MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
3, 33, 1995

AVERAGING OF MECHANICAL SYSTEMS BY MEANS OF NON-SMOOTH PERIODIC FUNCTIONS

VALERIY PILIPCHUK

Higher Mathematics Department

Ukraine State Technology and Chemistry University, Dniepropetrovsk

EUGENIUSZ ŚWITOŃSKI

Technical Mechanics Department

Silesian Technical University, Gliwice

The concept of averaging applied to mechanical systems subject to impulsive excitation is created in the paper. The process is realized by means of special transformation of variables including the pair of non-smooth periodic function.

1. Introduction

The impulses acting upon mechanical systems are usually modelled by:

- Additional conditions imposed on coordinates and velocities, which show the character of impulse influence on the system in a neighbourhood of their application point e.g., application of velocities transition to external impact
- Introducing singular terms of the Dirac function type into the equation of motion.

Great adventage of the first way of modelling is the fact that differential equations representing the systems are the same as in the case with no impulses imposed (cf Samoilenko (1979)). However we have to consider a sequence of impulses action.

The second way of modelling yields the system uniform for all intervals without introducing the aforementioned conditions upon variables but it calls for further consideration within the framework of distribution theory. It is known that such investigation is quite difficult, especially for non-linear systems. The corresponding single-impulse case for quasilinear equations has been considered by Liu Zheng-Roung (1987).

In the present paper the method allowing one to eliminate singular periodic terms from equations and obtain a solution in the uniform, analytic form for the whole interval is described. The method is based on a sawtooth periodic argument application and a corresponding transformation of differential equations. It is shown that the transformed equations correspond to those obtained for the periodic solution presented earlier (cf Pilipchuk (1992)). The method will be described here for dynamical systems of the general form, but as an example, the transversely loaded beam resting on the periodic set of linear-elastic springs will be considered, for which the spatial coordinate will play the role of time. For similar problems see the survey by Manevich et al. (1989).

2. System description

The differential equation of mechanical system motion subject to periodic excitation (including discontinuous and impulsive ones) may be written in the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\varphi) + \boldsymbol{p}\tau''(\varphi) \qquad \boldsymbol{x} \in R^n \qquad \varphi = \omega t \tag{2.1}$$

where $\boldsymbol{p}$ is n-dimensional vector; the regular component of the right-hand side part $\boldsymbol{f}(\boldsymbol{x},\varphi)$ is supposed to be continuous as a function of $\boldsymbol{x}$ and piece-wise continuous and periodic with the period $T = 4$ as a function of φ; there are discontinuities of the first kind at the points where the periodically affecting δ-impulses are placed

$$\tau''(\varphi) = 2\sum_{k=-\infty}^{\infty}\Big[\delta(\varphi+1-4k) - \delta(\varphi-1-4k)\Big]$$

represented as generalized second order derivative of the sawtooth, piece-wise continuous function $\tau(\varphi)$, which has the unit amplitude and the period equal to four (the normalization considered is convenient since $\tau'^2 = 1$), Fig.1, (cf Pilipchuk (1988)).

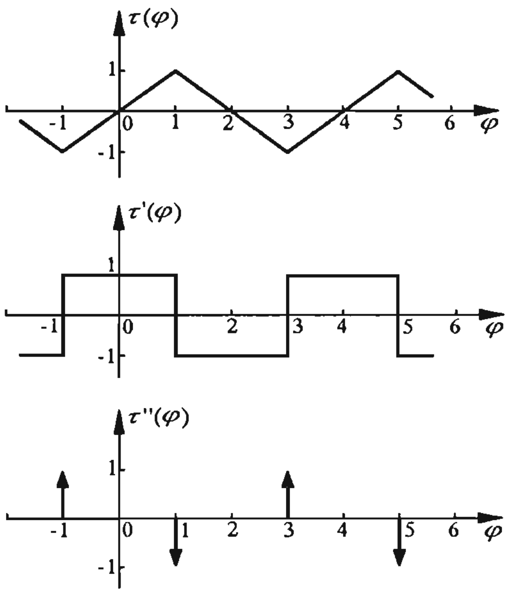
τ(φ)
1
-1
-1
0
1
2
3
4
5
6
φ
τ'(φ)
1
-1
-1
0
2
3
4
5
6
φ
τ''(φ)
1
-1
-1
0
1
2
3
4
5
6
φ

Fig. 1.

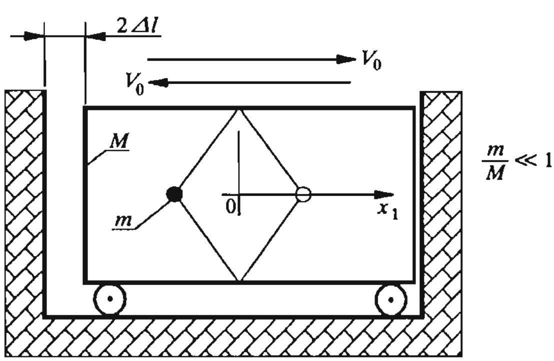
2Δl
V0
V0
M
m
0
x1
m/M ≪ 1

Fig. 2.

Let us consider Eq (2.1) with the following initial data

$$\boldsymbol{x}\Big|_{t=0} = \boldsymbol{x}^0 \tag{2.2}$$

Hence, one have the initial problem represented by Eqs(2.1) and (2.2).

There are numerous mechanical models which can be represented by Eq (2.1). To demonstrate the mechanical sense of Eq (2.1) we shall present the following two examples.

Example 1. Substituting

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \boldsymbol{f} = \begin{bmatrix} x_2 \\ -k^2x_1 - \beta x_1^3 \end{bmatrix} \qquad \boldsymbol{p} = \begin{bmatrix} 0 \\ q_0 \end{bmatrix}$$

into Eq (2.1) we shall have the Duffing oscillator under the periodic impulsive excitation

$$\ddot{x}_1 + k^2x_1 + \beta x_1^3 = q_0\tau''(\varphi)$$

where k, β, q_0 are constant. The possible mechanical interpretation of this equation is shown in Fig.2.

Example 2. Denoting $t \equiv y$ (the spatial independent variable) and substituting

$$\boldsymbol{x} = \begin{bmatrix} u(y) \\ v(y) \end{bmatrix} \qquad \boldsymbol{f} = \begin{bmatrix} \frac{v}{EF(1+\alpha\tau')} \\ 0 \end{bmatrix}$$

$$\boldsymbol{p} = \begin{bmatrix} 0 \\ \frac{q}{2a} \end{bmatrix} \qquad \omega = \frac{1}{a} \qquad (\dot{\ }) \equiv \frac{d}{dy}$$

we shall have the spatial periodic structure shown in Fig.3

$$\frac{d}{dy}\Big[EF(1+\alpha\tau')\frac{du}{dy}\Big] = \frac{q}{2a}\tau''\Big(\frac{y}{a}\Big)$$

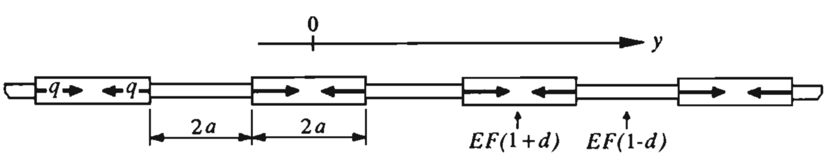

Fig. 3.

3. Averaging

The transformations described by Pilipchuk (1992) can be applied to the *periodic motion regimes taking place for the specific initial data. In the present* work the general initial problem represented by Eqs (2.1) and (2.2) will be examined using the transformations mentioned above combined with the averaging method of two-scales (cf Kuzmak (1959)). Therefore, the *oscillating time* $\tau = \tau(\omega t)$ will be considered as a fast variable in comparison with $t^0 \equiv t$. It means that the following assumption can be accepted

$$\omega^{-1} \equiv \epsilon \ll 1 \tag{3.1}$$

i.e., the system is subject to high-frequency excitation.

The solution to the initial problem represented by Eqs (2.1) and (2.2) is searched in the form

$$\boldsymbol{x} = \boldsymbol{X}(\tau, t^0) + \boldsymbol{Y}(\tau, t^0)\tau' \qquad\qquad \tau = \tau\Big(\frac{t}{\epsilon}\Big) \tag{3.2}$$

Hence, the representation of periodic solutions (cf Samoilenko (1979)) is *deformed* by the slow variable of time t^0 (cf Pilipchuk (1988)).

Let us assume that the *slow time* can be introduced into the right-hand side of Eq (2.1)

$$\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}, \varphi, t^0) \qquad\qquad \boldsymbol{p} = \boldsymbol{p}(t^0)$$

Substituting Eq (3.2) into Eq (2.1) yields

$$\frac{\partial \boldsymbol{Y}}{\partial \tau} + \epsilon\Big(\frac{\partial \boldsymbol{X}}{\partial t^0} - \boldsymbol{R}_f\Big) + \Big[\frac{\partial \boldsymbol{X}}{\partial \tau} + \epsilon\Big(\frac{\partial \boldsymbol{Y}}{\partial t^0} - \boldsymbol{I}_f\Big)\Big]\tau' + \underline{(\boldsymbol{Y} - \epsilon \boldsymbol{p})\tau''} = \boldsymbol{0}$$

where

$$\boldsymbol{R}_f = \frac{1}{2}[\boldsymbol{f}(\boldsymbol{X} + \boldsymbol{Y}, \tau, t^0) + \boldsymbol{f}(\boldsymbol{X} - \boldsymbol{Y}, 2 - \tau, t^0)]$$

$$\boldsymbol{I}_f = \frac{1}{2}[\boldsymbol{f}(\boldsymbol{X} + \boldsymbol{Y}, \tau, t^0) - \boldsymbol{f}(\boldsymbol{X} - \boldsymbol{Y}, 2 - \tau, t^0)]$$

Eliminating the periodic singular term and comparing separately the real and imaginary parts to zero, one obtains

$$\begin{aligned} &\frac{\partial \boldsymbol{Y}}{\partial \tau} + \epsilon\Big(\frac{\partial \boldsymbol{X}}{\partial t^0} - \boldsymbol{R}_f\Big) = \boldsymbol{0} \\ &\frac{\partial \boldsymbol{X}}{\partial \tau} + \epsilon\Big(\frac{\partial \boldsymbol{Y}}{\partial t^0} - \boldsymbol{I}_f\Big) = \boldsymbol{0} \qquad\qquad \boldsymbol{Y}\Big|_{\tau=\pm 1} = \epsilon \boldsymbol{p} \end{aligned} \tag{3.3}$$

Now the system of equations does not contain the singular terms and averaging techniques can be properly used.

The corresponding solution can be found in the form of power series with respect to $\epsilon = 0$

$$\boldsymbol{X} = \sum_{i=0}^{\infty} \epsilon^i \boldsymbol{X}^i(\tau, t^0) \qquad \boldsymbol{Y} = \sum_{i=0}^{\infty} \epsilon^i \boldsymbol{Y}^i(\tau, t^0) \tag{3.4}$$

where the functions $\boldsymbol{X}^i$, $\boldsymbol{Y}^i$ are to be defined. Substituting Eq (3.4) into Eq (3.3) and setting the coefficients of corresponding powers of ϵ equal to zero, one obtains the sequence of equations under the boundary conditions

– for ϵ^0

$$\frac{\partial \boldsymbol{X}^0}{\partial \tau} = \boldsymbol{0} \qquad \frac{\partial \boldsymbol{Y}^0}{\partial \tau} = \boldsymbol{0} \qquad \boldsymbol{Y}^0\Big|_{\tau=\pm 1} = \boldsymbol{0}$$

– for ϵ^1

$$\frac{\partial \boldsymbol{X}^1}{\partial \tau} = -\frac{\partial \boldsymbol{Y}^0}{\partial t^0} + \boldsymbol{I}_f^0$$

$$\frac{\partial \boldsymbol{Y}^1}{\partial \tau} = -\frac{\partial \boldsymbol{X}^0}{\partial t^0} + \boldsymbol{R}_f^0 \qquad \boldsymbol{Y}^1\Big|_{\tau=\pm 1} = \boldsymbol{p}$$

where $\boldsymbol{R}_f^0 = \boldsymbol{R}_f\Big|_{\epsilon=0}$, $\boldsymbol{I}_f^0 = \boldsymbol{I}_f\Big|_{\epsilon=0}$, ...

The equations of zero-order approximation are solved as follows

$$\boldsymbol{X}^0 = \boldsymbol{A}^0(t^0) \qquad \boldsymbol{Y}^0 \equiv \boldsymbol{0} \tag{3.5}$$

where $\boldsymbol{A}^0$ is an arbitrary vector-function of *slow time*. Taking into account Eq (3.5), in the next step the first order approximation is given by

$$\begin{aligned} \boldsymbol{X}^1 &= \int_0^\tau \boldsymbol{I}_f^0 \, d\tau + \boldsymbol{A}^1(t^0) \\ \boldsymbol{Y}^1 &= \int_{-1}^\tau \boldsymbol{R}_f^0 \, d\tau - \frac{d\boldsymbol{A}^0}{dt^0}(\tau + 1) + \boldsymbol{p} \end{aligned} \tag{3.6}$$

where A^1 is an arbitrary vector-function. In the second equation the arbitrary vector-function is chosen in the way ensuring the boundary condition at the point $\tau = -1$ to be satisfied. As a result the boundary condition at the second point $\tau = 1$ can be rewritten as

$$\frac{dA^0}{dt^0} = \langle R_f^0(A^0, \tau, t^0)\rangle \tag{3.7}$$

where $\langle\cdots\rangle$ stands for the averaging operator with respect to τ. Hence, the averaging operator in a given case arrives as the result of the boundary conditions for fast (oscillating) time fulfillment. Eqs (3.7) are to be solved for the following initial data: $A^0(0) = x^0$.

Note, that the second expression in Eqs (3.6), taking Eqs (3.7) into consideration can be written as

$$Y^1 = \int_{-1}^{\tau} (R_f^0 - \langle R_f^0\rangle)\, d\tau + p \tag{3.8}$$

The arbitrary function $A^1(t^0)$ will be defined at the next approximation step resulting from the boundary conditions satisfaction for the value of Y^2. The corresponding equation is

$$\frac{dA^1}{dt^0} - \Big\langle \frac{\partial R_f^0}{\partial A^0} \Big\rangle A^1 = F^1(A^0, t^0) \tag{3.9}$$

where

$\partial R_f^0 / \partial A^0$	–	matrix of partial derivatives
F^1	–	known function.

Note, that equations in the functions A^2, A^3,... will have the analogous structures.

In the case of one-directed impulses Eqs (3.8) and (3.9) can be written as

$$\frac{dA^1}{dt^0} - \Big\langle \frac{\partial R_f^0}{\partial A^0} \Big\rangle A^1 = F^1(A^0, t^0) + p$$

$$Y^1 = \int_{-1}^{\tau} \Big(R_f^0 - \langle R_f^0\rangle\Big)\, d\tau - p\tau$$

4. Second order equation

Let us consider the system of second order equations

$$\begin{aligned} \ddot{\boldsymbol{x}} &= -\Big[\mathbf{q}(\varphi,t) + \mathbf{p}\tau''(\varphi)\Big]\boldsymbol{x} + \boldsymbol{g}(\varphi,t) + \boldsymbol{r}(t)\tau''(\varphi) \\ \varphi &= \frac{t}{\epsilon} \qquad\qquad \boldsymbol{x} \in R^n \end{aligned} \tag{4.1}$$

where

ϵ – small parameter, $\epsilon \ll 1$

$\mathbf{q},\mathbf{p}$ – $n \times n$-matrices

$\boldsymbol{g},\boldsymbol{r}$ – n-dimensional vector-functions.

Assuming the form of solution as given by Eq (3.2), we have the following set of equations and the boundary conditions

$$\begin{aligned} \frac{\partial^2 \boldsymbol{X}}{\partial\tau^2} &= -2\epsilon\frac{\partial^2 \boldsymbol{Y}}{\partial\tau\partial t^0} - \epsilon^2\Big(\frac{\partial^2 \boldsymbol{X}}{\partial t^{02}} + \mathbf{Q}\boldsymbol{X} + \mathbf{P}\boldsymbol{Y} - \boldsymbol{G}\Big) \\ \frac{\partial^2 \boldsymbol{Y}}{\partial\tau^2} &= -2\epsilon\frac{\partial^2 \boldsymbol{X}}{\partial\tau\partial t^0} - \epsilon^2\Big(\frac{\partial^2 \boldsymbol{Y}}{\partial t^{02}} + \mathbf{Q}\boldsymbol{Y} + \mathbf{P}\boldsymbol{X} - \boldsymbol{F}\Big) \end{aligned} \tag{4.2}$$

$$\frac{\partial \boldsymbol{X}}{\partial\tau}\Big|_{\tau=\pm1} = \epsilon^2(\boldsymbol{r} - \mathbf{p}\boldsymbol{X})\Big|_{\tau=\pm1} \qquad\qquad \boldsymbol{Y}\Big|_{\tau=\pm1} = \mathbf{0} \tag{4.3}$$

where $\mathbf{Q}$, $\mathbf{P}$, $\boldsymbol{G}$, $\boldsymbol{F}$ appear after the substition: $\mathbf{q} = \mathbf{Q} + \mathbf{P}\tau'$, $\boldsymbol{g} = \boldsymbol{G} + \boldsymbol{F}\tau'$.

We take here $\tau'\tau'' = 0$. Note that the exact solution to Eqs (4.2) and (4.3) is a special mathematical problem of the distribution theory. Assuming the solution to Eqs (3.4) in a power series form with respect to ϵ and then setting the corresponding coefficients powers of ϵ equal to zero, one obtains the set of equations and boundary conditions. The corresponding solutions can be written as follows

$$\boldsymbol{X}^0 = \boldsymbol{B}^0(t^0) \qquad \boldsymbol{Y}^0 \equiv \mathbf{0} \qquad \boldsymbol{X}^1 \equiv 0 \qquad \boldsymbol{Y}^1 \equiv \mathbf{0}$$

$$\boldsymbol{X}^2 = \int\limits_{-1}^{\tau}(\tau-\xi)\Big[\boldsymbol{G} - \langle\boldsymbol{G}\rangle - (\mathbf{Q} - \langle\mathbf{Q}\rangle)\boldsymbol{B}^0\Big]\;d\xi + (\boldsymbol{r} - \mathbf{p}\boldsymbol{B}^0)\tau + \boldsymbol{B}^2$$

$$\boldsymbol{Y}^2 = \int\limits_{-1}^{\tau}\Big[(\tau-\xi)(\boldsymbol{F} - \mathbf{p}\boldsymbol{B}^0) - \langle(1-\xi)(\boldsymbol{F} - \mathbf{p}\boldsymbol{B}^0)\rangle\Big]\;d\xi$$

...

where the function $\boldsymbol{B}^0(t^0)$ is described by the "oscillating", average time equation

$$\frac{d^2 B^0}{dt^{02}} + \langle \mathsf{Q} \rangle B^0 = \langle G \rangle \tag{4.4}$$

the function $\boldsymbol{B}^0$ will be defined in the next step of the procedure. The boundary conditions imposed upon $\boldsymbol{X}$ for one-directed impulses have the following form

$$\frac{\partial \boldsymbol{X}}{\partial \tau}\Big|_{\tau=\pm 1} = \mp \epsilon^2 (\boldsymbol{r} - \mathbf{p}\boldsymbol{X})\Big|_{\tau=\pm 1}$$

In this case one have the new formula for $\boldsymbol{X}^2$

$$\boldsymbol{X}^2 = \int\limits_{-1}^{\tau} (\tau - \xi)\Big[G - \langle G \rangle - (\mathsf{Q} - \langle \mathsf{Q} \rangle)B^0\Big]\, d\xi - \frac{\tau^2}{2}(\boldsymbol{r} - \mathbf{p}B^0) + \boldsymbol{B}^2$$

and the averaged equation takes the following form

$$\frac{d^2 B^0}{dt^{02}} + (\langle \mathsf{Q} \rangle + \mathbf{p})B^0 = \langle G \rangle + \boldsymbol{r} \tag{4.5}$$

Example 3. Consider the transversely loaded beam resting on the periodic set of linear-elastic springs. The corresponding equation of equilibrium is

$$D\frac{d^4 w}{dy^4} + \frac{cw}{a} \sum_{k=-\infty}^{\infty} \delta\Big(\frac{y}{a} - 1 - 2k\Big) = q\Big(\frac{y}{L}\Big) \qquad -\infty < y < \infty$$

Let us introduce the nondimentional quantities

$$\xi = \frac{y}{L} \qquad \bar{w} = \frac{w}{a} \qquad \bar{q} = \frac{qL^4}{aD}$$

$$\gamma = \frac{cL^4}{aD} \qquad \varphi = \frac{\xi}{\epsilon}$$

and assume that $\epsilon \ll 1$. The equation of equilibrum can be written in the form of Eq (4.1) by putting

$$\boldsymbol{x} = \begin{bmatrix} \bar{w} \\ \frac{d^2\bar{w}}{d\xi^2} \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 0 & 0 \\ \frac{\gamma}{2} & 0 \end{bmatrix}$$

$$\boldsymbol{g} = \begin{bmatrix} 0 \\ \bar{q}(\xi) \end{bmatrix} \equiv \boldsymbol{G} \qquad \mathbf{q} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \equiv \mathbf{Q}$$

$$\mathbf{P} \equiv \boldsymbol{F} \equiv \mathbf{0} \qquad \boldsymbol{r} \equiv \mathbf{0} \qquad t \equiv \xi$$

For the given case one obtains

$$X^2 = \frac{\tau^2}{2}\mathbf{p}B^0 + B^2 \qquad\qquad Y^2 \equiv 0$$

Hence, we have

$$x = B^0(\xi) + \epsilon^2\Big[\frac{\tau^2}{2}\mathbf{p}B^0(\xi) + B^2(\xi)\Big] + \dots$$

$$\tau = \tau\Big(\frac{\xi}{\epsilon}\Big) \qquad\qquad \epsilon = \frac{a}{L}$$

where the vector-function B^0 is defined by Eq (4.5) having the following form in the components of matrix $B^0 = [B_1^0, B_2^0]^\top$

$$\frac{d^4 B_1^0}{d\xi^4} + \frac{\gamma}{2}B_1^0 = \bar{q}(\xi) \qquad\qquad B_2^0 = \frac{d^2 B_1^0}{d\xi^2}$$

This is the averaged equation for the elastic beam resting on a continuous elastic foundation.

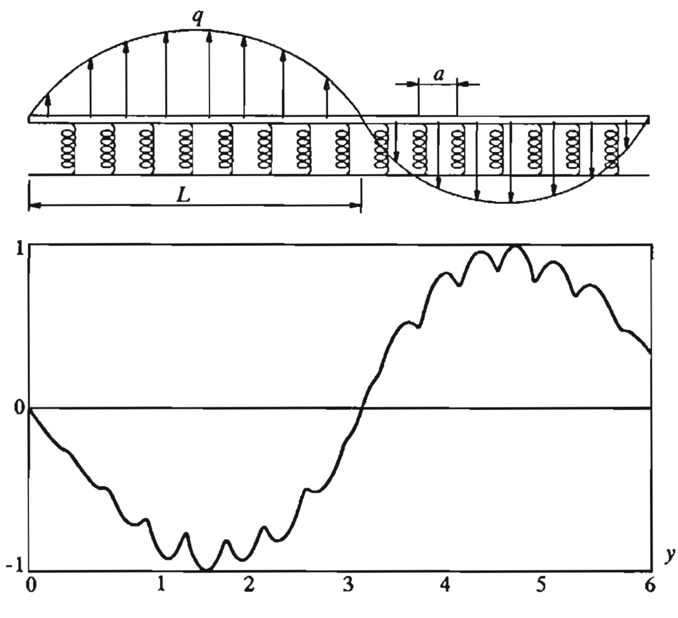

Fig. 4.

To illustrate the presented asymptotic $(\epsilon \to 0)$ solution, consider the beam under the harmonic transverse force, $q\left(\frac{y}{L}\right) = q_0 \sin\frac{y\pi}{L}\ q_0 =$ constant. Taking into account only the main *slow* and *fast* components of the expansion we obtain the following formula for the moment

$$M(y) = D\frac{d^2w}{dy^2} = -q_0\frac{2\pi^2L^2}{2\pi^4+\gamma}\Big[1-\gamma\frac{\epsilon^2}{4\pi^2}\tau^2\Big(\frac{y}{a}\Big)\Big]\sin\frac{y\pi}{L}$$

Fig.4 shows the moment chart, which has been calculated for the following values: $L = \pi$; $a = 0.2$ ($\epsilon = 0.063\ldots$); $\gamma = 1947$; $q_0 = 11$.

5. Conclusions

So the presented example of solution has a uniform analytic form for the entire independent variable interval. The advantage of it when applied to calculation and research into the system properties is connected with the sawtooth argument presence. The presented technique consists in treating the sawtooth argument in combination with the averaging method. The role of sawtooth argument is to exclude external periodic impulses (or spatially localized periodic irregularities of 1D elstic media) from the corresponding equations.

The possible alternative methods for solving the same problems have been presented in the survey by Manevich et al. (1989). The homogenization procedure, which has been described by Benosussan et al. (1978), should be noted. This procedure gives the averaged equation in the slow spatial scale and a *cell problem* in the fast one. The *cell* in Example 3 is viewed as a part of the beam between arbitrary taken two springs. The corresponding *cell solutions* should be *pasted together* to get a global description of the system in both slow and fast spatial scales. The employed here pair of non-smooth functions allows one to get *automatically* both the *cell problem* solution and the global description of the system, which is very convenient for the researcher.

References

1. SAMOILENKO A.M., 1979, The Averaging Method for Systems with Pushes, in: *Mathematical Physics*, Kiev, Naukova Dumka, 63-96, (in Rus.)
2. LIU ZHENG-ROUNG, 1987, Discontinuous and Impulsive Excitation, *Applied Mathematics and Mechanics*, **8**, 1, 31-35
3. PILIPCHUK V.N., 1992, On Calculation of Periodic Processes in Mechanical Systems with Impulsive Excitation, *Zeszyty Naukowe Pol. Śl., Ser. Mechanika*, **107**, 335-343, (in Rus.)

4. MANEVICH L.I., MIKHLIN YU.V., PILIPCHUK V.N., 1989, *Method of Normal Vibrations for Essentially Nonlinear Systems,* Moskow, Nauka, (in Rus.)

5. KUZMAK G.E., 1959, Asymptotic Solutions of Non-Linear Second-Order Differential Equations with Variable Coefficients, *Appl. Math. and Mech., (PMM),* **23**, 3, 515-526, (in Rus.)

6. PILIPCHUK V.N., 1988, The Vibrating Systems Transformation by Means of the Pair of Non-Smooth Periodic Functions, *Ukrainian Acad. of Sci. Reports,* Ser.A, 4, 37-40, (in Rus.)

7. BENOSUSSAN A., LION J.L., PAPANICOLAOU G., 1978, *Asymptotic Analysis for Periodic Structures,* North Holland, Amsterdam

8. ANDRIANOV I.V., MANEVICH L.I., 1983, The Averaging Method for Computation of Shells, *Advances in Mechanics,* **6**, 3/4

Uśrednianie układów mechanicznych przy pomocy nie-gładkich funkcji okresowych

Streszczenie

W pracy przedstawiono koncepcję uśredniania ukladów mechanicznych z wymuszeniem impulsowym. Uśrednianie realizowane jest przy pomocy specjalnej transformacji zmiennych, zawierającej pary nie-gladkich funkcji okresowych.

Manuscript received July 7, 1994; accepted for print March 14, 1995