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WAVES GENERATED BY AN INCIDENT SHOCK IN A PECULIAR HYPERELASTIC MATERIAL[1]

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A finite transverse shock wave propagates through an unbounded medium and is reflected at a plane boundary or reflected-refracted at a plane interface between two joined elastic half-spaces. The problem is not solvable in general, and we look for some kind of degeneracy in it, which make the solution possible. Two cases: clamped boundary and frictionless-rigid boundary are considered particular numerically for different assumptions. Both reflection patterns assumed here: either shock or simple wave, give the unique result in its range of admittace. There are two different critical angles: for shock as well as for simple wave, but in contrast with the second one, the first one has the pure geometrical meaning only.

1. Introduction

We apply the *semi-inverse* method (Wright (1971)) to examination of the *reflection problem* of oblique finite elastic plane shock wave at a plane boundary of nonlinear elastic solids and the *reflection-refraction problem* for a plane shock wave propagating in an unbounded medium consisting of two joined elastic half-spaces of different material properties, in the direction oblique to the interface. In such a homogeneous or composite medium, systems of additional waves can be superposed to represent the incident shock in conjunction with reflection at the boundary (reflection and refraction at the interface separating the two media). These additional waves are called reflected (and refracted) waves.

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If the medium ahead of the propagating shock has a given (undisturbed) state, then for a given incident shock, the region immediately behind the shock has the known state. The problem now is to fit the reflected (and refracted) waves so as to connect the state just fixed with some state at the boundary that is compatible with the boundary (interface) condition. It is assumed that the constant state behind the wave and the state at the boundary (interface) are connected by means of a sequence of centered simple waves, and undisturbed state regions. The assumption that the reflected (and refracted) waves are simple waves will reduce this problem to determining the distribution of the wavelets by means of ordinary differential equations. In some cases it may be necessary to modify the assumed reflection (and transmission) pattern, to include shocks as well; for shocks, the reflection problem is then reduced to solving a system of algebraic equations for the direction of propagation and strength of the reflected shocks.

We assume further that the elastic material is a special kind of idealised incompressible rubber, and that the oblique incident wave is a plane transverse shock. Since in such cases the motion is restricted to one dimension, there are only two (nontrivial) conditions to be met at the boundary (interface); hence; the assumed reflection (and refraction) pattern will include a single reflected wave (and a single refracted wave) only.

Section 2 contains a summary of the adequate theory, and derivation of the propagation condition for simple waves in incompressible materials. Since the reflected (and refracted) waves can be simple waves or shocks, we present in Section 2 differential equations for simple waves and jump conditions for shock waves. The reflection (and refraction) paterns are considered in Sections 4, 5 and the solutions is discussed in Sections 6 and 7.

2. Basic equations

We use here traditional symbols for deformation gradient its inverse and the particle velocity: $F_{i\alpha} = x_{i\alpha}$, $X_{\alpha i}$, $\dot{x}_i = u_i$. is assumed that the material is homogeneous, elastic and incompressible. The incompressibility condition requires that

$$J = \det[x_{i\alpha}] = 1 \tag{2.1}$$

It is assumed that the material is homogeneous and hyperelastic. The Piola-Kirchhoff stress tensor for such material is

$$T_{Ri\alpha} = \rho_R \frac{\partial \sigma}{\partial x_{i\alpha}} + pX_{\alpha i} \tag{2.2}$$

where

σ – internal energy per unit mass in the reference congiguration B_R

ρ_r – density

p – an arbitrary scalar function (hydrostatic pressure), $p = p(X_\alpha)$.

If the stress and velocity fields are differentiable, then the equations expressing balance of momentum are the equations of motion

$$T_{Ri\alpha,\alpha} = \rho_R \dot{u}_i \tag{2.3}$$

If the functions $x_i(X_\alpha, t)$ are continuous everywhere but have discontinuous first derivatives on some propagating surface $\Sigma(X,t)$, the equations (2.3) must be replaced by the jump conditions on this surface (cf Truesdell and Noll (1965))

$$\begin{aligned} [\![T_{Ri\alpha}]\!] N_\alpha &= -\rho_R V [\![u_i]\!] \\ [\![x_{i\alpha}]\!] = H_i N_\alpha & \qquad [\![u_i]\!] = -H_i V \end{aligned} \tag{2.4}$$

where

$$H_i = m d_i \qquad\qquad m = \sqrt{H_i H_i} > 0$$

and

N_α – components of a material unit normal to the wave

V – speed of propagation along N_α

H_i – components of the amplitude vector of the jump.

The bold square brackets indicate the jump in the quantity enclosed across Σ (cf Truesdell and Noll (1965)). Such a surface is called shock wave and is assumed to be stable (cf Lax (1957)). Eliminating the velocity jump from Eqs $(2.4)_1$ we obtain for the shock speed

$$[\![T_{Ri\alpha}]\!] N_\alpha H_i = \rho_R V^2 [\![x_{j\beta}]\!] N_\beta H_j \tag{2.5}$$

Simple waves are defined by Varley (1965) to be regions of space-time in which all field quantities are continuous functions of a single parameter, say $\gamma = G(X_\alpha, t)$. This means that in the region of a simple wave all field quantities can be expressed as functions of one of them. Hence, if one of the field quantities is constant in this region, the remaining quantities are also constant throughout this region. Regions of constant γ are propagating surfaces, called wavelets, with unit normal and normal velity in B_R given by

$$N_\alpha = \frac{G_{,\alpha}}{|\nabla G|} \qquad\qquad U(\gamma) = \frac{\dot{G}}{|\nabla G|} \tag{2.6}$$

Combining the equations of motion (2.3) with the compatibility condition $u_{j\alpha} = \dot{x}_{j\alpha}$ we obtain the propagation condition for simple waves in incompressible materials and equation which describe the change of $p(\gamma)$ in the simple wave region (cf Varley (1965))

$$(Q^*_{ij} - \rho_R U^2 \delta_{ij}) u'_j = 0 \tag{2.7}$$

$$p'(\gamma) = \frac{u}{U^2} Q_{ij} u'_j n_i \tag{2.8}$$

where n_i is a unit normal and u the speed of propagation in the present configuration B, the prime indicates differentiation with respect to γ, and

$$\begin{aligned} Q_{ij} &= \rho_R \frac{\partial^2 \sigma}{\partial x_{i\alpha} \partial x_{j\beta}} N_\alpha N_\beta \\ Q^*_{ij} &= \rho_R \frac{\partial T_{Ri\alpha}}{\partial x_{j\beta}} N_\alpha N_\beta = Q_{ij} - Q_{kj} n_k n_i \end{aligned} \tag{2.9}$$

are called the acoustic and the reduced acoustic tensors, respectively.

3. Incident shock

The condition of incompressibility restricts the propagating waves to transverse wave only. In general, a propagating wave incident on a boundary (interface) of an elastic medium does not meet the boundary (interface) conditions. If it is the only wave, the medium is not in a state of dynamic equilibrium; this is the reason for some additional waves, called reflected (and refracted) waves, being formed in association with the incident wave. In the main, the reflection (and refraction) problem may have no solution in the terms of simple waves, as there are at most two possible families of reflected waves (and two families of refracted waves) in such a case; this means that there are two (four) free parameters, with three bondary (six interfacial continuity) conditions to be meet. However, solutions may exist for some types of incompressible materials, with particular deformation and boundary conditions. In this paper we examine such particular cases.

Suppose the incident wave is a plane shock wave, and it is propagating through an elastic half-space $X_2 > 0$. The angle of incidence $\Theta_0 \in (0, \Theta_c)$ on the boundary $X_2 = 0$, and the shock strength m0 are known. Thus,

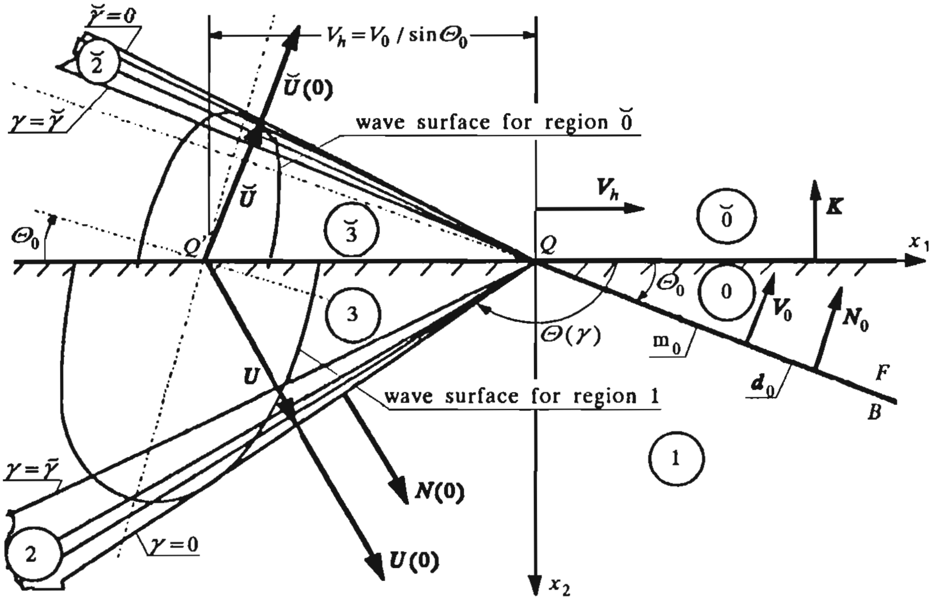

Fig. 1. Incident shock and assumed reflection patterns

this traveling discontinuity surface belongs to one-parameter family of parallel planes with normals

$$N_0 = [\sin\Theta_0, -\cos\Theta_0, 0] \qquad\qquad d_0 = [0,0,1] \tag{3.1}$$

Furthermore, we assume that the direction of polarisation, given by unit vector d_0, is parallel to the X_3-axis. The reflection line (point Q in Fig.1) moves along the boundary with constant speed $V_h = V_0 \sin\Theta_0$ where V_0 is the incident shock speed. It is assumed that the reflected wave is a simple wave (region 2). The material region 0 (and $\breve{0}$) ahead of the incident shock is unstrained and at rest

$$F^F_{i\alpha} = \delta_{i\alpha} \qquad\qquad \dot{u}^F_i = T_{i\alpha} = 0 \tag{3.2}$$

Regions 1 and 3 have constant state. Since all waves are centered at the point Q (cf Wright (1971)), we have for the reflected wave

$$\begin{aligned} N(\gamma) &= [\sin\Theta(\gamma), -\cos\Theta(\gamma), 1] \\ U(\gamma) &= V_h \sin\Theta(\gamma) \end{aligned} \tag{3.3}$$

where $\Theta(\gamma)$ is the angle of reflection and γ is the reflected wave parameter.It is expected that the reflected waves will propagate away from the boundary, hence $\pi/2 < \Theta(\gamma) < \pi$.

Let us denote

$$\tau = \cot \Theta(\gamma) \tag{3.4}$$

We have then from equations (2.7) and (3.3)

$$\widehat{\boldsymbol{N}} = \frac{1}{\sin \Theta(\gamma)} \boldsymbol{N} = [1, -\tau, 0] \tag{3.5}$$

$$\begin{aligned} &(\widehat{Q}^*_{ij} - \rho_R V_h^2 \delta_{ij}) u'_j = 0 \\ &V_h F'_{i\alpha} + u'_i N_\alpha = 0 \qquad\qquad i,j = 1,2 \end{aligned} \tag{3.6}$$

where $\widehat{Q}^*_{ij} = \sigma_{i\alpha j\beta} \widehat{N}_\alpha \widehat{N}_\beta$ and V_h is independent of $\Theta(\gamma)$.

At every point in the simple wave region the following condition must be satisfied

$$\pi(\tau) = \det(\widehat{Q}^*_{ij} - \rho_R V_h^2 \delta_{ij}) = 0 \qquad\qquad u'_i = k r_i \tag{3.7}$$

where $\pi(\tau)$ is a fourth degree polynomial in τ, $\boldsymbol{r}$ is a right eigenvector of the acoustic tensor $Q^* = \widehat{Q}^* \sin^2 \Theta(\gamma)$ associated with a particular root Θ and k is a scalar function of the deformation gradient; it is convenient to assume that $\boldsymbol{r}$ is a unit vector. The corresponding eigenvalue of $\mathbf{Q}^*$, the characteristic speed of the simple wave, is $U^2 = \rho_R V_h^2/(1+\tau^2)$. Thus, if τ and $\boldsymbol{r}$ correspond to the reflected simple wave under consideration, then we have

$$\boldsymbol{u}' = k\boldsymbol{r} \qquad\qquad \boldsymbol{F}' = -\frac{k}{V_h} \boldsymbol{r} \otimes \boldsymbol{N} \tag{3.8}$$

Each simple wave is completely described by a one-parameter set of functions the variation of which is governed by the above system of ordinary differential equations. Since the velocity and the deformation gradient are continuous throughout the regions behind the incident shock, the initial values for Eqs (3.8) are the constant values of the region in front of the wave. The undisturbed state of the region just behind the wave is fixed by the values at the trailing edge of the wave.

A detailed discussion and geometric interpretation of the roots of $\pi(\tau)$ can be found in Wright (1971). Here we shall only state that for the simple wave to propagate $\tau(\gamma)$ must be a real decreasing function of $\gamma \in [0, \widetilde{\gamma}]$, when γ changes from its initial value 0 to the extreme value $\widetilde{\gamma}$ (which may be negative); this means that its wavelets (rays) diverge with increasing γ (cf

Eq (3.4)). If $\tau(\gamma)$ increases, then the assumed reflection pattern should be modified to include shocks as well.

It is assumed that both material solids are isotropic incompressible and are characterized by the constitutive equations

$$\begin{aligned} W(I_1,I_2) &= \rho_R\sigma(I_1,I_2) = C_1(I_1-3)+C_2(I_2-3)+C_3(I_1^2-9) \quad \text{for } X_2>0 \\ \check{W}(\check{I}_1,\check{I}_2) &= \check{\rho}_R\sigma(\check{I}_1,\check{I}_2) = \check{C}_1(\check{I}_1-3)+\check{C}_2(\check{I}_2-3)+\check{C}_3(\check{I}_1^2-9) \quad \text{for } X_2<0 \end{aligned} \tag{3.9}$$

proposed by Isihara et al. (1952), where $I_1 = B_{ii}$, $I_2 = (B_{ii} - B_{ij}B_{ij})/2$, are the invariants of the left Cauchy-Green strain tensor B_{ij}. The set of values C_1, C_2, C_3 – represents the material elastic constants. The symbol $(\check{\cdot})$ serves here to label the field quantities and the field equations in the half-space $X_2 < 0$. Approximation (3.9) of the strain energy function W is valid for rubber-like materials under moderate strain. Experimental investigations (cf Zahorski (1962)) indicate that the constant C_3, important in the following discussion is positive.

Since the medium in front of the shock is unstrained and at rest, the jump condition across the incident shock Eq (2.4) become now

$$\begin{aligned} [\![x_{31}]\!] &= x_{31}^B = m_0 \sin\Theta_0 \\ [\![x_{32}]\!] &= x_{32}^B = -m_0\cos\Theta_0 \\ [\![\dot{x}_3]\!] &= -m_0V_0 \end{aligned} \tag{3.10}$$

Substituting Eqs (3.1) and (3.10) into Eq (2.5) we obtain the equation relating the shock speed V_0 and the shock strength m_0

$$V_0^2 = c^2(1+\eta m_0^2) \tag{3.11}$$

where

$$c^2 = \frac{2(C_1+C_2+6C_3)}{\rho_R} \qquad \eta = \frac{4C_3}{\rho_R c^2}$$

The state behind the propagating shock wave (region 1) is now completely determined by the shock speed V_0 and its strength m_0. Eqs (3.10) determine the deformation gradient and its inverse

$$\left[x_{i\alpha}^B\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hat{\nu}_1 & \hat{\nu}_2 & 1 \end{bmatrix} \qquad \left[X_{\alpha i}^B\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\hat{\nu}_1 & -\hat{\nu}_2 & 1 \end{bmatrix} \tag{3.12}$$

and the particle velocity

$$\boldsymbol{u} = [0,0,u_3] \qquad u_3 = -m_0V_0 \tag{3.13}$$

We denote here $x^B_{31} = \hat{\nu}_1$, $x^B_{32} = \hat{\nu}_2$, $u^B_3 = u$, $x_{31}(\gamma) = \nu_1$, $x_{32}(\gamma) = \nu_2$.

The Piola-Kirchhoff stress components $T_{Ri\alpha}$ and stress components $\sigma^{\alpha\beta}_{ik}$ required in this paper are then evaluated in regions 1 and $\breve{0}$ (cf Fig.1)

$$
\begin{aligned}
&T_{R11} = 2\rho_R[\sigma_1 + \sigma_2(2+\nu_2^2)] + p && T_{R13} = -2\rho_R\sigma_2\nu_1 - p\nu_1 \\
&T_{R22} = 2\rho_R[\sigma_1 + \sigma_2(2+\nu_1^2)] + p && T_{R31} = 2\rho_R(\sigma_1+\sigma_2)\nu_1 \\
&T_{R33} = 2\rho_R(\sigma_1 + 2\sigma_2) + p && T_{R23} = -2\rho_R\sigma_2\nu_1 - p\nu_1 \\
&T_{R12} = T_{R21} = -2\rho_R\sigma_2\nu_1\nu_2 && T_{R32} = 2\rho_R(\sigma_1+\sigma_2)\nu_2
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
&\sigma^{11}_{33} = 2(\sigma_1+\sigma_2) + 4\sigma_{11}\nu_1^2 && \sigma^{22}_{33} = 2(\sigma_1+\sigma_2) + 4\sigma_{11}\nu_2^2 \\
&\sigma^{12}_{33} = \sigma^{21}_{33} = 4\sigma_{11}\nu_1\nu_2 && \sigma^{22}_{13} = -2\sigma_2\nu_1 \qquad \sigma^{11}_{23} = -2\sigma_2\nu_2
\end{aligned}
\tag{3.15}
$$

where

$$
\begin{aligned}
&\sigma_1 = \frac{\partial\sigma}{\partial I_1} = \frac{1}{\rho_R}(C_1 + 2C_3I_1) && \sigma_2 = \frac{\partial\sigma}{\partial I_2} = \frac{C_2}{\rho_R} \\
&\sigma_{11} = \frac{\partial^2\sigma}{\partial I_1^2} = \frac{2C_3}{\rho_R} && I_1 = I_2 = 3 + \nu_1^2 + \nu_2^2
\end{aligned}
$$

Eqs (3.12) imply the motions under consideration are restricted to the x_3 direction. By Eqs (3.14), (3.15) and on the assumption that u_3 is not equal zero, the propagation condition $(Q^*_{ij} - \rho_R U^2\delta_{ij})u'_j = 0$ is reduced to the set of equations

$$
(Q^*_{i3} - \rho_R U^2\delta_{i3})u'_3 = 0 \Rightarrow \tag{3.16}
$$

$$
Q^*_{33} - \rho_R U^2 = 0 \qquad \wedge \qquad Q^*_{13} = Q^*_{23} = 0 \tag{3.17}
$$

The last two equations in expanded form are

$$
\begin{aligned}
Q^*_{13} &= Q_{13} - Q_{13}n_1n_1 - Q_{23}n_2n_1 = 0 \\
Q^*_{23} &= Q_{23} - Q_{13}n_1n_2 - Q_{23}n_2n_2 = 0
\end{aligned}
\tag{3.18}
$$

and they form a homogeneous system of algebraic equations

$$
\begin{bmatrix} 1 - n_1n_1 & -n_2n_1 \\ -n_1n_2 & 1 - n_2n_2 \end{bmatrix}
\begin{bmatrix} Q_{13} \\ Q_{23} \end{bmatrix} =
\begin{bmatrix} 0 \\ 0 \end{bmatrix}
\tag{3.19}
$$

For the assumed deformation and motion the components of the wave normal are identical

$$
n_1 = N_1 \qquad\qquad n_2 = N_2 \qquad\qquad n_3 = N_3 = 0 \tag{3.20}
$$

The determinant of the system of equations (3.19) is equal zero, and there exist non-triviale solutions of the system for $Q_{23} = -\tau Q_{13}$. On which conditions this relation is safisfied?

The resulting equation (3.18) in compact form reads

$$Q_{i3} - Q_{j3}N_jN_i = 0 \tag{3.21}$$

which says that $\widetilde{Q}_{i3}$ ist proportional to N_i. If this result is hold for arbitrary $\boldsymbol{N}$, which it must, then taking first $\boldsymbol{N} = [1,0,0]$ and then $\boldsymbol{N} = [0,1,0]$ it turns out that for the assumed class of deformations, namely that all motion is restricted to x_3 direction we have

$$\begin{aligned} \boldsymbol{N} = [1,0,0] \quad &\Rightarrow \quad Q_{13} = Q_{13} \quad \wedge \quad Q_{23} = \sigma_{23}^{11} = -2\sigma_2\nu_2 = 0 \\ \boldsymbol{N} = [0,1,0] \quad &\Rightarrow \quad Q_{23} = Q_{23} \quad \wedge \quad Q_{13} = \sigma_{13}^{22} = -2\sigma_2\nu_1 = 0 \end{aligned} \tag{3.22}$$

This means that the condition $Q_{23} = -\tau Q_{13}$ is satisfied for both (3.22) if $\sigma_2 = 0$ and this in turn implies that the strain energy does not depend on the second invariant of the Cauchy-Green strain tensor. This condition is fullfiled also in two particular cases: for $\sigma_2 \neq 0$ and $\nu_1 = 0$ (normal incidence) as well as for $\sigma_2 \neq 0$ and $\nu_1 = 0$ (grazing incidence). Taking $\boldsymbol{N}$ in more general form $\boldsymbol{N} = [\alpha, \beta, 0]$ in Eq (3.21) we obtain the same results as above.

4. Reflection-refraction pattern ($\sigma_2 = 0$)

We assume that the reflected wave is a single simple plane wave (both the reflected and the refracted waves are single simple plane waves).

The propagation condition is reduced to a single equation, and the last two equations below are also satisfied identically

$$Q_{33}^* - \rho_R U^2 = 0 \qquad \text{and} \qquad Q_{13}^* = Q_{23}^* = 0 \tag{4.1}$$

Because $\widehat{Q}_{33}^* = Q_{33}\widehat{N}_\alpha \widehat{N}_\beta = \sigma_{33}^{\alpha\beta}\widehat{N}_\alpha \widehat{N}_\beta$ the above equation takes the form

$$\sigma_{33}^{\alpha\beta}\widehat{N}_\alpha \widehat{N}_\beta - V_h^2 = 0 \tag{4.2}$$

we can rewrite the propagation condition as a quadratic equation in τ

$$\sigma_{33}^{22}\tau^2 - 2\sigma_{33}^{12}\tau + (\sigma_{33}^{11} - V_h^2) = 0 \tag{4.3}$$

Its smaller (greater) root indicates the planes of reflected (refracred) wavelets respectively

$$\tau = \frac{\sigma_{33}^{12}}{\sigma_{33}^{22}} \pm \sqrt{\Big(\frac{\sigma_{33}^{12}}{\sigma_{33}^{22}}\Big)^2 - \frac{\sigma_{33}^{11} - V_h^2}{\sigma_{33}^{22}}} \tag{4.4}$$

and stress components $\sigma_{ik}^{\alpha\beta}$ required in Eq (4.4) are evaluated corresponding by in material region $X_2 > 0$ $(\sigma_{ik}^{\alpha\beta})$ or $X_2 < 0$ $(\breve{\sigma}_{ik}^{\alpha\beta})$.

The requirement that the roots are real gives the condition for the critical angle Θ_c: it is the largest angle Θ_0 for which the following inequalities hold

$$\begin{aligned} &\sigma_{33}^{12} - \sigma_{33}^{22}(\sigma_{33}^{11} - V_h^2) > 0 \\ &\breve{\sigma}_{33}^{12} - \breve{\sigma}_{33}^{22}(\breve{\sigma}_{33}^{11} - V_h^2) > 0 \end{aligned} \tag{4.5}$$

The wave surface configuration for the both materials (Fig.2) shows that the point Q moves toward it as the incidence angle increases, and intersects first the below wave sheet. In the reversed materials combination the point Q will intersect the top wave surface first. Evaluating $(4.5)_1$ in region 1 we obtain

$$\sin\Theta_c \leq \sqrt{\frac{1 + 3\eta m_0^2}{1 + 5\eta m_0^2}} \tag{4.6}$$

Substituting the components of the deformation gradient into the propagation condition $Q_{33}^* - \rho_R U^2 = 0$ for arbitrary angle of incidence Θ_0, we obtain the expression for the velocity surface in region 1

$$U^2 = c^2\Big(1 + \eta m_0^2[1 + 2\cos^2(\Theta - \Theta_0)]\Big) \tag{4.7}$$

The wave surface geometry for the leading edge of the reflected simple wave depends on the deformation gradient in region 1 which in turn depends on Θ_0, m_0. In region 1 the wave surface for deformation (3.10) is ellipse.

For the fixed incident angle Θ_0 (fixed amplitude m_0), the point Q moves toward the wave surface as the amplitude m_0 of the incident wave (the incident angle Θ_0) increases. All waves in the configuration are centered at the point Q. The limit value for the cotangent of the reflection angle corresponding with the leading wavelet: $\cot\Theta_{lim}$ is given by $(4.8)_2$

$$\begin{aligned} V_h &= \frac{V_0}{\sin\Theta_0} = \frac{U(0)}{\sin\Theta} \\ \cot\Theta_{lim} &= -\frac{2\eta m_0^3\sqrt{2\eta(1 + 3\eta m_0^2)}}{1 + 3\eta m_0^2(2 + 3\eta m_0^2)} \end{aligned} \tag{4.8}$$

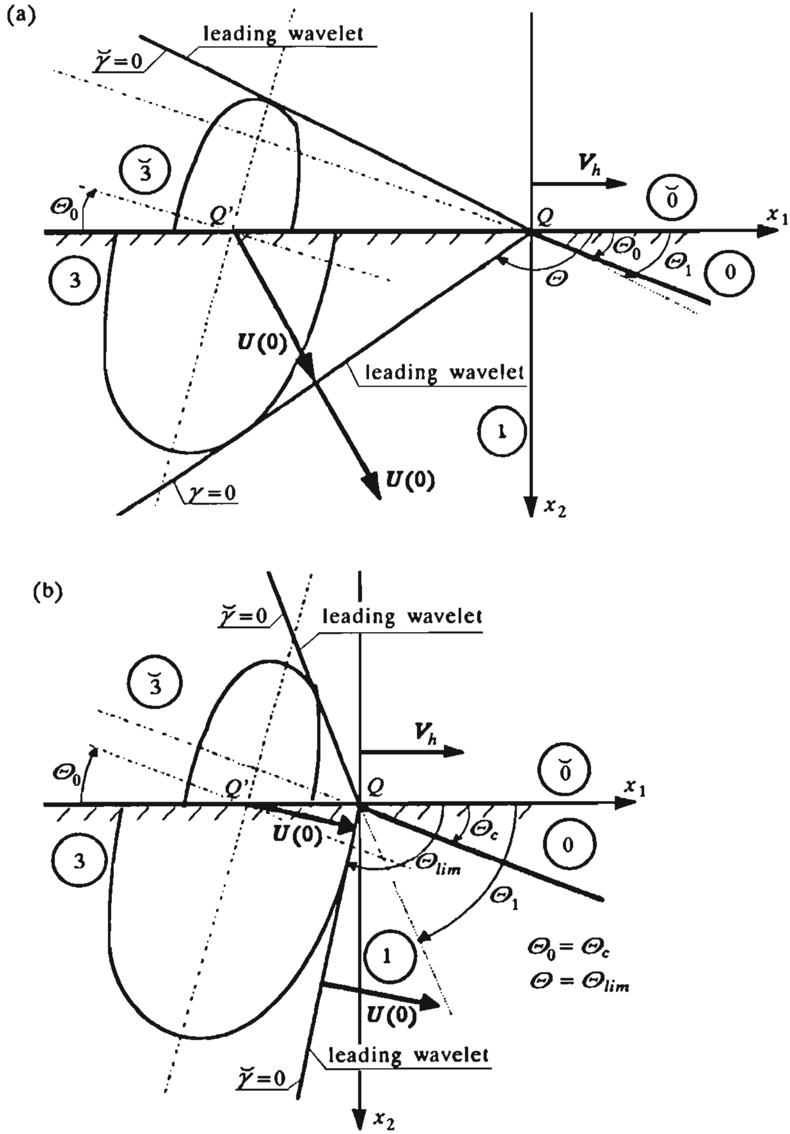

Fig. 2. Geometry of the wave surface in regions 1 and $\breve{0}$

It is easy to calculate the both semi-axes of the velocity surface (cf Eq (4.7))

$$d_1 = c\sqrt{1 + 3\eta m_0^2} \qquad\qquad d_2 = c\sqrt{1 + \eta m_0^2} \tag{4.9}$$

5. Boundary and interface conditions ($\sigma_2 = 0$)

The simple wave is completely described by a one-parameter set of functions given by the ordinary differential equations (3.8) and satysfying the initial and boundary conditions. Equations (3.8) are now (cf Eqs (3.5), (3.14) and (3.15))

$$u_3' = V_h f(x_{i\alpha}) \qquad\qquad x_{31}' = -f(x_{i\alpha}) \qquad\qquad x_{32}' = \tau f(x_{i\alpha}) \tag{5.1}$$

The components of the deformation gradient and velocity behind the shock are given by Eq (3.10); hence, they are, the initial condition for Eqs (5.1).

5.1. Clamped boundary

Let us assume that the incident shock is reflected from a ridigly constrained boundary; this means that

$$u_3 = 0 \qquad \text{on} \qquad X_2 = 0 \tag{5.2}$$

To meet this condition it is convenient to choose in Eq (5.1) $f(x_{i\alpha}) = -V_h^{-1}$, for then the system (5.1) becomes

$$u_3' = -1 \qquad\qquad x_{31}' = \frac{1}{V_h} \qquad\qquad x_{32}' = -\frac{\tau}{V_h} \tag{5.3}$$

Integrating the first two conditions, with the initial condition (3.10), we obtain

$$u_3 = -\gamma - m_0 V_0 \qquad\qquad x_{31} = \frac{\gamma}{V_h} + m_0 V_0 \tag{5.4}$$

and the condition (5.2) is satisfied when $\widetilde{\gamma} = -m_0 V_0$. For $\gamma = \widetilde{\gamma}$ also vanishes $x_{31}(\widetilde{\gamma}) = 0$. Substitution for x_{31} and τ (given by Eq (4.4)) into Eq $(5.3)_3$ leads to a nonlinear differential equation for x_{32} which can be solved only numerically.

5.2. Frictionless-rigid boundary

Let us consider a case of "mixed" boundary conditions on the plane $X_2 = 0$, when the normal displacement and the shearing stresses T_{R12}, T_{R32} are zero. Since the motions under consideration are restricted to X_3-axis direction, the displacement condition is satisfied identically. The stress conditions (cf Eqs (3.14)) are met when

$$x_{32} = 0 \qquad \text{on} \qquad X_2 = 0 \tag{5.5}$$

To satisfy this condition it is convenient to choose in Eqs (5.1), $f(x_{i\alpha}) = -\tau^{-1}$ for then the system (5.1) becomes

$$x'_{31} = \frac{1}{\tau} \qquad x'_{32} = -1 \qquad u'_3 = -V_h x'_{31} \tag{5.6}$$

integrating the last two equations, with the initial conditions (3.10), we obtain

$$\begin{aligned} x_{32} &= -\gamma - m_0 \cos \Theta_0 \\ x_{31} &= -V_h(x_{31} - m_0 \sin \Theta_0) - m_0 V_0 \end{aligned} \tag{5.7}$$

and the condition (5.5) is met when $\widetilde{\gamma} = -m_0 \cos \Theta_0$.

Substitution for x_{32} and τ (cf Eq Eq (4.4)) into Eq $(5.6)_1$ gives a nonlinear differential equation for x_{31}. This problem, however, can be solved only numerically.

5.3. Free boundary condition

Consider a case in which the stress vector $t_i = T_{Ri\alpha}K_\alpha$ ($K = [0, -1, 0]$), vanishes on the plane $X_2 = 0$. This means that for $X_2 = 0$, $T_{R12} = T_{R32} = T_{R22} = 0$. The first two equations (cf Eqs (3.14)) led to the condition (5.5). The third equation

$$T_{R22} = 2\rho_R\Big(\sigma_1 + \sigma_2[2 + (x_{31})^2]\Big) + p = 0 \tag{5.8}$$

which must be satisfied on $X_2 = 0$ in both regions 0 and 3 determines the hydrostatic pressure in region 0 and in region 3, respectively :

$$p_0 = -c^2 \rho_R \qquad p_3 = -c^2 \rho_R\Big(1 + \eta[x_{31}(\widetilde{\gamma})]^2\Big) \tag{5.9}$$

where the $x_{31}(\tilde{\gamma})$ are the values of the deformation gradient at the trailing wavelet of region 2.

The function $p(\gamma)$ is continuous throughout regions $1 \div 3$, but it suffers a jump across the shock surface that separates regions 0 and 1. To find p_1 in region 1 we use the jump conditions (2.4). The first two equations $[\![T_{Ri\alpha}]\!]N_\alpha = 0$, or equivalently $[\![T_{R11}]\!] = \tau^2[\![T_{R22}]\!]$, together with Eqs (3.10) and (3.14), give the jump of p

$$[\![p]\!] = p_1 - p_0 = -4C_3 m_0^2 \tag{5.10}$$

and the function p_1 in region 1 of constant state is

$$p_1 = p_0 = -4C_3 m_0^2 \tag{5.11}$$

for an arbitrary value of the incident angle Θ_0.

In region 2 (simple wave) the deformation gradient and velocity are completely determined by Eqs (5.6) and (5.7) as continuous functions of the wave parameter in the interval $\langle\tilde{\gamma}, 0\rangle$. Eqs (5.6) and (5.7) are consistent with the two conditions $T_{R12} = T_{R32} = 0$. As for the propagation condition we can use the equations of motion and compatibility condition with the Piola-Kirchhoff dtress tensor (3.14) to establish the equation describing the change of the function $p(\gamma)$ in the region of the simple wave (2.8) .The differential equation (2.8), after substitution for x_{32}, x_{31} and u_3, determines the hydrostatic pressure $p(\gamma)$ up to a constant in the interval $\langle\tilde{\gamma}, 0\rangle$. Direct integration with the aid of Eqs (5.6), gives

$$p(\gamma) = -c^2 \rho_R \eta \Big[\Big(x_{31}(\gamma)\Big)^2 + \Big(x_{32}(\gamma)\Big)^2 \Big] + p_0 \tag{5.12}$$

Due to continuity throughout regions $1 \div 3$, the function $p(\gamma)$ satisfies two conditions, $p(0) = p_1$ and $p(\tilde{\gamma}) = p_3$ where p_1 and p_3 are given by Eqs (5.9).

5.4. Reflection and refraction at the interface ($\sigma_2 = 0$ and $\sigma_2 \neq 0$)

Let two nonlinear elastic materials (described by Eqs (3.9)) differing on elastic properties are rigidly coupled at the interface $X_2 = 0$. There are three conditions for stresses and one for velocity to consider at $X_2 = 0$

$$u_i = \check{u}_i \qquad t_i = \check{t}_i \qquad t_i = T_{Ri\alpha} K_\alpha \qquad \boldsymbol{K} = [0, -1, 0] \tag{5.13}$$

$$t_i = \check{t}_i \Rightarrow T_{R12} = \check{T}_{R12} \qquad T_{R22} = \check{T}_{R22} \qquad T_{R32} = \check{T}_{R32} \tag{5.14}$$

5.4.1. Reflected simple wave, $x_2 > 0$ (see Fig.1)

The equations in region 1 are assumed in the form

$$u_3' = -1 \qquad x_{31}' = \frac{1}{V_h} \qquad x_{32}' = -\frac{\tau}{V_h} \qquad f(x_{i\alpha}) = -\frac{1}{U_h} \tag{5.15}$$

integrating we obtain

$$\begin{aligned} u_3 &= -\gamma + u_3(0) \qquad x_{31} = \frac{\gamma}{V_h} + x_{31}(0) \\ x_{32} &= -\int_0^\gamma \frac{\tau}{V_h}\, d\gamma + x_{32}(0) \end{aligned} \tag{5.16}$$

5.4.2. Refracted simple wave, $x_2 < 0$ (see Fig.1)

Analogous in the case of the refracted wave $\breve{u}_3(0) = \breve{x}_{31}(0) = \breve{x}_{32}(0) = 0$, because the region $\breve{0}$ is unstrained and at rest

$$\breve{u}_3' = -\breve{\gamma} \qquad \breve{x}_{31}' = \frac{\breve{\gamma}}{V_h} \qquad \breve{x}_{32}' = \int_0^{\breve{\gamma}} \frac{\breve{\tau}}{V_h}\, d\breve{\gamma} \qquad \breve{f}(x_{i\alpha}) = -\frac{1}{V_h} \tag{5.17}$$

At the interface $X_2 = 0$ **and for** $X_1 > 0$, between regions 0 and $\breve{0}$ the three conditions are satisfied identically

$$u_3 = \breve{u}_3 = 0 \qquad T_{R12} = \breve{T}_{R12} = 0 \qquad T_{R32} = \breve{T}_{R32} = 0 \tag{5.18}$$

the fourth equation

$$T_{R22} = \breve{T}_{R22} \Rightarrow c^2\rho_R + p_0 = \breve{c}^2\breve{\rho}_R + \breve{p}_0 \tag{5.19}$$

relates the pressures p_0 and $\breve{p}_0$ in regions 0 and $\breve{0}$ across the interface.

At the interface $X_2 = 0$ **and for** $X_1 < 0$, between regions 3 and $\breve{3}$ we obtain four nontrivial equations involving the final values of the wave parameters $\tilde{\gamma}$ and $\tilde{\breve{\gamma}}$

$$\begin{aligned} &u_3 = \breve{u}_3 \Rightarrow -\tilde{\gamma} + u_3(0) = -\tilde{\breve{\gamma}} \\ &\frac{1}{V_h} \neq 0 \Rightarrow u_3 = \breve{u}_3 \Leftrightarrow x_{31} = \breve{x}_{31} \end{aligned} \tag{5.20}$$

$$T_{R12} = \breve{T}_{R12} \Rightarrow C_2 x_{32} x_{31} = \breve{C}_2 \breve{x}_{32} \breve{x}_{31} \Rightarrow C_2 x_{32} = \breve{C}_2 \breve{x}_{32} \qquad (x_{31} = \breve{x}_{31} \neq 0) \tag{5.21}$$

$$T_{R22} = \breve{T}_{R22} \Rightarrow C_2 (x_{31})^2 = \breve{C}_2 (\breve{x}_{31}^2 \tag{5.22}$$

$$T_{R32} = \breve{T}_{R32} \Rightarrow c^2 \rho_R \Big(1 + \eta[(x_{31})^2 + (x_{32})^2]\Big) x_{32} = = \breve{c}^2 \breve{\rho}_R \Big(1 + \breve{\eta}[(\breve{x}_{31})^2 + (\breve{x}_{32})^2]\Big) \breve{x}_{32} \tag{5.23}$$

There are four nontrivial algebraic equations involving two unknowns $\widetilde{\gamma}$, $\breve{\widetilde{\gamma}}$.

We can reduce this system of equation in two ways: the first is $\sigma_2 = \breve{\sigma}_2 = 0 \Rightarrow C_2 = \breve{C}_2 = 0$ and leads to two nontrivial equation for $\widetilde{\gamma}, \breve{\widetilde{\gamma}}$

$$\begin{aligned} &-\widetilde{\gamma} + u_3(0) = -\breve{\widetilde{\gamma}} \\ &c^2 \Big(1 + \eta[(x_{31}(\widetilde{\gamma}))^2 + (x_{32}(\widetilde{\gamma}))^2]\Big) x_{32}(\widetilde{\gamma}) = \\ &= \breve{c}^2 \Big(1 + \breve{\eta}[(\breve{x}_{31}(\breve{\widetilde{\gamma}}))^2 + (\breve{x}_{32}(\breve{\widetilde{\gamma}}))^2]\Big) \breve{x}_{32}(\breve{\widetilde{\gamma}}) \end{aligned} \tag{5.24}$$

or the other one $\sigma_2 = C_2 \neq 0$ ($\breve{\sigma}_2 = \breve{C}_2 \neq 0$) with $x_{31} = \breve{x}_{31} = 0$ (cf Duszczyk et al (1986), Kosiński and Duszczyk (1989))

$$\begin{aligned} &-\widetilde{\gamma} + u_3(0) = -\breve{\widetilde{\gamma}} \\ &c^2[1 + \eta(x_{32}(\widetilde{\gamma}))^2] x_{32}(\widetilde{\gamma}) = \breve{c}^2[1 + \breve{\eta}(\breve{x}_{32}(\breve{\widetilde{\gamma}}))^2] \breve{x}_{32}(\breve{\widetilde{\gamma}}) \end{aligned} \tag{5.25}$$

Both cases correspond to oblique and normal incidence of the shock, respectively.

6. Reflected shock waves

The reflection solution was assumed in a form of a sequence of simple waves and undisturbed state regions. If $\tau(\gamma)$ increases with γ, the travelling pencil of wavelets converges to the leading wavelet, thus forming a shock wave (cf Eqs (3.4)). Now we assume that the reflected wave is a shock and will investigate in which cases such a wave is stable. We restrict our attention to the reflection problem at the *clamped* and *frictionless-rigid* boundary.

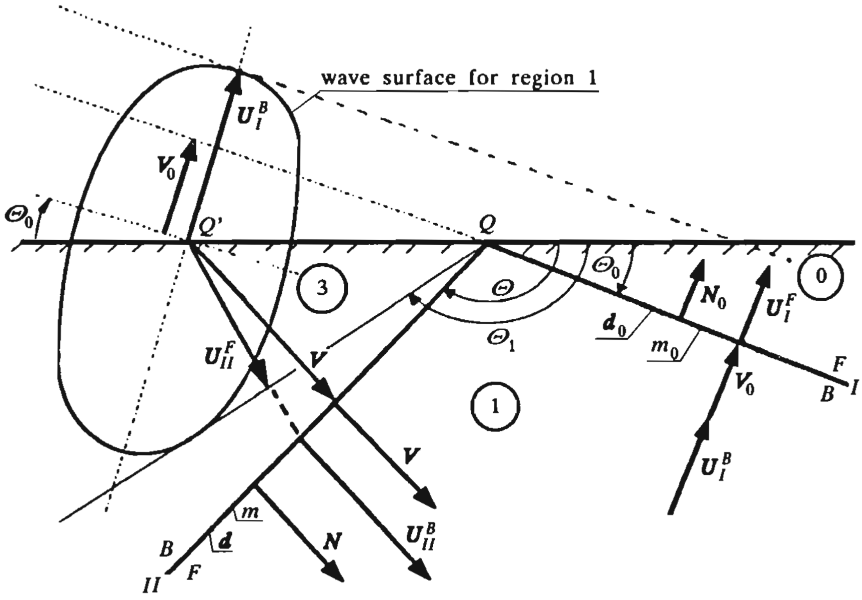

Fig. 3. Configuration of the reflected shocks

According to Lax (1957), for a shock wave to be stable it is necessary that it must travel faster than the corresponding type of acoustic wave ahead of the shock and slower than the corresponding type of acoustic wave behind the shock. Thus

$$U_I^F \le V_0 \le U_I^B \qquad\qquad U_{II}^F \le V \le U_{II}^B \tag{6.1}$$

In order to the incident shock wave be stable it must intersect the wave surface which is extended through the material boundary. The leading wavelet of a simple wave at angle Θ_1 must be tangent to the wave surface computed for region 1, a reflected shock wave must lie at an angle less than Θ_1 since for reason of stability it must have a higher speed of propagation V than that speed U_{II}^F of an acoustic wave of the same type in the region ahead of the shock. The components of the deformation gradient and velocity in region 3 are

$$\begin{aligned}(x_{31})_{II}^B &= (x_{31})_{II}^F + md_3N_1\\ (x_{32})_{II}^B &= (x_{32})_{II}^F + md_3N_2\\ (u_3)_{II}^B &= (u_3)_{II}^F - md_3N_1\end{aligned} \tag{6.2}$$

- **Clamped boundary**

For the clamped boundary condition $(u_3)^B_{II} \Rightarrow -m_0V_0 = -md_3V$, $(m_0 > 0, m > 0)$ and $d_3 = -1$. The polarisation vector $\boldsymbol{d}$ of the reflected shock wave has also the opposite direction then $\boldsymbol{d}_0$. From the boundary and centered wave conditions $(5.2)_3$, $(4.8)_1$ we obtain

$$m = m_0\frac{\sin\Theta_0}{\sin\Theta} \qquad (x_{31})^B_{II} = 0$$
$$(x_{32})^B_{II} = m_0(\sin\Theta_0\cot\Theta - \cos\Theta_0) \tag{6.3}$$

- **Frictionless-rigid boundary**

According to Eqs (6.2) and boundary condition (5.5) $x_{32} = 0 \Rightarrow -m_0\cos\Theta_0 - md_3\cos\Theta$ and $(m_0 > 0,\ m > 0,\ \cos\Theta < 0) \Rightarrow d_3 = 1,\ \Rightarrow \boldsymbol{d} = \boldsymbol{d}_0$ both polarisation vectors have the same direction. Analogously as for the previous case we have

$$m = -m_0\frac{\cos\Theta_0}{\cos\Theta} \qquad (x_{32})^B_{II} = 0$$
$$(x_{31})^B_{II} = m_0(\sin\Theta_0 - \cos\Theta_0\tan\Theta) \tag{6.4}$$

Substituting the components of the deformation gradient given by Eqs (3.10), (6.3), (6.4) into Eq (2.5) we obtain the velocities for the incident and reflected shock, and the acoustic waves. For the velocities in region 1 we obtain

$$U^F_I = c \qquad V_0 = c\sqrt{1+\eta m_0^2} \qquad U^B_I = c\sqrt{1+3\eta m_0^2} \tag{6.5}$$

The Lax stability condition for the incident shock is always safisfied, and for the reflected shock it is satisfied if the following conditions are satisfied:

- **Clamped boundary**

$$V \geq U^F_{II} \Rightarrow \frac{\sin\Theta_0}{\sin\Theta} > \cos(\Theta - \Theta_0)$$
$$U^B_{II} \geq V \Rightarrow \frac{\sin\Theta_0}{\sin\Theta} > \frac{3}{2}\cos(\Theta - \Theta_0) \tag{6.6}$$

Both inequalities are satisfied for the values $\cot\Theta$

$$\cot\Theta < \xi_1 = \frac{3}{2}\cot\Theta_0 - \sqrt{\Big(\frac{3}{2}\cot\Theta_0\Big)^2 + 2} \tag{6.7}$$

The reflected shock wave speed can be computed from Eqs (2.5) and (6.3)

$$V^2 = c^2\Big[1+\eta m_0^2\Big(1+2\cos^2(\Theta-\Theta_0)+\frac{\sin^2\Theta_0}{\sin^2\Theta}-3\frac{\sin\Theta_0}{\sin\Theta}\cos(\Theta-\Theta_0)\Big)\Big] \tag{6.8}$$

Comparing Eq (6.8) with the result from centered wave condition $V^2 = V_0^2\sin^2\Theta/\sin^2\Theta_0$ (see Fig.3), which is analogous to Eq $(4.8)_1$, and with Eq (3.11) we obtain

$$\kappa_1 = \frac{1+\eta m_0^2}{\eta m_0^2} = \frac{2\cos^2(\Theta-\Theta_0)+\frac{\sin\Theta_0}{\sin\Theta}\Big(\frac{\sin\Theta_0}{\sin\Theta}-3\cos(\Theta-\Theta_0)\Big)}{\Big(\frac{\sin\Theta}{\sin\Theta_0}\Big)^2-1} \tag{6.9}$$

The left hand side of the above equation is positive, $\sin\Theta_0/\sin\Theta > 0$ and numerator of the right hand side of the above expression is positive (cf Eq (6.6)), for this reason the denominator is positive if

$$\sin\Theta > \sin\Theta_0 \Rightarrow \frac{\pi}{2} < \Theta < \pi-\Theta_0 \qquad \text{and} \qquad \cot\Theta < \xi_1 \tag{6.10}$$

$$m_0 = \frac{1}{\sqrt{\eta(\kappa_1-1)}} \qquad \kappa_1 > 1 \tag{6.11}$$

- **Frictionless-rigid boundary**

The analogous calculations give the following results in this case

$$V \geq U_{II}^F \;\wedge\; U_{II}^B \geq V \Rightarrow \frac{\cos\Theta_0}{\cos\Theta} < 3\cos(\Theta-\Theta_0) \tag{6.12}$$

$$\tan\Theta < \xi_2 = \frac{3}{2}\tan\Theta_0 - \sqrt{\Big(\frac{3}{2}\tan\Theta_0\Big)^2+2} \tag{6.13}$$

$$\kappa_2 = \frac{1+\eta m_0^2}{\eta m_0^2} = \frac{2\cos^2(\Theta-\Theta_0)+\frac{\cos\Theta_0}{\cos\Theta}\Big(\frac{\cos\Theta_0}{\cos\Theta}-3\cos(\Theta-\Theta_0)\Big)}{\Big(\frac{\sin\Theta}{\sin\Theta_0}\Big)^2-1} \tag{6.14}$$

$$\kappa_2 > 0 \qquad \frac{\cos\Theta_0}{\cos\Theta} < 0 \Rightarrow \frac{\cos\Theta_0}{\cos\Theta} < 3\cos(\Theta-\Theta_0) \tag{6.15}$$

$$\sin\Theta > \sin\Theta_0 \Rightarrow \frac{\pi}{2} < \Theta < \pi-\Theta_0 \qquad \text{and} \qquad \tan\Theta < \xi_2 \tag{6.16}$$

$$m_0 = \frac{1}{\sqrt{\eta(\kappa_2-1)}} \qquad \kappa_2 > 1 \tag{6.17}$$

It is interesting that in the presented approximation both Eqs (6.7) and (6.13) are independent of the material constants and incident shock strength and they have the *pure geometrical meaning* only.

7. Numerical calculations

The reflection solutions discussed in Sections 5.1, 5.2 and 6 are examined numerically for the material (3.9) with constants $C_1 = 0.352$ MPa, $C_2 = 0$, $C_3 = 0.023$ MPa (cf Zahorski (1962)). Since the material region behind the incident shock should remain elastic, the discontinuity jumps can not be arbitrary, and the appropriate estimates for the shock strength should be established. In this paper we use the estimation for $m_0 \approx 4$. Comparison of the Fig.4a with Eq (6.10) shows that the possible angles of reflection Θ appear in the segment AB smaller then $\pi/2 \leq \Theta \leq \pi$. The function $\xi_1^* = \xi_1(\pi - \Theta_0)$ is depicted here for convenience only. Both conditions (6.10) are satisfied in the region marked with dots only.

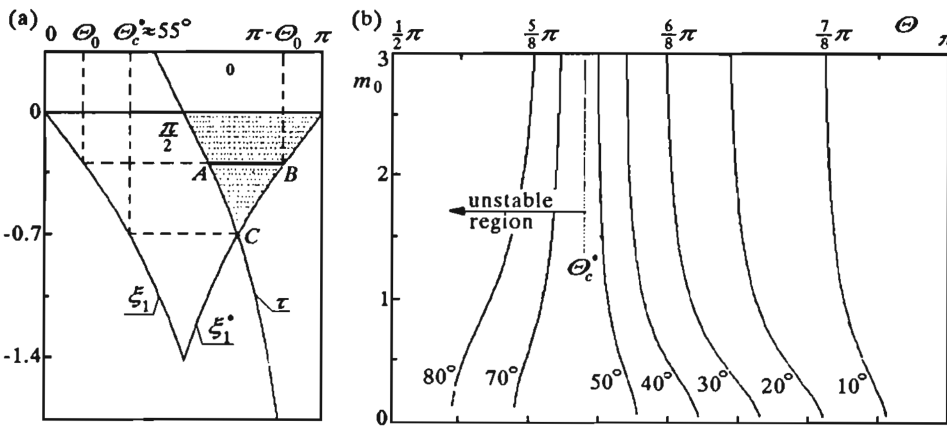

Fig. 4. Clamped boundary conditions. Reflected shock wave

If the angle of incidence exceeds certain critical value $\Theta_0 = \Theta_c^* \approx 55°$ (point C) both segments for Θ which correspond with Eqs (6.10) are disconnected. For this reason Θ_c^* is the critical angle for the incident shock wave and it is *independent of the incident shock wave strength* and *constitutive relations.* This stands in contrast with the reflection pattern in the form of a simple wave (4.6), for which the critical angle depends on m_0 and η (see Fig.6). Fig.4b presents for comparison all contour lines of expression (6.11) obtained for $\Theta_0 \in \langle 1o°, 90° \rangle$ (also for completness in the range larger then addmited), but the reflected shock is stable for $\Theta_0 \leq \Theta_c^*$ only. The contour lines do not tend to infinity, but for $m_0 \approx 10$ they take their finite values a little higher above than it is depicted in Fig.4b.

The results for frictionless-rigid boundary are shown in Fig.5. The three

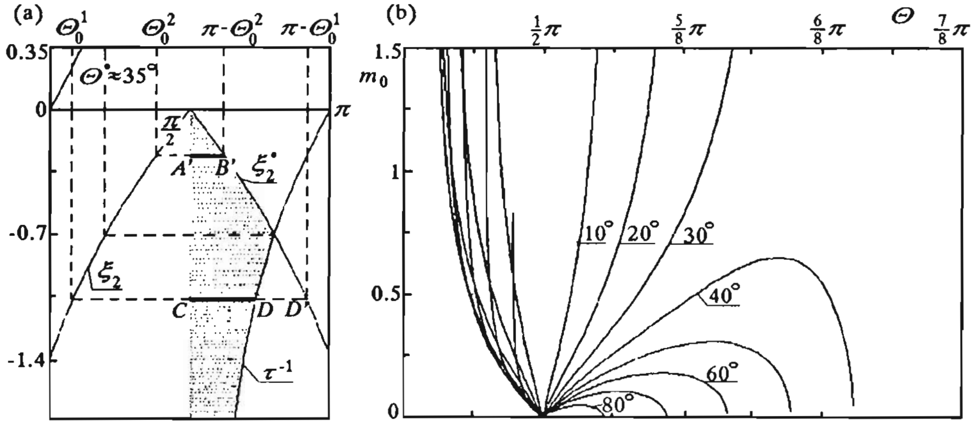

Fig. 5. Frictionless-rigid boundary. Reflected shock wave

functions τ^{-1}, ξ_2 and for the simplicity only, the function $\xi_2^* = \xi_2(\pi - \Theta_0)$ are plotted in Fig.5a. It is easy to see that for small angles of incidence $\Theta_0 < \Theta^* \approx 35°$ the segments for the reflection angles are smaller than segment CD' which equals $\pi/2 \leq \Theta \leq \pi$, if the incident angle goes beyond Θ^* the full segment $\pi/2 \leq \Theta \leq \pi$ for the angles of reflection can be taken into consideration. Figure 5b presents the contour lines for expression (6.16), they are originated by cutting the surface (6.16) with the planes Θ_0 =const (see Fig.7). All contour lines for incidence angles $\Theta_0 = 10°, 20°, ..., 90°$ are shown in Fig.5b there is any critical angle in this case. Both conditions (6.15) are satisfied in the dotted region. If the angle of incidence increases from zero, the length of the segment for admitted angles of reflection increases, if Θ_0 goes over Θ^* it decreases to zero.

Fig.7 presents the full 3D graph for Eq (6.16). Relation between the critical angle and the incident shock wave strength is presented in Fig.6. for the reflected simple wave Θ_c and for the shock wave Θ_c^* in the case of clamped boundary.

It is very important to prove the uniqueness of the reflection pattern for the material constants: C_1, C_3 (cf Zahorski (1962)) used here. Examining Eqs (5.3) and (5.6) we can rule out the possibility that reflected wave can be *simultaneously* stable shock or simple wave. The solutions, however, can be proved only numerically. According to the previous remarks for the simple wave (see page 5) $\tau(\gamma)$ must be a real monotonic *decreasing* function of γ, when γ changes from 0 to its extreme value $\tilde{\gamma}$ (which in both cases here $(5.4)_1$ and $(5.7)_1$ is *negative* (monotone decreasing γ). The special attention

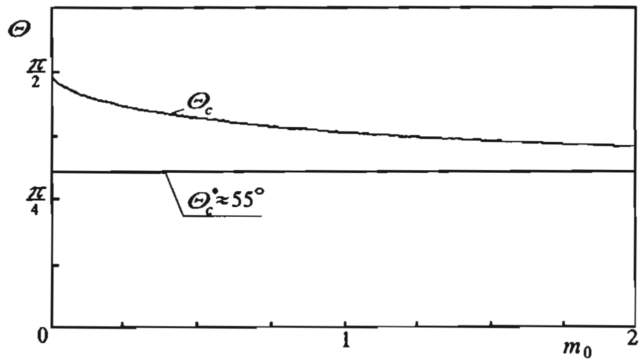

Fig. 6. Critical angles of incidence versus of m_0

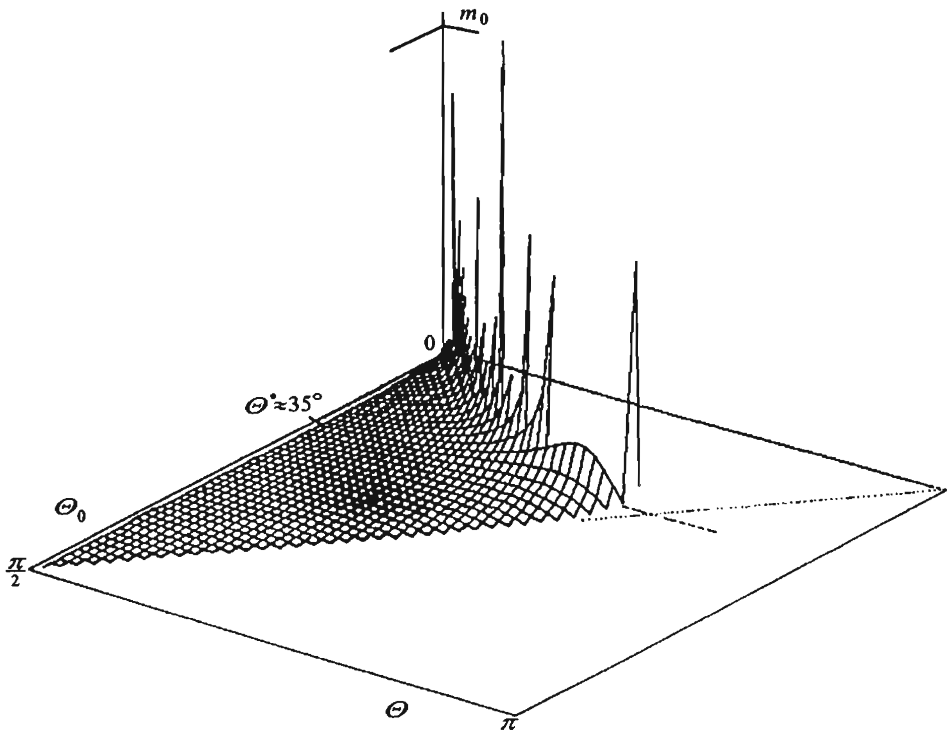

Fig. 7. Incident shock wave strength m_0 versus the incident angle Θ_0 and admitted reflection angle Θ

should be called this fact, because such function $\tau(\gamma)$ is in fact monotonic *increasing* with *increasing* γ (from $\tilde{\gamma}$ to 0). Geometrically it means that, if $\tau(\gamma)$ increases (decreases) with γ (changing the other way round from $\tilde{\gamma}$ to 0) the travelling pencil of wavelets diverges from (converges to) the leading wavelet thus forming a simple (shock) wave. Analysing and differentiating $(5.3)_3$ we obtain (for increasing γ)

$$\begin{aligned} &\text{simple wave} \quad \frac{d\tau}{d\gamma} > 0 \Rightarrow -V_h \frac{d^2 x_{32}}{d\gamma^2} > 0 \Rightarrow \frac{d^2 x_{32}}{d\gamma^2} < 0 \\ &\text{shock wave} \quad \frac{d\tau}{d\gamma} < 0 \Rightarrow \frac{d^2 x_{32}}{d\gamma^2} > 0 \end{aligned} \tag{7.1}$$

Similarly for the frictionless-rigid boundary from Eq $(5.6)_1$ appears

$$\begin{aligned} &\text{simple wave} \quad \frac{d\tau}{d\gamma} > 0 \Rightarrow -\tau^2 \frac{d^2 x_{31}}{d\gamma^2} > 0 \Rightarrow \frac{d^2 x_{31}}{d\gamma^2} < 0 \\ &\text{shock wave} \quad \frac{d\tau}{d\gamma} < 0 \Rightarrow \frac{d^2 x_{31}}{d\gamma^2} > 0 \end{aligned} \tag{7.2}$$

Fig.8 shows the regions for the stable reflected shock or reflected simple wave for both the types of boundary conditions. The below curve in Fig.8b follows directly from the Fig.5b and indicates the limit of strength m_0 at the fixed incident angle Θ_0 for the stable reflected shock. We choose now three points A, B, C in Fig.8a with common coordinate $m_0 = 1$ and different coordinates for Θ_0 equal to $60°$, $45°$ and $30°$ respectively. According to the previous interpretation of Fig.6, in the case of the clamped boundary the points B, C are characteristic for the stable reflected shock and point A for the reflected simple wave. The Runge-Kutta method is used to solve the initial problem with data for $x_{31}(0)$, $x_{32}(0)$ at points A, B, C. For the numeric data matrix obtained in such a way, we use two procedures which approximate it to the first and second derivatives, respectively. The second derivative at the points B, C has the sign compatible with Eq (7.1) (see Fig.10). Another situation is at the point A, x''_{32} changes the sign with changing γ, this means that the so called *composite* reflected wave is created (first $x''_{32}(0) < 0$ also simple wave is formed with the trailing wavelet in the form of the shock wave $x''_{32}(\tilde{\gamma}) > 0$ (see Fig.10)). In the case of the frictionless-rigid boundary the sign of the second derivative is also consistent with (7.2). Such analysis have been made for many others points, always with *unique* result. The typical diagrams for x_{32}, x_{31}, x''_{32} and clamped boundary are presented in Fig.9 and Fig.10.

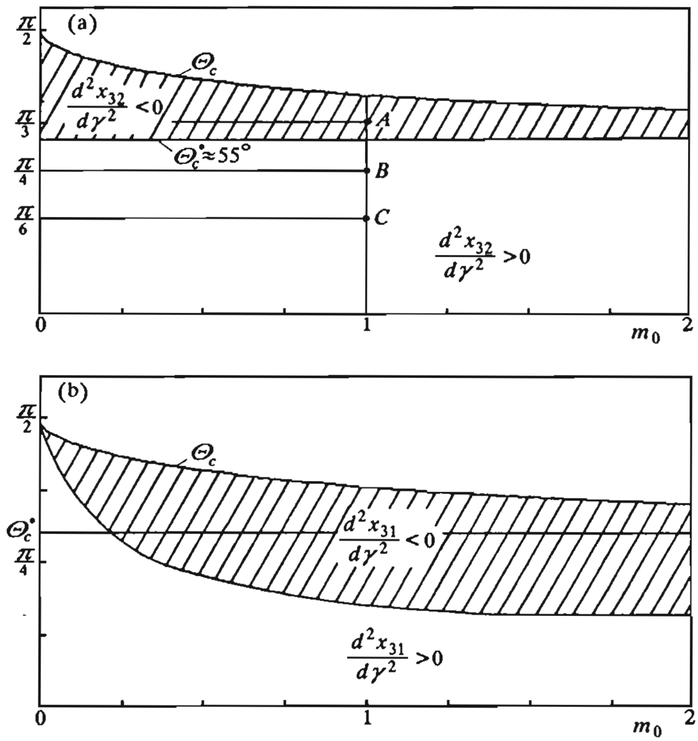

Fig. 8. Admissible regions for stable reflected shock or simple wave (a) clamped and (b) frictionless-rigid boundary

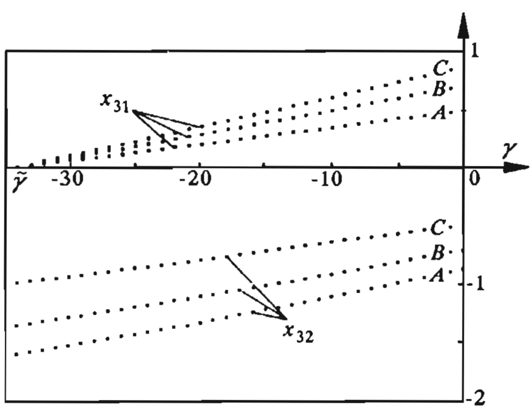

Fig. 9. Components of the deformation gradient at the points A, B, C

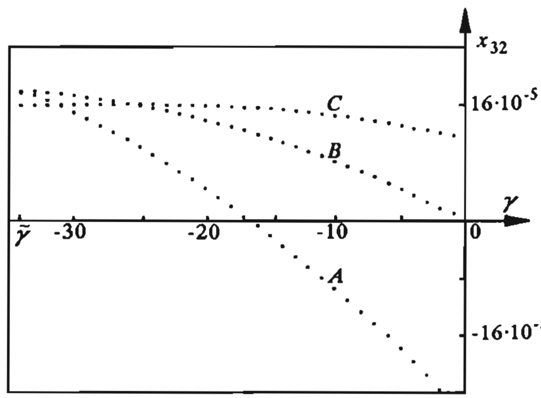

Fig. 10. Second derivative $x''_{32}(\gamma)$ at the points A, B, C

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Fale wywołane przez padającą falę uderzeniową w pewnym szczególnym materiale hipersprężystym

Streszczenie

Płaska poprzeczna fala uderzeniowa propaguje się w nieskończonym ośrodku sprężystym i ulega odbiciu od płaskiego brzegu lub odbiciu i załamaniu na płaszczyźnie rozdziału dwóch połączonych ze sobą półprzestrzeni sprężystych. W przypadku ogólnym zagadnienie nie ma rozwiązania, poszukiwane są warunki szczególne dla jego istnienia. Rozpatrzono szczegółowo oraz wykonano obliczenia numeryczne dla dwóch typów warunków brzegowych: utwierdzenia oraz dla mieszanych warunków brzegowych. Odbita fala prosta lub uderzeniowa jest rozwiązaniem jednoznacznym w swoim obszarze stateczności. Występują dwa różne kąty krytyczne związane z różnymi postaciami rozwiązania, przy czym dla odbitej fali uderzeniowej wyrażenie dla kąta krytycznego ma charakter typowo geometryczny.

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