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ON THE IMPULSE RESPONSE FUNCTION FOR WATER WAVES GENERATED IN FLUID OF CONSTANT DEPTH

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The paper deals with the problem of surface gravitational waves generation in fluid of constant depth. The main goal of the investigations is to find the impulse response function for the generator-fluid system. Two independent formulations of the original task are considered. These two formulations lead to different formulae describing the same characteristic feature of the system. One on these formulations however is more advantageous then the other one since it takes shorter time of computer calculations.

1. Introduction

In analysis of water waves generated in fluid of constant depth we often deal with the initial value problem of fluid motion starting from rest. An example of such a case is the generation of water waves in a hydraulic flume where the generator-fluid system starts to move at a given moment of time. Usually, laboratory experiment results describe the transmission of the generator motion into the free surface elevation measured at a chosen point. In a general case, the data obtained in experiments has the form of a sequence of numbers corresponding to the discrete sequence of time steps. In theoretical description of the aforementioned problems a very useful tool is the impulse response function of the system considered. Assuming that the generator-fluid system is the linear, time-invariant one, a solution for an arbitrary excitation of the fluid motion may be obtained by superposing unit impulse solutions in the time domain.

In the present paper the problem of generation of water waves in a semi-infinite layer of fluid of constant depth is considered. The waves are generated by a piston-type wave-maker placed at the beginning of the layer. The aim of analysis is to find the impulse response function for the system mentioned. In order to obtain the function two independent formulations of the problem are considered. The first one is based on the complex frequency response function for the generator-fluid system. The second one adapts direct solution to the initial motion of the system starting from rest. These two formulations lead to solutions which are expressed by different formulae. Apparently, these solutions seem to be different from each other but careful numerical computations show that they provide identical results.

The initial value problem of fluid motion starting from rest has been studied by many authors. A number of problems of this kind was discussed by Lamb (1975). Within the framework of the linear theory Stoker (1957) obtained closed form analytical solutions to water wave problem due to disturbances originated at the free surface points. Some general theorems associated with the initial value problems in hydrodynamics may be found in Wehausen and Laitone (1960). As concerns the generation of water waves in a flume of constant depth, the important contribution has been given by Biésel and Sequet (1951). These authors formulated the classical wave-maker theory for steady-state harmonic generation of waves. The generation of water waves by a piston type wave-maker starting from rest was investigated by Madsen (1970). The theoretical results obtained were compared with experimental data. The experiments showed second order effects in wave amplitudes and thus an approximate second order wave-maker theory was presented. For a sinusoidally moving wave-maker, the same author (Madsen 1971) extended the classical linear theory to a second order solution. Recently, Hudspeth and Sulisz (1991) have presented a complete second-order solution for the two-dimensional wave motion forced by a sinusidally moving generic planar wave-maker. They shown that the first-order part of solution, represented by evanescent eigenseries, cannot be neglected when computing the amplitude of the second-order free wave. Wilde and Szmidt (1993) investigated properties of random generation of water waves in a layer of fluid. The discussion was confined to the linear transformation of a certain class of stochastic processes describing the generator motion into the free surface elevation processes. The impulse generation of waves in fluid of finite depth was discussed by Szmidt (1993). The solution for the output response function was obtained by means of the Fourier transformation technique. The contents of that article is closely related to the problem investigated in the present paper.

2. Formulation of the problem

We will focus our attention on the plane problem of generation of water-waves by a piston type wave-maker as it is shown in Fig.1. The fluid flow is induced by horizontal motion of the rigid wall $0A$. It is assumed that the fluid is inviscid, incompressible and the velocity field is potential.

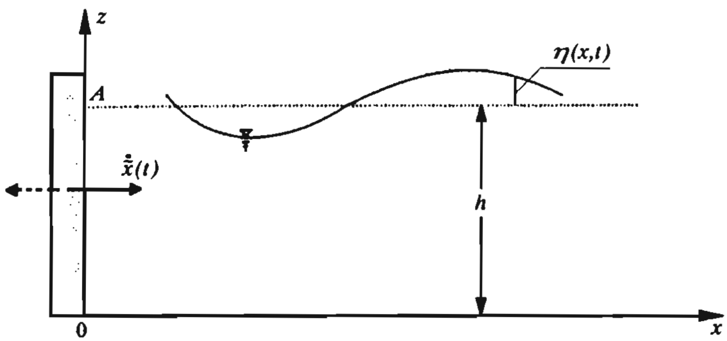

Fig. 1. Definition sketch for the generator-fluid system

For the assumed irrotational motion, the linearized equations governing the fluid motion generated by the wave-maker are

$$\nabla^2\Phi(x,z,t) = 0 \qquad x \geq 0 \qquad 0 \leq z \leq h \qquad t \geq 0 \tag{2.1}$$

$$\left.\frac{\partial\Phi}{\partial z}\right|_{z=0} = 0 \tag{2.2}$$

$$\left.\left(\dot{\eta} - \frac{\partial\Phi}{\partial z}\right)\right|_{z=h} = 0 \tag{2.3}$$

$$\left.(\dot{\Phi} + g\eta)\right|_{z=h} = 0 \tag{2.4}$$

where

$\Phi(x,z,t)$ – velocity potential

$\eta(x,t)$ – surface elevation

g – gravitational acceleration, and the dot denotes differentiation with respect to time.

The third and the fourth equations describe kinematic and dynamic conditions for the free surface, respectively. At the wave-maker the boundary condition reads

$$\left.\frac{\partial\Phi}{\partial x}\right|_{x=0} = \dot{x}(t) \tag{2.5}$$

where $\dot{\tilde{x}}(t)$ is the horizontal velocity of the wave-maker. For the case of steady-state motion, the boundary conditions (2.2)÷(2.5) are supplemented by the Sommerfeld condition describing the potential behaviour at $x \to \infty$. For the case of initial motion starting from rest, the velocity potential and its derivatives die out when going to infinity. For the latter case the initial conditions of the system should be specified. Without loss of generality it will be assumed that

$$\Phi(x, z, t \le 0) \equiv 0 \qquad\qquad \eta(x, t \le 0) \equiv 0 \tag{2.6}$$

3. Complex frequency response and impulse response functions

As it is known (see Crandal and Mark (1973)), for the linear time-invariant systems the complex frequency response function $H(\sigma)$ and the impulse response function $h(t)$ are mutual Fourier transforms, namely

$$\begin{aligned} H(\sigma) &= \int\limits_{-\infty}^{+\infty} h(t) e^{-i\sigma t}\, dt \\ h(t) &= \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} H(\sigma) e^{i\sigma t}\, d\sigma \end{aligned} \tag{3.1}$$

Knowing the complex frequency response function $H(\sigma)$, the impulse response function $h(t)$ my be calculated from the second one of Eqs (3.1) by performing integration in the frequency domain. Such a method of solution of the problem considered may be found in Szmidt (1993). To make the discussion unbiased, some important results obtained in this paper are summarized herein. In order to obtain the $H(\sigma)$ function the steady-state harmonic generation of the waves is considered. Thus, let the velocity of the generator be expressed in the form

$$\dot{\tilde{x}}(t) = A e^{i\sigma t} \tag{3.2}$$

where

A – constant

σ – angular frequency of vibrations.

The solution to the Laplace equation (2.1) satisfying all the boundary conditions prescribed is given by the formula (cf Szmidt (1993))

$$
\begin{aligned}
\Phi(x,z,t) &= \Big[\frac{4\mathrm{i}}{k_0}\frac{\sinh(k_0h)\cosh(k_0z)}{2k_0h+\sinh(2k_0h)}\mathrm{e}^{-\mathrm{i}k_0x} + \\
&- \sum_{j=1}^{\infty}\frac{4}{k_j}\frac{\sin(k_jh)\cos(k_jz)}{2k_jh+\sin(2k_jh)}\mathrm{e}^{-k_jx}\Big]A\mathrm{e}^{\mathrm{i}\sigma t}
\end{aligned}
\tag{3.3}
$$

where $k_0, k_1, k_2, ...$ are eigenvalues and $\cosh(k_0z)$, $\cos(k_nz)$,... are eigenfunctions of the boundary-value problem considered. The eigenvalues must satisfy the following dispersion relations

$$
\begin{aligned}
&\sigma^2 = gk_0\tanh(k_0h) \\
&\sigma^2 = -gk_j\tan(k_jh) \qquad j = 1,2,...
\end{aligned}
\tag{3.4}
$$

To save the space it is convenient to introduce the dimensionless variable

$$
\alpha = \frac{\sigma^2 h}{g}
\tag{3.5}
$$

and to rewrite the dispersion relations as

$$
\begin{aligned}
&\alpha = \beta_0\tanh\beta_0 \\
&\alpha = -\beta_j\tan\beta_j \qquad j = 1,2,...
\end{aligned}
\tag{3.6}
$$

where $\beta_0 = k_0h$ and $\beta_j = k_jh$ $(j = 1,2,...)$. It is seen that for small values of α $(\alpha \to 0)$, $\beta_j \to j\pi$, while with growing values of α $(\alpha \gg 1)$, β_j tend to $(2j-1)\pi/2$. With respect to the solution (3.3), the free surface elevation may be obtained

$$
\begin{aligned}
\eta(x,t) &= 2\Big[\frac{\sigma}{gk_0}\frac{\sinh(2\beta_0)}{2\beta_0+\sinh(2\beta_0)}\mathrm{e}^{-\mathrm{i}k_0x} + \\
&+ \mathrm{i}\sum_{j=1}^{\infty}\frac{\sigma}{gk_j}\frac{\sin(2\beta_j)}{2\beta_j+\sin(2\beta_j)}\mathrm{e}^{-k_jx}\Big]A\mathrm{e}^{\mathrm{i}\sigma t}
\end{aligned}
\tag{3.7}
$$

The first part of the solution describes the surface gravitational wave propagating from the wave-maker to infinity. The second part of the solution, represented by the infinite series, describes the standing wave which die out when going to infinity.

The complex frequency response function $H(\sigma)$ is defined by the quotient of Eqs (3.7) and (3.2). Simple calculations give

$$H(\sigma,x) = \frac{4}{\sigma}\Big[F_1(\sigma)e^{-ik_0x} - iF_2(\sigma,x)\Big] \tag{3.8}$$

where

$$\begin{aligned} F_1(\sigma) &= \frac{\sinh^2\beta_0}{2\beta_0+\sinh(2\beta_0)} \\ F_2(\sigma,x) &= \sum_{j=1}^{\infty}\frac{\sin^2\beta_j}{2\beta_j+\sin(2\beta_j)}e^{-k_jx} \end{aligned} \tag{3.9}$$

As $\sigma \to 0$, we have

$$\lim_{\sigma\to 0} F_1(\sigma) = 0 \qquad\qquad \lim_{\substack{\sigma\to 0\\ x\geq 0}} F_2(\sigma,x) = 0 \tag{3.10}$$

At the same time, the following relations hold (cf Szmidt (1993) and (1984))

$$\begin{aligned} \lim_{\sigma\to\infty} F_1(\sigma) &= \frac{1}{2} \\ \lim_{\substack{\sigma\to 0\\ x\geq 0}} F_2(\sigma,x) &= \frac{1}{\pi}\sum_{j=1}^{\infty}\frac{e^{-k_j^*x}}{2j-1} = \frac{1}{2\pi}\ln\Big(\coth\frac{\pi x}{4h}\Big) \end{aligned} \tag{3.11}$$

where

$$k_j^* = \frac{2j-1}{2h}\pi \qquad\qquad j=1,2,...$$

For a chosen value of x, from substitution of Eq (3.8) into the second one of Eqs (3.1) it follows

$$h(t) = J_1(t) + J_2(t) \tag{3.12}$$

where

$$\begin{aligned} J_1(t) &= \frac{1}{\pi}\int_0^{\infty}\frac{\tanh(sh)}{s}\cos(sx-rt)\,ds \\ J_2(t) &= \frac{4}{\pi}\int_0^{\infty}\frac{\sin(\sigma t)}{\sigma}\sum_{j=1}^{\infty}\frac{\sin^2\beta_j}{2\beta_j+\sin(2\beta_j)}e^{-k_jx}\,d\sigma \end{aligned} \tag{3.13}$$

and

$$r^2 = gs\tanh(sh) \qquad \alpha = -\beta_j \tan\beta_j \qquad j = 1, 2, ... \tag{3.14}$$

For greater value of x the second term of the right hand side of Eq (3.12) may be neglected and thus, the impulse response function may be assumed in the following form

$$h(t) \cong \frac{1}{\pi} \int_0^\infty \frac{\tanh(sh)}{s} \cos(sx - rt)\, ds \tag{3.15}$$

4. Direct solution to the initial value problem

Let us assume that for $t \leq 0$ the whole system is at rest. At the instant $t = 0$ the generator begins to move with the velocity $\dot{\bar{x}}(t)$. In order to solve the problem at hand it is convenient to split the velocity potential into two parts

$$\Phi(x, z, t) = \phi(x, z, t) + \psi(x, z, t) \tag{4.1}$$

where the components satisfy the Laplace equations

$$\nabla^2\phi = 0 \qquad \nabla^2\psi = 0 \tag{4.2}$$

and the appropriate boundary and initial conditions. The boundary conditions are

$$\begin{aligned}
&\frac{\partial\phi}{\partial x}\Big|_{x=0} = 0 \qquad && \frac{\partial\phi}{\partial z}\Big|_{z=0} = 0 \\
&\frac{\partial\psi}{\partial x}\Big|_{x=0} = \dot{\bar{x}}(t) \qquad && \frac{\partial\psi}{\partial z}\Big|_{z=0} = 0 \qquad && \psi\Big|_{z=h} = 0
\end{aligned} \tag{4.3}$$

Combination of the boundary conditions (2.3) and (2.4) and Eq (4.1) leads to the boundary condition for the upper surface of layer

$$\Big(\ddot{\phi} + g\frac{\partial\phi}{\partial z} + g\frac{\partial\psi}{\partial z}\Big)\Big|_{z=h} = 0 \tag{4.4}$$

With respect to the linear combination (4.1) and the boundary conditions (4.3) the solution for the potential $\psi(x, z, t)$ may be found independently from the second part of the potential $\Phi(x,z,t)$. The Fourier method of separation of variables leads to the result

$$\psi(x,z,t) = -\sum_{j=1}^{\infty} A_j(t)\frac{1}{k_j}\mathrm{e}^{-k_j x}\cos(k_j z) \tag{4.5}$$

where

$$k_j = \frac{2j-1}{2h}\pi \qquad\qquad j = 1,2,... \tag{4.6}$$

The functions $A_j(t)$ entering the solution are found from the boundary condition at the wave-maker. Simple transformations yield

$$A_j(t) = \frac{2(-1)^{j+1}}{k_j h}\dot{\tilde{x}}(t) \tag{4.7}$$

and, finally

$$\psi(x,z,t) = -\frac{8h\dot{\tilde{x}}(t)}{\pi^2}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{(2j-1)^2}\mathrm{e}^{-k_j x}\cos(k_j z) \tag{4.8}$$

It is seen, that the solution obtained expresses the standing wave only. In order to find a solution for the second part of the velocity potential we apply the one side cosine Fourier transform according to the formulae (Nowacki (1972))

$$\begin{aligned}
\phi^*(s,z,t) &= \int_0^{\infty} \phi(x,z,t)\cos(sx)\,dx \\
\phi(x,z,t) &= \frac{2}{\pi}\int_0^{\infty} \phi^*(s,z,t)\cos(sx)\,ds
\end{aligned} \tag{4.9}$$

where s is the parameter of the transforms. The Fourier transform of the Laplace equation $\nabla^2\phi = 0$ leads to the ordinary differential equation

$$\frac{d^2\phi^*(s,z,t)}{dz^2} - s^2\phi^*(s,z,t) = 0 \tag{4.10}$$

The solution to the equation satisfying the bottom boundary condition is

$$\phi^*(s,z,t) = A(s,t)\cosh(sz) \tag{4.11}$$

where $A(s,t)$ is a constant of integration. The Fourier transform of the boundary condition (4.4) gives

$$\Big(\ddot{\phi}^* + g\frac{\partial\phi^*}{\partial z} + g\frac{\partial\psi^*}{\partial z}\Big)\Big|_{z=h} = 0 \tag{4.12}$$

In order to obtain the last term in this equation let us calculate the relevant derivative of Eq (4.8). With respect to the second of Eqs (3.11), this derivative may be expressed in the closed analytical form

$$\frac{\partial \psi^*}{\partial z}\Big|_{z=h} = -\frac{4\dot{\tilde{x}}}{\pi}\sum_{j=1}^{\infty}\frac{e^{-k_j x}}{2j-1} = \frac{2\dot{\tilde{x}}}{\pi}\ln\Big(\coth\frac{\pi x}{4h}\Big) \tag{4.13}$$

Accordingly, the Fourier transform of the result obtained is (cf Bateman (1954))

$$\begin{aligned}\frac{\partial \psi^*}{\partial z}\Big|_{z=h} &= \frac{2\dot{\tilde{x}}}{h}\int_0^{\infty}\ln\Big(\coth\frac{\pi x}{4h}\Big)\cos(sx)\,dx = \\ &= \frac{\dot{\tilde{x}}}{hs}\int_0^{\infty}\frac{\sin(sx)}{\sinh\frac{\pi x}{2h}}\,dx = \dot{\tilde{x}}\frac{\tanh(sh)}{s}\end{aligned} \tag{4.14}$$

Finally, from substitution of Eqs (4.11) and (4.14) into the Eq (4.12) it follows

$$\ddot{A}\cosh(sh) + g\Big[As\sinh(sh) + \dot{\tilde{x}}\frac{\tanh(sh)}{s}\Big] = 0 \tag{4.15}$$

or, in a more concise form

$$\ddot{A} + r^2 A + F(s)\dot{\tilde{x}} = 0 \tag{4.16}$$

where the following substitutions have been made

$$r^2 = gs\tanh(sh) \qquad\qquad F(s) = \frac{g\tanh(sh)}{s\cosh(sh)} \tag{4.17}$$

The solution to Eq (4.16) takes the form

$$A(s,t) = A_0(s)\cos(rt) + B_0(s)\sin(rt) - \frac{F(s)}{r}\int_0^t \dot{\tilde{x}}(\tau)\sin[r(t-\tau)]\,d\tau \tag{4.18}$$

where $A_0(s)$ and $B_0(s)$ are constants of integration.

On inserting Eq (4.18) into Eq (4.11) and then carrying out the inverse transform we arrive at the following solution

$$\begin{aligned}\phi(x,z,t) &= \frac{2}{\pi}\int_0^{\infty}\Big\{A_0(s)\cos(rt) + B_0(s)\sin(rt) + \\ &- \frac{F(s)}{r}\int_0^t \dot{\tilde{x}}(\tau)\sin[r(t-\tau)]\,d\tau\Big\}\cosh(sz)\cos(sx)\,ds\end{aligned} \tag{4.19}$$

Taking the initial conditions (2.6) into account one can obtain

$$A_0(s) = B_0(s) = 0 \tag{4.20}$$

and, finally

$$\phi(x,z,t) = -\frac{2}{\pi}\int_0^\infty \frac{F(s)}{r}\Big\{\int_0^t \dot{\ddot{x}}(\tau)\sin[r(t-\tau)]\,d\tau\Big\}\cosh(sz)\cos(sx)\,ds \tag{4.21}$$

Following the solution obtained the free surface elevation is expressed by the formula

$$\eta(x,t) = \int_0^t \dot{\ddot{x}}(t-\tau)\Big[\frac{2}{\pi}\int_0^\infty \frac{\tanh(sh)}{s}\cos(r\tau)\cos(sx)\,ds\Big]\,d\tau \tag{4.22}$$

The function in square brackets is simply the impulse response function

$$h^*(t) = J_1^*(t) + J_2^*(t) \tag{4.23}$$

where, like in the previous formulation, the following substitutions have been made

$$\begin{aligned} J_1^*(t) &= \frac{1}{\pi}\int_0^\infty \frac{\tanh(sh)}{s}\cos(sx - rt)\,ds \\ J_2^*(t) &= \frac{1}{\pi}\int_0^\infty \frac{\tanh(sh)}{s}\cos(sx + rt)\,ds \end{aligned} \tag{4.24}$$

From comparison of the results (3.13) and (4.24) it may be seen that the second terms of these solutions $\big(J_2(t)$ and $J_2^*(t)\big)$ are expressed by different formulae. It is not a simple task to prove that these formulae describe the same functions and thus, each of them may be converted into the other one. To investigate the solutions obtained, numerical computations have been made. Some of the results obtained in this way are presented in Fig.2 where the plots of $J_1(t) = J_1^*(t)$ and $J_2(t) = J_2^*(t)$ for chosen values of x/h are given. From these graphs it is seen that with growing values of x the function $J_2(t)$ is decreasing. Moreover, for large values of time (say $t > 1$ sec) the effect of $J_2(t)$ on the value of the impulse response function is very small and may be ignored in practical calculations.

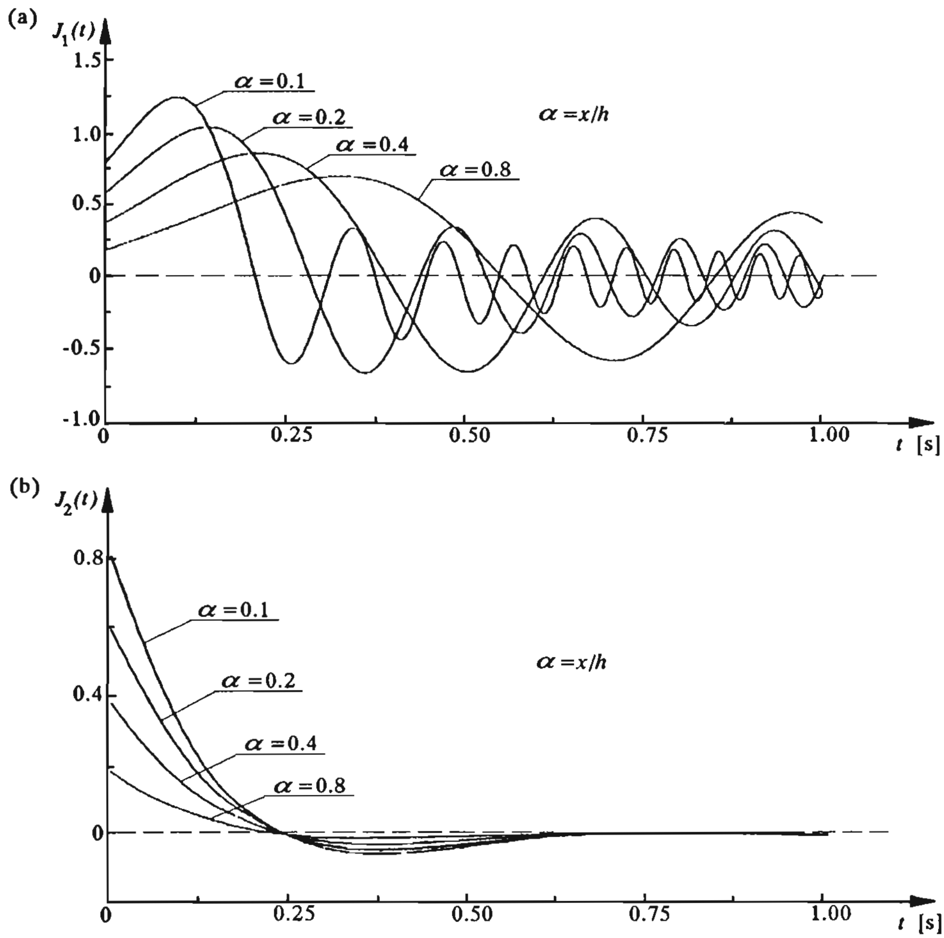

Fig. 2. Components (a) $J_1(t)$ and (b) $J_2(t)$ of the impulse response function

5. Concluding remarks

We have obtained the impulse response function for water waves in fluid of constant depth by means of two different formulations. The resulting final formulae for each case considered have different shapes but in fact they describe the same inherent feature of the system mentioned. Although we have not proved that the formulae may be converted into each other, numerical tests performed show that they provide identical results. From the computational point of view more convenient is the formulation based on the complex frequency response function of the generator-fluid system since it needs less time of computer calculations. The choice of formulation is a matter of convenience, but one should be aware that each of them may lead to different formulae and it may be difficult to convert from one formula to another.

References

1. BATEMAN H., 1954, *Tables of Integral Transforms,* **1**, McGraw Hill, New York
2. BIÉSEL F, SUQUET F., 1951, Les appareils generateures de houle en laboratoire, *La Hauille Blanche,* 157-165, 475-496, 723-727
3. CRANDALL S.H., MARK W.D., 1973, *Random Vibration in Mechanical Systems,* Academic Press, New York
4. HUDSPETH R.T., SULISZ W., 1991, Stokes Drift in Two-Dimensional Wave Flumes, *J.Fluid Mech.,* **230**, 209-229
5. LAMB H., 1975, *Hydrodynamics,* Cambridge University Press
6. MADSEN O.S., 1970, Waves Generated by a Piston-Type Wavemaker, *Proc. Twelfth Conf. Coastal Eng.,* 589-607
7. MADSEN O.S., 1977, On the Generation of Long Waves, *J. of Geophysical Research,* **76**, 36, 8672-8683
8. NOWACKI W., 1972, *Dynamika Budowli,* Arkady, Warszawa
9. STOKER J.J., 1957, *Water Waves,* Intersciences Publishers, New York
10. SZMIDT J.K., 1984, Problemy dyskretyzacji plaskich zagadnień hydrosprężystości, *Prace IBW PAN,* 12, Gdańsk
11. SZMIDT K., 1993, Impulse Generation of Waves in Fluid of Finite Depth, *Zesz. Naukowe Politechniki Gdańskiej,* (in print)
12. WILDE P., SMIDT K., 1993, Properties of Random Generation of Water Waves in a Layer of Fluid, *Zesz. Naukowe Politechniki Gdańskiej,* (in print)
13. WEHAUSEN J.V., LAITONE E.V., 1960, *Surface Waves,* in *Encyclopaedia of Physics,* Springer Verlag, Berlin

O funkcji odpowiedzi impulsowej dla fal wodnych generowanych w cieczy o stałej głębokości

Streszczenie

W pracy rozważa się problem generacji powierzchniowych fal grawitacyjnych w cieczy o stałej głębokości. Glównym celem badań jest wyznaczenie odpowiedzi impulsowej dla układu generator-ciecz. Rozważa się dwa niezależne sformułowania problemu. Te dwa sformułowania prowadzą do różnych wzorów opisujących tę samą właściwość układu. Jedno z tych sformułowań jest lepsze z uwagi na to, że wymaga ono mniejszego czasu obliczeń numerycznych.

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