bodies a_{ijkl} = const and we obtain $< a_{ijkl}\ h_{a,l} >_A \ = \ a_{ijkl} < h_{a,l} >_A \ = \ 0$ and hence $A_{aijk} \ = \ 0$; in this case all correctors Q_i^a are equal to zero. Thus we have proved that the correctors describe (together with micro-shape functions) an effect of micro-heterogeneity on a macro-behaviour of the composite under consideration.

3. Concluding remarks

The formulation of resulting macro-relations (2.7), (2.8) and (2.11) requires an introduction of appropriate micro-shape functions $h_a(\cdot)$ and computation of macro fields $A_{ijkl}(\cdot)$, $A_{aijk}(\cdot)$, $A_{abij}(\cdot)$, $M(\cdot)$, that can be done by an application of interpolation formulas

$$
\begin{aligned}
A_{ijkl}(x) &= \sum_K < a_{ijkl} >_K \eta^K(x) \\
A_{aijk}(x) &= \sum_K < a_{ijkl} h_{a,l} >_K \eta^K(x) \\
A_{abij}(x) &= \sum_K < a_{ikjl} h_{a,k} h_{b,l} >_K \eta^K(x) \\
M(x) &= \sum_K < \rho >_K \eta^K(x) \qquad\qquad x \in \Omega
\end{aligned}
\tag{3.1}
$$

where $\eta^K(\cdot)$ are suitable interpolation functions and index K runs over a certain subsequence of sequence $1, ..., R$. Hence also shape functions can be introduced only in m.v.e. V^K in Eqs (3.1). The general discussion of the resulting relations and the related boundary-value problems is similar to that concerning the results of Woźniak (1992) and will be not repeated here. Applications of the obtained models of macro-heterogeneous composites to special problems including a numerical analysis will be presented in a forthcoming paper.

References

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Modelowanie makro-niejednorodnych kompozytów

Streszczenie

W pracy Woźniak (1992) przedstawiono asymptotyczną metodę modelowania makro-niejednorodnych liniowo-sprężystych kompozytów. W tym komunikacie proponujemy alternatywne podejście do problemu modelowania, które ma prostą postać analityczną i jasną interpretację fizyczną. Otrzymane równania stanowią dogodną podstawę zarówno teoretycznej jak i numerycznej analizy makro-niejednorodnych struktur kompozytowych.

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