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ON THE ELASTODYNAMICS OF THIN MICROPERIODIC PLATES[1]

JAROSŁAW JĘDRYSIAK

Department of Structural Mechanics
University of Technology, Łódź

CZESŁAW WOŹNIAK

Institute of Fundamental Technological Research, Warsaw

The aim of this paper is to propose and investigate a new modelling approach to thin elastic plates having micro-periodic structure in planes parallel to the midplane. The main feature of this approach is that it describes the effect of the microstructure length dimensions on the macro-behaviour of the plate. This effect is neglected in the asymptotic homogenization approaches leading to the known effective stiffness models. It is shown that in micro-dynamic problems the aforementioned length scale effect plays a crucial role and cannot be neglected.

1. Introduction

The composite plates having a micro-periodic structure in their midplanes are usually described using homogenized models. These models from a formal point of view represent certain homogeneous plate structures with constant effective stiffnesses and averaged mass densities; the pertinent modelling approaches were investigated by Duvaut and Metellus (1976), Caillerie (1984), Kohn and Vogelius (1984), Matysiak and Nagórko (1989), Lewiński (1991) and others. In this paper it will be shown that these effective stiffness plate theories are not able to describe some important features of the dynamic plate

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behaviour. To this end we propose a certain new refined theory of periodically structured thin elastic plates, which takes into account the microstructure length scale effects on the dynamic plate response and hence is able to describe both macro- and micro-dynamic plate behaviour. The approach is based on the refined method of modelling for micro-periodic elastic materials and structures, given by Woźniak (1993), Woźniak et al. (1993) and developed in a series of related contributions (cf the papers by Wierzbicki and M.Woźniak in this issue).

The considerations will be based on the Kirchhoff plate theory assumptions and carried out within a framework of the linear elasticity theory. An alternative refined model of a plate, in which the Reissner-Hencky – type hypotheses are used, will be presented separately in the forthcoming paper by Baron and Woźniak.

Denotations. Throughout the paper subscripts $\alpha, \beta, ...$ run over $1,2$ being related to the orthogonal cartesian coordinates x_1, x_2 parametrizing the plate midplane. Non-tensorial superscripts $a, b, ...$ run over $1, ..., n$ and are related to the postulated a priori micro-shape functions, the meaning of which will be explained in the subsequent section. Summation convention holds for all aforementioned indices. Setting $\boldsymbol{x} \equiv (x_1, x_2)$ and denoting by z the cartesian coordinate in the direction normal to the midplane, we assume that the plate under consideration in its underformed configuration occupies the region $\Omega := \{(\boldsymbol{x}, z) : -h(\boldsymbol{x}) < z < h(\boldsymbol{x}),\ \boldsymbol{x} \in \Pi\}$, where Π is the region of midplane and $2h(\boldsymbol{x})$ is the plate thickness at a point $\boldsymbol{x} \in \Pi$. Denoting by $A := (0, l_1) \times (0, l_2)$ the region on $0x_1x_2$ plane, where l_1, l_2 are the length dimensions sufficiently small compared to the minimum characteristic length dimension of Π, we shall assume that $h(\cdot)$ is the A-periodic function of $\boldsymbol{x}$. All material and inertial properties of the plate are also A-periodic functions of $\boldsymbol{x}$. For an arbitrary integrable A-periodic function $f(\cdot)$ we shall denote by

$$
< f >= \frac{1}{l_1 l_2} \int\limits_A f(\boldsymbol{x})\, da \qquad\qquad da \equiv dx_1 dx_2
$$

its averaged (constant) value. Moreover, the time coordinate will be denoted by τ. Setting $x_3 = z$ we introduce the cartesian coordinate system $0x_1x_2x_3$ in the physical space; subscripts i, j related to this system run over $1, 2, 3$.

2. General formulation of the Kirchhoff plate theory

The starting point of analysis is a general formulation of the Kirchhoff plate equations. Let u_i, ε_{ij}, σ_{ij} stand for displacements, strains and stresses, p^+, p^- be tractions (in the x_3-axis direction) on upper and lower plate boundaries, respectively, t be tractions on the plate cross-sections along the boundary $\partial\Pi$ of the midplane Π and b be the constant body force (in the x_3-axis direction). Moreover, by $\rho = \rho(\boldsymbol{x}, z)$ and $C_{ijkl} = C_{ijkl}(\boldsymbol{x}, z)$ we denote mass density and elastic modulae of the plate material and assume that z =const are material symmetry planes; at the same time $\rho(\cdot)$ and $C_{ijkl}(\cdot)$ are assumed to be even functions of z and A-periodic functions of $\boldsymbol{x}$.

The formulation of the linear elastic plate theory is given by the strain-displacement equations

$$\varepsilon_{ij} = u_{(i,j)} \tag{2.1}$$

by the stress-strain relations for $\sigma_{\alpha\beta}$ restricted by the plain strain assumption $\sigma_{33} = 0$ and hence given by

$$\sigma_{\alpha\beta} = D_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta} \tag{2.2}$$

where $D_{\alpha\beta\gamma\delta} := C_{\alpha\beta\gamma\delta} - C_{\alpha\beta33}C_{\gamma\delta33}(C_{3333})^{-1}$, and by the virtual work principle

$$\begin{aligned}\int\limits_{\Pi}\int\limits_{-h}^{h}\left(\sigma_{\alpha\beta}\delta u_{(\alpha,\beta)} + 2\sigma_{\alpha3}\delta u_{(\alpha,3)}\right) dz da = \int\limits_{\Pi}\left(p^+\delta u_3\Big|_h + p^-\delta u_3\Big|_{-h}\right) da + \\ +b\int\limits_{\Pi}\int\limits_{-h}^{h}\rho\delta u_3\, dz da - \int\limits_{\Pi}\int\limits_{-h}^{h}\rho\ddot{u}_i\delta u_i\, dz da + \oint\limits_{\partial\Pi}\int\limits_{-h}^{h} t\delta u_3\, dz ds\end{aligned} \tag{2.3}$$

which has to be satisfied for every admissible virtual displacement field δu_i.

Moreover, within the framework of the Kirchhoff kinematic hypothesis we assume that

$$u_3 = u_3(\boldsymbol{x}, \tau) \qquad u_\alpha = -z u_{3,\alpha}(\boldsymbol{x}, \tau) \tag{2.4}$$

and hence $\delta u_3 = \delta u_3(\boldsymbol{x})$, $\delta u_\alpha = -z\delta u_{3,\alpha}(\boldsymbol{x})$ in Eq (2.3). As it is known, Eqs (2.1) $\div$ (2.4) lead to the Kirchhoff plate theory in which $u_3(\boldsymbol{x}, \tau)$ has to satisfy the known partial differential equation of the fourth order. However, for the micro-periodic plates under consideration, this equation involves highly oscillating coefficients which are A-periodic functions of $\boldsymbol{x} = (x_1, x_2)$. Equations of this form do not constitute the proper analytical basis for a computational analysis of special problems. That is why different homogenization macro-modelling approaches have been proposed in order to approximate the problem and describe the behaviour of the periodic heterogeneous plates in terms of partial differential equations with constant coefficients. The known macro-modelling approaches are often based on the scaling the microstructure down by the formal asymptotic assumption that $l_1 \to 0$ and $l_2 \to 0$ but l_1/l_2 holds constant. Hence the resulting equations neglect the effect of the size of microstructure (described by the length parameters l_1, l_2) on the plate behaviour. In order to retain this effect we shall propose the alternative approach which will be referred to as the refined macro-modelling of the plates under consideration.

3. Refined macro-modelling of Kirchhoff plates

The refined macro-modelling of micro-periodic composite materials and structures takes into account two auxiliary concepts, cf Woźniak (1993). The first is the concept of a macro-function, related to the microstructure length parameter l (in the problem under consideration we can assume that $l = \sqrt{(l_1)^2 + (l_2)^2}$) and to a certain numerical accuracy parameter. Function F, defined on Π (which can also depend on τ), will be called *macro-function* if for every $\boldsymbol{x}, \boldsymbol{y} \in \Pi$ condition $\|\boldsymbol{x} - \boldsymbol{y}\| < l$ implies $|F(\boldsymbol{x}) - F(\boldsymbol{y})| < \lambda_F$, where λ_F is a numerical accuracy parameter related to F. If F is a regular function and the similar conditions (with the pertinent numerical accuracy parameters) hold also for all derivatives of F (including time derivatives provided that F depends also on time), then F will be called the regular macro-function. Generally speaking, by the regular macro-function (related to a certain micro-periodic structure and suitable numerical accuracy parameters) we understand the function which together with all its derivatives in an arbitrary but fixed periodicity cell $A(\boldsymbol{x})$, $A(\boldsymbol{x}) := \boldsymbol{x} + A$, $A(\boldsymbol{x}) \subset \Pi$, suffers oscillations which from the computational viewpoint can be neglected. It is easy to see that the macro-description of micro-periodic materials and structures have to be realized by means of macro-functions.

The second auxiliary concept in the refined macro-modelling is the micro-shape function system. It is a system of n linear-independent continuous functions $g^a(\boldsymbol{x})$ which have continuous first and second order derivatives, are A-periodic and satisfy the conditions: $< g^a{}_{,\alpha} >= 0$, $g^a(\boldsymbol{x}) \in \mathcal{O}(l^2)$, $g^a{}_{,\alpha}(\boldsymbol{x}) \in \mathcal{O}(l)$, and the values of the second derivatives of g^a are independent of l. Moreover, every linear combination of *micro-shape* functions in an arbitrary but fixed periodicity cell $A(\boldsymbol{x})$, $A(\boldsymbol{x}) \subset \Pi$, has to describe expected disturbances of the plate deflections $u_3(\boldsymbol{y}, \tau)$, $\boldsymbol{y} \in A(\boldsymbol{x})$, caused by the plate inhomogeneity. Hence the choice of micro-shape functions depends on the problem under consideration and accuracy of modelling. As a simple example of these functions we can take n functions of the form $l^2 \sin(p\pi x_1/l_1) \sin(r\pi x_2/l_2)$, where p, r are positive integers.

The refined macro-modelling approach to the Kirchhoff plate theory will be based on Eqs (2.1) ÷ (2.4) and on the following three hypotheses:

- *Macro-Kinematic Hypothesis.* The deflections $u_3(\boldsymbol{x},\tau)$, $\boldsymbol{x} \in \Pi$, of the Kirchhoff plate with micro-periodic structure can be assumed in the form

$$u_3(\boldsymbol{x},\tau) = w(\boldsymbol{x},\tau) + g^a(\boldsymbol{x})q^a(\boldsymbol{x},\tau) \tag{3.1}$$

  where $g^a(\cdot)$ are postulated a priori micro-shape functions and $w(\cdot,\tau)$, $q^a(\cdot,\tau)$ are arbitrary linear-independent macro-functions.

- *Virtual Work Hypothesis.* The principle of virtual work (2.3), where $\delta u_3 = \delta u_3(\boldsymbol{x})$, $\delta u_\alpha = -z\delta u_{3,\alpha}(\boldsymbol{x})$, holds for

$$\delta u_3 = \delta w(\boldsymbol{x}) + g^a(\boldsymbol{x})\delta q^a(\boldsymbol{x}) \tag{3.2}$$

  where $\delta w(\cdot), \delta q^a(\cdot)$ are arbitrary regular and linear-independent macro-functions.

- *Macro-Modelling Approximation.* In Eq (2.3) combined with Eqs (2.1), (2.2), (2.4), (3.1) and (3.2) terms $\mathcal{O}(\lambda)$ can be neglected, where λ stands for the pertinent numerical accuracy parameter related to macro-functions w, q^a, δw, δq^a and their derivatives.

Macro-functions w, q^a represent the new basis unknown kinematic fields of the refined theory of micro-periodic plates and are called macro-deflections and microstructural variables (or inhomogeneity correctors), respectively. The term $g^a(\boldsymbol{x})q^a(\boldsymbol{x},\tau)$ describes the expected disturbances in the plate deflections caused by the micro-periodic structure of the plate subjected to time dependent loadings or to free vibrations. For more detailed discussion of the refined method of macro-modelling the reader is referred to Woźniak (1993) and Woźniak et al. (1993).

It can be shown that Eqs (2.1) ÷ (2.4) combined with the three aforementioned modelling hypotheses lead to the system of equations in w and q^a. Setting aside rather lengthy calculations we shall confine ourselves to the final equations representing the proposed refined macro-theory of Kirchhoff plates with micro-periodic structure. It has to be emphasized that this theory has a physical meaning only for micro-periodic plates in which stresses and deformations can be described (with a sufficient accuracy) by the Kirchhoff assumptions (2.2) and (2.4), respectively. Assumptions of this form may not hold for the plates with rapidly varying discontinuous thickness.

4. Refined theory

In order to write down governing equations of the refined macro-theory of plates with periodic structure we shall introduce the following A-periodic functions

$$M(\boldsymbol{x}) \equiv \int\limits_{-h(x)}^{h(x)} \rho(\boldsymbol{x},z)\,dz \qquad\qquad B_{\alpha\beta\gamma\delta}(\boldsymbol{x}) \equiv \int\limits_{-h(x)}^{h(x)} z^2 D_{\alpha\beta\gamma\delta}(\boldsymbol{x},z)\,dz$$

For the sake of simplicity we shall also neglect the rotational inertia terms involving the A-periodic function

$$J(\boldsymbol{x}) \equiv \int\limits_{-h(x)}^{h(x)} z^2 \rho(\boldsymbol{x},z)\,dz$$

Under this approximation, the macro-modelling procedure based on the assumptions formulated in sections 2 and 3 yields the following system of equations in macrodeflections $w(\boldsymbol{x},\tau)$ and microstructural variables $q^a(\boldsymbol{x},\tau)$

$$\begin{aligned}
&< B_{\alpha\beta\gamma\delta} > w_{,\alpha\beta\gamma\delta} + < B_{\alpha\beta\gamma\delta} g^a{}_{,\gamma\delta} > q^a{}_{,\alpha\beta} + < M > \ddot{w} + \underline{< M g^a >} \ddot{q}^a = \\
&= < p > + b < M > \\
&< B_{\alpha\beta\gamma\delta} g^a{}_{,\alpha\beta} > w_{,\gamma\delta} + < B_{\alpha\beta\gamma\delta} g^a{}_{,\gamma\delta}\, g^b{}_{,\alpha\beta} > q^b + \underline{< M g^a >} \ddot{w} + \\
&+ \underline{< M g^a g^b >} \ddot{q}^b = \underline{< p g^a >} + b \underline{< M g^a >}
\end{aligned} \tag{4.1}$$

where $p := p^+ + p^-$.

Thus, we have arrived at the system of $n+1$ differential equations with constant coefficients. Hence Eqs (4.1) can be used as a basis for computational analysis of the micro-periodic plates under consideration. The values of underlined terms in Eqs (4.1) depend on the size of the microstructure (on the microstructure length parameter l) and hence describe the scale length effect on the plate behaviour. The characteristic feature of the obtained result is the fact that equations for microstructural variables q^a, $a = 1, ..., n$, are ordinary differential equations involving exclusively time derivatives of q^a. It means that the microstructural variables are independent of the boundary conditions and Eqs (4.1) have to be considered together with two boundary conditions for macrodeflections w, two initial conditions for w and two initial conditions for every microstructural variable q^a. It has to be emphasized that solutions to the pertinent boundary-initial value problems have a physical sense only if w and q^a are regular macro-functions.

At the end of this section let us consider a thin plate made of a homogeneous isotropic material and having the A-periodic thickness. In this case

$$B_{\alpha\beta\gamma\delta} = \frac{h^3}{12}\Big(\delta_{\alpha\gamma}\delta_{\beta\delta}\frac{E}{1+\nu} + \delta_{\alpha\delta}\delta_{\beta\gamma}\frac{\nu E}{1-\nu^2}\Big)$$

and under denotations

$$B \equiv \frac{Eh^2}{12(1-\nu^2)}$$

$$D^a_{\alpha\beta} \equiv < B(1-\nu)g^a{}_{,\alpha\beta} + \delta_{\alpha\beta}\nu B g^a{}_{,\gamma\gamma} >$$

$$D \equiv < B(1-\nu)g^a{}_{,\alpha\beta}\, g^a{}_{,\alpha\beta} + \nu B g^a{}_{,\alpha\alpha}\, g^a{}_{,\beta\beta} >$$

the system of governing equations (4.1) will take the form

$$\begin{gathered} < B > w_{,\alpha\alpha\beta\beta} + D^a_{\alpha\beta} q^a{}_{,\alpha\beta} + < M > \ddot{w} + \underline{< Mg^a >\ddot{q}^a} = < p > + b < M > \\ D^{ab}q^b + D^a_{\alpha\beta} w_{,\alpha\beta} + \underline{< Mg^a g^b >\ddot{q}^b} + \underline{< Mg^a >\ddot{w}} = \underline{< pg^a >} + b\underline{< Mg^a >} \end{gathered} \tag{4.2}$$

where the underlined terms have the same meaning as those in Eq (4.1).

5. Effective stiffness theories

Scaling the microstructure down by setting $l \to 0$, we arrive at the asymptotic approximation of the refined theory. Neglecting the underlined terms in Eqs (4.1) we obtain for q^a the system of linear algebraic equations

$$< B_{\alpha\beta\gamma\delta} g^a{}_{,\gamma\delta}\, q^b{}_{,\alpha\beta} > q^b = - < B_{\alpha\beta\gamma\delta} q^a{}_{,\alpha\beta} > w_{,\gamma\delta} \tag{5.1}$$

It can be shown that the $n \times n$ matrix of elements $< B_{\alpha\beta\gamma\delta} g^a{}_{,\gamma\delta}\, q^b{}_{,\alpha\beta} >$ is non-singular. Denoting by G^{ab} elements of the inverse matrix we can eliminate microstructural variables from the governing equations by means of

$$q^a = -G^{ab} < B_{\alpha\beta\gamma\delta} q^b{}_{,\alpha\beta} > w_{,\gamma\delta} \tag{5.2}$$

Setting

$$B^{\text{eff}}_{\alpha\beta\gamma\delta} \equiv < B_{\alpha\beta\gamma\delta} > - < B_{\alpha\beta\mu\nu} g^a{}_{,\mu\nu} > G^{ab} < B_{\gamma\delta\pi\rho} g^b{}_{,\pi\rho} > \tag{5.3}$$

we arrive at the following equation

$$B^{\rm eff}_{\alpha\beta\gamma\delta} w_{,\alpha\beta\gamma\delta} + <M> \ddot{w} = <p> + b <M> \tag{5.4}$$

where $B^{\rm eff}_{\alpha\beta\gamma\delta}$ are called the effective stiffnesses of the micro-periodic plate under consideration. Let us observe that for a plate homogeneous in $0x_1x_2$ plane and having constant thickness we obtain $B_{\alpha\beta\gamma\delta}$=const, and by means of $<q^a{}_{,\alpha\beta}>= 0$, from Eq (5.2) it follows that $q^a = 0$, $a = 1,...,n$. Thus we conclude that the micro structural variables describe the effect of the inhomogeneous periodic structure on the plate behaviour.

Theories of micro-periodic Kirchhoff plates governed by Eq (5.4) are called the effective stiffness plate theories. The formulae for the effective stiffnesses $B^{\rm eff}_{\alpha\beta\gamma\delta}$ can be obtained using different procedures (cf references mentioned in the Introduction) independently of the asymptotic approximation of Eqs (4.1) leading to Eq (5.3). The effective stiffness theories neglect the effect of the size of the unit cell A on the macro-behaviour of the body, being independent of the microstructure length parameter l. Let us observe that for quasi-stationary problems we can also eliminate variables q^a from Eqs (4.1) and arrive at the governing equation in w which has the form similar to that of Eq (5.4) with the extra term of the order $O(l^2)$ on the right-hand side of this equation. It follows that for quasi-stationary problems we can use the effective stiffness theory represented by Eq (5.4) instead of the refined theory governed by Eqs (4.1). For dynamic problem the situation is different; to show this fact we shall consider in the subsequent section a certain special problem.

At the end of the section we shall pass to the effective stiffness theory for isotropic homogeneous plates with A-periodic thickness. The pertinent governing equations can be derived directly from Eqs (4.2) by neglecting the underlined terms. Denoting by G^{ab} elements of $n \times n$ matrix which is inverse to the matrix of elements D^{ab}, we obtain $q^a = -G^{ab}D^a_{\alpha\beta}w_{,\alpha\beta}$ and after denotation

$$B^{\rm eff}_{\alpha\beta\gamma\delta} \equiv <B> \delta_{\alpha\beta}\delta_{\gamma\delta} - D^a_{\alpha\beta}G^{ab}D^b_{\gamma\delta}$$

we arrive at the governing equation, the form of which coincides with that of Eq (5.4).

6. Refined versus effective stiffness theories

In order to compare the scope of applicability of the refined and effective stiffness plate theories we shall analyse vibrations of a simply supported rectangular plate made of an isotropic homogeneous material and having the A-periodic thickness. It will be assumed that the unit plate element (based on the unit cell $A \equiv (0,l_1) \times (0,l_2)$) has two symmetry planes: $x_1 = l_1/2$ and $x_2 = l_2/2$. For the sake of simplicity we confine ourselves to one micro-shape function $g = g^1 = l^2[\cos(2\pi x_1/l_1)\cos(2\pi x_2/l_2) + c]$, where the constant c is defined by the condition $< Mg >= 0$. Denoting $\Pi \equiv (0,L_1) \times (0,L_2)$, $k_1 \equiv 2\pi/L_1$, $k_2 \equiv 2\pi/L_2$ we shall assume that $p = p_0 \sin(k_1x_1)\sin(k_2x_2)\cos(\breve{\omega}\tau)$, where p_0 is an arbitrary constant, $p_0 \neq 0$. At the same time the effect of the body forces on the plate vibrations will be neglected. Since $l_1 \ll L_1$, $l_2 \ll L_2$, then p can be treated as a macro-function and the term $< pg >$ on the right-hand side of Eqs (4.2) will be neglected. Hence, setting $q \equiv q^1$, $D_{\alpha\beta} \equiv D^1_{\alpha\beta}$, $D \equiv D^{11}$, from Eqs (4.2) we obtain

$$\begin{gathered} < B > w_{,\alpha\alpha\beta\beta} + D_{\alpha\beta}q_{,\alpha\beta} + < M > \ddot{w} = p \\ Dq + D_{\alpha\beta}w_{,\alpha\beta} + < M(g)^2 > \ddot{q} = 0 \end{gathered} \tag{6.1}$$

with $p = p_0 \sin(k_1x_1)\sin(k_2x_2)\cos(\breve{\omega}\tau)$.

Let us observe that now $D_{12} = D_{21} = 0$. Solution to Eqs (6.1), satisfying boundary conditions for the simply supported plate, can be assumed in the form

$$\begin{aligned} w(x_1,x_2,\tau) &= A_w \sin(k_1x_1)\sin(k_2x_2)\cos(\breve{\omega}\tau) \\ q(x_1,x_2,\tau) &= A_q \sin(k_1x_1)\sin(k_2x_2)\cos(\breve{\omega}\tau) \end{aligned} \tag{6.2}$$

where A_w, A_q are vibration amplitudes, $A_wA_q \neq 0$. Substituting the right-hand sides of Eqs (6.2) into Eqs (6.1) we obtain the system of two linear algebraic equations for A_w, A_q

$$\begin{bmatrix} < B > (k_1^2 + k_2^2)^2 - < M > \breve{\omega}^2 & -\Big(D_{11}k_1^2 + D_{22}k_2^2 + < Mg > \breve{\omega}^2\Big) \\ -\Big(D_{11}k_1^2 + D_{22}k_2^2 + < Mg > \breve{\omega}^2\Big) & D - < M(g)^2 > \breve{\omega}^2 \end{bmatrix} \cdot \tag{6.3}$$

$$\cdot \begin{bmatrix} A_w \\ A_q \end{bmatrix} = \begin{bmatrix} p_0 \\ 0 \end{bmatrix}$$

Under denotations

$$\begin{aligned} \lambda_1^2 &= \frac{1}{2}\Big[\frac{D}{< M(g)^2 >} + \frac{< B >}{< M >}(k_1^2 + k_2^2)^2\Big] + \\ &- \frac{1}{2}\sqrt{\Big[\frac{D}{< M(g)^2 >} - \frac{< B >}{< M >}(k_1^2 + k_2^2)^2\Big]^2 + 4\frac{(D_{11}k_1^2 + D_{22}k_2^2)^2}{< M >< M(g)^2 >}} \end{aligned}$$

$$\lambda_2^2 = \frac{1}{2}\Big[\frac{D}{<M(g)^2>} + \frac{<B>}{<M>}(k_1^2+k_2^2)^2\Big] + \\ + \frac{1}{2}\sqrt{\Big[\frac{D}{<M(g)^2>} - \frac{<B>}{<M>}(k_1^2+k_2^2)^2\Big]^2 + 4\frac{(D_{11}k_1^2+D_{22}k_2^2)^2}{<M><M(g)^2>}} \tag{6.4}$$

the solution to Eqs (6.3) can be written down in the form

$$\begin{aligned} A_w &= \frac{p_0[D- <M(g)^2> \breve{\omega}^2]}{<M><M(g)^2> (\breve{\omega}^2-\lambda_1^2)(\breve{\omega}^2-\lambda_2^2)} \\ A_q &= \frac{p_0(D_{11}k_1^2+D_{22}k_2^2)}{<M><M(g)^2> (\breve{\omega}^2-\lambda_1^2)(\breve{\omega}^2-\lambda_2^2)} \end{aligned} \tag{6.5}$$

Substituting the formulae for amplitudes A_w, A_q into Eqs (6.2) we obtain the solution to the problem under consideration. From Eqs (6.5) it can be easily seen that λ_1, λ_2 are lower and higher resonance frequencies, respectively, for the vibration problem under consideration.

The above analysis was carried out within a framework of the refined theory. On passing to the effective stiffness theory we shall use Eq (5.4), where now

$$B^{\text{eff}}_{\alpha\beta\gamma\delta} = <B> \delta_{\alpha\beta}\delta_{\gamma\delta} - \frac{1}{D}D_{\alpha\beta}D_{\gamma\delta} \tag{6.6}$$

After neglecting the body forces as well as assuming that $<p> \cong p$ we obtain

$$B^{\text{eff}}_{\alpha\beta\gamma\delta}w_{,\alpha\beta\gamma\delta} + <M> \ddot{w} = p \tag{6.7}$$

where $p = p_0\sin(2\pi x_1/L_1)\sin(2\pi x_2/L_2)\cos(\breve{\omega}\tau)$.

The solution to the pertinent boundary value problem can be assumed in the form

$$w(x_1,x_2,\tau) = A_0\sin(k_1x_1)\sin(k_2x_2)\cos(\breve{\omega}\tau) \tag{6.8}$$

where A_0 is the vibration amplitude. Substituting the right-hand side of Eq (6.8) into Eq (6.7) and bearing in mind Eq (6.6), after denotation

$$\lambda^2 \equiv \frac{<B>}{<M>}(k_1^2+k_2^2)^2 - \frac{1}{<M>D}(D_{11}k_1^2+D_{22}k_2^2)^2 \tag{6.9}$$

we obtain the following well known formula for the vibration amplitude

$$A_0 = \frac{p_0}{<M>(\lambda^2-\breve{\omega}^2)} \tag{6.10}$$

with the resonance frequency λ given by Eq (6.9).

In order to compare the lower resonance vibration frequency obtained from the first of Eqs (6.4) and that given by Eq (6.9), let us denote

$$\beta \equiv \frac{<B>}{<M>}(k_1^2+k_2^2)^2 \qquad\qquad \varepsilon \equiv <M(g)^2>$$

$$\gamma \equiv \frac{(D_{11}k_1^2+D_{22}k_2^2)^2}{<M>}$$

Under the above notations the first one from Eqs (6.4) takes the form

$$\lambda_1^2 = \frac{1}{2}\Big[\frac{D}{\varepsilon}+\beta\Big] - \frac{D}{2\varepsilon}\sqrt{1+\Big(\frac{4\gamma}{D^2}-\frac{2\beta}{D}\Big)\varepsilon+\frac{\beta^2}{D^2}\varepsilon^2}$$

Let us observe that since $\varepsilon \in \mathcal{O}(l^4)$ then the constant ε can be treated as a small parameter. Representing the square root in the formula for λ_1 in the form of the power series with respect to ε, we obtain

$$\lambda_1^2 = \beta - \frac{\gamma}{D} + \mathcal{O}(\varepsilon)$$

Taking into account Eq (6.9) and bearing in mind the definitions of β and γ, we arrive finally at the interrelation

$$\lambda_1^2 = \lambda^2 + \mathcal{O}(l^4)$$

between the values of resonance frequencies λ_1 and λ obtained within frameworks of the refined and effective stiffness theories, respectively.

Summarizing results of this section, the following conclusions related to the both aforementioned theories can be formulated:

- The squares of lower resonance frequencies related to the effective stiffness theory are approximations of the order $\mathcal{O}(l^4)$ of the resonance frequencies derived from the refined theory.

- Higher resonance frequencies are caused by the effect of the microstructure length dimension on the dynamic behaviour of the plate and cannot be obtained within a framework of the effective stiffness theory.

- For the analysis of forced vibrations at high frequencies the refined plate theory has to be used.

The above conclusions are similar to those obtained by Baron and Woźniak (1995), which where related to the theories of periodic composite plates based on the Reissner-type plate theory.

7. Concluding remarks

The above example is a simple illustration of the proposed refined macro-theory of Kirchhoff plates with micro-periodic structure; the aim of the example was to show that the refined theory is able to describe, on the macro-level, certain micro-dynamical aspects of the plate behaviour. On the other hand, these aspects cannot be investigated within a framework of the known effective stiffness theories. This result is strictly related to the fact that Eqs (4.1) involve terms describing the effect of the microstructure length dimensions on the macro-behaviour of the plates under consideration. For the sake of simplicity in the example the have cofined ourselves to one micro-shape function; in order to carry out more detailed investigations of the microdynamical plate behaviour we have to introduce two or more micro-shape functions. Problems related to various applications of Eqs (4.1) to dynamics of micro-periodic plate as well as the possible generalizations of these equations are reserved for separate papers.

References

1. Baron E., Woźniak C., 1995, On the Micro-Dynamics of Composite Plates, *Arch. Appl. Mech.*, in press

2. Caillerie D., 1984, Thin Elastic and Periodic Plates, *Math. Meth. in the Appl. Sci.*, **6**, 159-191

3. Duvaut G., Metellus A.M., 1976, Homogéneisation d'une plaque mince en flexion des structure périodique et symmetrique, *C.R. Acad. Sci.*, Paris, **283**(A), 947-950

4. Kohn R.V., Vogelius M., 1984, A New Model for Thin Plates with Rapidly Varying Thickness, *Int. J. Solids Structures*, **20**, 333-350

5. Lewiński T., 1992, Homogenizing Stiffnesses of Plates with Periodic Structure, *Int. J. Solids Structures*, **21**, 309-326

6. Matysiak S.J., Nagórko W., 1989, Microlocal Parameters in the Modelling of Micro-Periodic Plates, *Ing. Arch.*, **59**, 434-444

7. Woźniak C., 1993, Refined Macrodynamics of Periodic Structures, *Arch. Mech.*, **45**, 295-304

8. Woźniak C., Woźniak M., Konieczny S., 1993, A Note on Dynamic Modelling of Periodic Composites, *Arch. Mech.*, **45**, 779-783

O elastodynamice cienkich mikroperiodycznych płyt

Streszczenie

Celem pracy jest opracowanie i analiza nowej metody modelowania cienkich sprężystych płyt o mikroperiodycznej strukturze w płaszczyznach równoległych do płaszczyzny środkowej płyty. Zasadniczą cechą otrzymanej teorii jest uwzględnienie wpływu wielkości mikrostruktury na dynamikę płyty. Efekt ten jest pomijany w znanych asymptotycznych teoriach płyt kompozytowych o periodycznej strukturze. W pracy wykazano, że wielkość mikrostruktury odgrywa istotną rolę przy badaniu procesów dynamicznych.

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