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DUAL FINITE ELEMENT METHODS IN MECHANICS OF COMPOSITE MATERIALS

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Application of the finite element method to homogenization of a fibrous, linearly elastic, composite material is considered in the paper. The finite element method is used in two approaches based on dual variational principles which give two approximate solutions: the kinematically and statically admissible ones. Having these two solutions one can easily evaluate the lower and upper bounds for effective moduli of a homogenized material. The paper contains the formulation of the method used to solve the considered problem, and some examples of numerical results.

1. Introduction

The problem of estimation of effective moduli for periodic, composite materials is important from the point of view of designing. A boundary value problem stated for the composite body, the basic cell of which has a very small size, when compared to the size of the whole body, cannot be solved directly, and some methods of macroscopic modelling have to be used for such a body. One way is to model the heterogeneous material by the homogeneous one with effective moduli. Such a body having anisotropic properties is called a homogenized material. Effective moduli of it can be calculated exactly only in some simple cases when a corresponding boundary value problem is one-dimensional. Since, in general case, the effective moduli can only be determined approximately, the problem of estimation of their lower and upper bounds is important. The first attempts at such an estimation were done by Voigt (1910) and Reuss (1929), however, differences between the lower and

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upper limits obtained by the proposed method are too large from the practical point of view. More accurate estimations have been done by Hashin and Shtrikman (1962), (1963), and Hill (1963)÷(1965) using dual variational principles of elasticity theory. The intervals for effective moduli obtained in their works are rather wide, especially in case of large differences between the moduli of components of a composite body. The precise definition of effective moduli has been introduced by Duvaut (1976), Bensoussan, Lions and Papanicolaou (1978) within a framework of the homogenization theory where the asymptotic analysis is used to state the effective behaviour of the homogenized material.

This work presents application of the finite element method to the problem of homogenization of a fibrous, linearly elastic, composite material. The bounds for effective moduli are obtained using the displacement and equilibrium models of the finite element method based on the dual variational principles of the theory of elasticity (cf Więckowski (1986)). The work starts with some basic results of the homogenization theory, afterwards the method of solution to the considered problem is described. Some examples of numerical results are included in the paper.

2. Problem of homogenization of periodic fibrous composite material

Let us consider a fibrous, composite body of periodic structure, the cross-section of which occupies an open bounded region $\Xi \subset \mathcal{R}^2$ with a regular boundary $\partial\Xi$ consisted of two parts $\partial\Xi_u$ and $\partial\Xi_\sigma$, where the displacement and stress boundary conditions are given, respectively. It is assumed that

$$\overline{\partial\Xi_u} \cup \overline{\partial\Xi_\sigma} = \partial\Xi \qquad \partial\Xi_u \cap \partial\Xi_\sigma = \emptyset$$
$$\text{meas}(\partial\Xi_u) \neq 0$$

where $\text{meas}(\partial\Xi_u)$ denotes the length of $\partial\Xi_u$.

Material constants are assumed to be periodic functions of coordinates x_α, $\alpha = 1,2$. Let $\Omega \in \mathcal{R}^2$ denote a region, in which the smallest repeatable part Ω^ε of Ξ can be mapped using the transformation

$$y_\alpha = \frac{1}{\varepsilon}(x_\alpha - a_\alpha)$$

where $\varepsilon > 0$, a_α is the position of the centre of a repeatable element. A typical periodic structures of a fibrous composite material. The meanings of the vector a_α and the number ε are shown in the same figure.

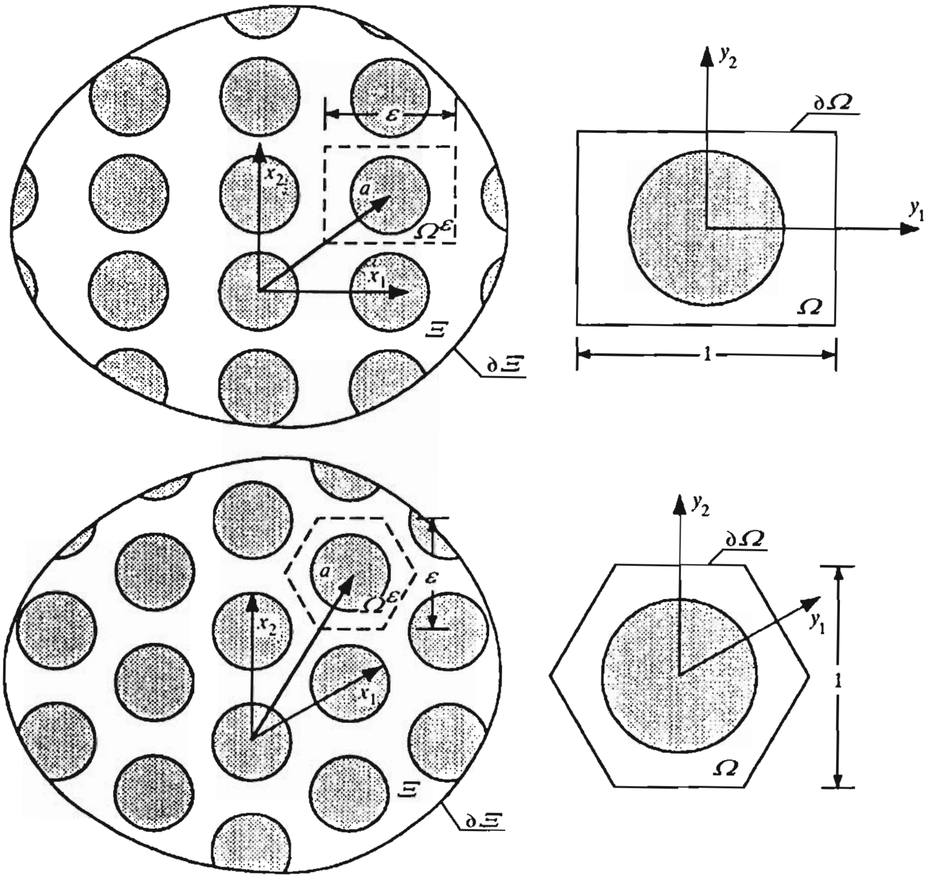

Fig. 1. Typical structures of composite material

Any Ω^ε-periodic function $G^\varepsilon(x)$ can be written in the form

$$G^\varepsilon(x) = G\left(\frac{x}{\varepsilon}\right)$$

if $G(y)$ is Ω-periodic function defined on the entire space $\mathcal{R}^2$.

Assuming that the elastic constants $A_{ijkl}(y)$ are Ω-periodic functions, we can write the following relation

$$A^\varepsilon_{ijkl}(x) = A_{ijkl}\left(\frac{x}{\varepsilon}\right)$$

Then an equilibrium problem for the linearly elastic, periodic, composite body can be stated as follows (e.g. Duvaut (1976), Bensoussan, Lions and Papanicolaou (1978), Sanchez-Palencia (1980), Suquet (1982)):

— Find $u^\varepsilon \in V_U$ such that

$$a^\varepsilon(u^\varepsilon, v - u^\varepsilon) = f(v - u^\varepsilon) \qquad \forall v \in V_U \tag{2.1}$$

where

$$a^\varepsilon(u,v) = \int\limits_{\Xi} \varepsilon_{ij}(u) A^\varepsilon_{ijkl}(x) \varepsilon_{kl}(v)\, dx$$

$$f(v) = \int\limits_{\Xi} f_i v_i\, dx + \int\limits_{\partial\Xi_\sigma} t_i v_i\, dx$$

$$V_U = \Big\{ u \in [H^1(\Omega)]^3 : \quad u_i = U_i \text{ on } \partial\Xi_u \Big\}$$

U_i and t_i are the vectors of displacements and stresses given on Ξ_U and Ξ_σ, respectively.

When the size of cell ε is small when compared to the diameter of the region Ξ, the approximate solution to Eq (2.1) cannot be obtained in the direct way. An efficient manner of solving the problem is provided by the homogenization method where we have to solve the problem corresponding to a homogenized material:

— Find $u \in V_U$ such that

$$a^{\text{eff}}(u, v - u) = f(v - u) \qquad \forall v \in V_U \tag{2.2}$$

where

$$a^{\text{eff}}(u,v) = \int\limits_{\Xi} \varepsilon_{ij}(u) A^{\text{eff}}_{ijkl} \varepsilon_{kl}(v)\, dx \tag{2.3}$$

The tensor of effective moduli A^{eff}_{ijkl}, introduced above, is evaluated from the relation (cf Duvaut (1976), Bensoussan, Lions and Papanicolaou (1978))

$$A^{\text{eff}}_{ijkl} = \frac{1}{\text{meas}(\Omega)} \int\limits_{\Omega} (A_{ijkl} - \chi^{pq}_{i,j} A_{pqrs} \chi^{rs}_{k,l})\, dy$$

where $\text{meas}(\Omega)$ is the area of Ω and χ^{pq}_i denotes the homogenization function being the solution to the following cell problem:

— Find $\chi^{pq} \in V$ such that

$$\int\limits_{\Omega} (\chi^{pq}_{i,j} + \delta_{pi}\delta_{qj}) A_{ijkl} v_{k,l}\, dy = 0 \qquad \forall v \in V \tag{2.4}$$

where

$$V = \Big\{ v \in [H^1(\Omega)]^3 : \quad \int\limits_{\Omega} v_i\, dy = 0 \quad v_i\, \Omega\text{-periodic} \Big\} \tag{2.5}$$

In the definition (2.5), the condition $\int_\Omega v_i \, dy = 0$ should be satisfied in order to avoid non-uniqueness of solution.

It can be proved easily by direct derivation that the formula (2.4) can be written in the following equivalent form:

— Find $w \in V$ such that

$$\int_\Omega (w_{i,j} + E_{ij}) A_{ijkl} v_{k,l} \, dy = 0 \qquad \forall v \in V \tag{2.6}$$

where

$$E_{ij} = \begin{cases} 1 & \text{if } i = p \wedge j = q \\ 0 & \text{otherwise} \end{cases}$$

The field w_i, being the solution to Eq (2.6), can be regarded as the periodic part of the displacement field in infinite, periodic, composite body, corresponding to the tensor of mean strains E_{ij}, which can be expressed as follows

$$u_i = E_{ij} y_j + w_i \tag{2.7}$$

where

$$E_{ij} = \tilde{\varepsilon}_{ij}$$

$$\widetilde{(\cdot)} = \frac{1}{\text{meas}(\Omega)} \int_\Omega (\cdot) \, dy$$

The following relation exists between both fields w_i and $\chi_i^{\varepsilon pq}$

$$w_i = \varepsilon E_{pq} \chi_i^{\varepsilon pq}$$

Eq (2.7) is shown in Fig.2

The formula (2.6) is somewhat more useful than Eq (2.4) in numerical calculations.

3. Solution to cell problem by displacement model of FEM

In case of fibrous, periodic, composite material, the displacements field w_i depends only on two variables x_1, x_2, therefore the following relation is true

$$w_{i,3} \equiv 0 \qquad i = 1,2,3 \tag{3.1}$$

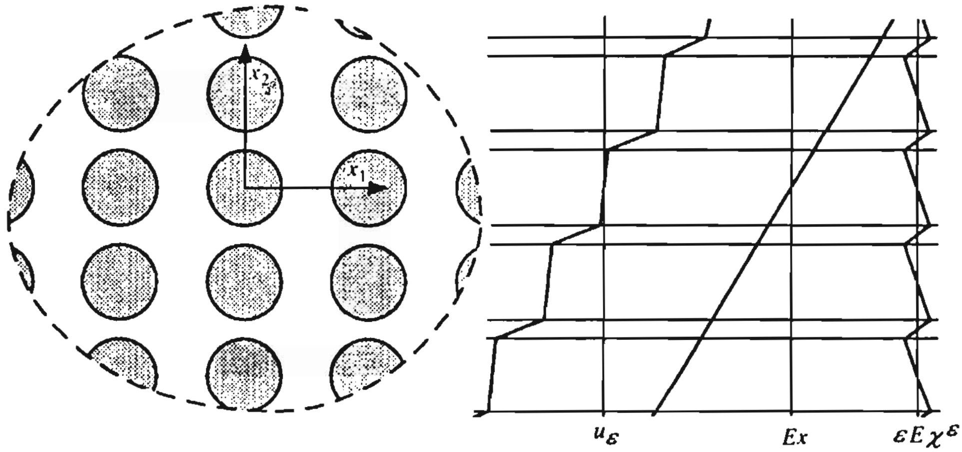

Fig. 2. Deformation of infinite, periodic, composite body

Using Eq (3.1), we can split the 3-dimensional problem (2.6) into two 2-dimensional problems:

The anti-plane shear problem: Find $w_3 \in V_1$ such that

$$\int_\Omega A_{3\alpha 3\beta} w_{3,\alpha} v_{,\beta}\, dx = -\int_\Omega A_{3\alpha 3\beta} E_{3\beta} v_{,\alpha}\, dx \qquad \forall v \in V_1 \tag{3.2}$$

where

$$V_1 = \Big\{ v \in H^1(\Omega): \quad v\,\Omega\text{-periodic} \quad \tilde{v} = 0 \Big\}$$

and $\alpha, \beta = 1, 2$.

The plane strain problem: Find $w_\alpha \in V_2$ such that

$$\int_\Omega A_{\alpha\beta\gamma\delta} w_{\gamma,\delta} v_{\alpha,\beta}\, dx = -\int_\Omega (A_{\alpha\beta\gamma\delta} E_{\gamma\delta} + A_{\alpha\beta 33} E_{33}) v_{\alpha,\beta}\, dx \qquad \forall v \in V_2 \tag{3.3}$$

where

$$V_2 = \Big\{ v \in [H^1(\Omega)]^2: \quad v_\alpha\,\Omega\text{-periodic} \quad \tilde{v}_\alpha = 0 \Big\}$$

Let us define two finite dimensional spaces V_{Kh}, $K = 1, 2$, being a subspaces of V_K

$$V_{1h} = \Big\{ u_h: \quad u_h(x) = \sum_{I=1}^{N} \varphi^I p^I(x) \quad p^I(x) \in C^0(\overline{\Omega})$$
$$p^I(x) \in H^1(\omega^I) \qquad u_h\,\Omega\text{-periodic} \Big\}$$

$$V_{2h} = \Big\{ u_{h\alpha} : \quad u_{hi}(x) = \sum_{I=1}^{N} \varphi_\alpha^I p^I(x) \quad p^I(x) \in C^0(\overline{\Omega}) \\ p^I(x) \in H^1(\omega^I) \quad u_{h\alpha}\, \Omega\text{-periodic} \Big\}$$

where

$\varphi^I, \varphi_\alpha^I$ – degrees of freedom

$p^I(x)$ – shape function

ω^I – support of the shape function p^I, $\omega^I \equiv \text{supp}(p^I)$.

The discrete problems corresponding to Eqs (3.2) and (3.3) can be written as follows:

— Find $u_h \in V_{Kh}$ such that

$$a(u_h, v - u_h) = f(v - u_h) \qquad \forall v \in V_{Kh} \qquad K = 1,2$$

The 3-node triangular and 4-node rectangular elements have been used to solve the considered problem in the displacement model of the finite element method.

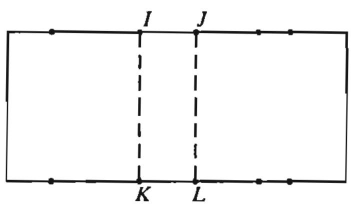

Fig. 3. Required position of nodes on the opposite sides of a cell

The problem of construction of kinematically admissible fields of displacements satisfying the condition of Ω-periodicity has been solved using the Lagrange multipliers method. The nodes laying on the two opposite sides of the cell have to be distributed in the same way, as it is shown in Fig.3, and for each pair of nodes corresponding to each other, the following condition must be fulfilled

$$\varphi^I - \varphi^K = 0 \qquad\qquad \varphi_\alpha^I - \varphi_\alpha^K = 0 \tag{3.4}$$

where φ_α^I and φ_α^K are the degrees of freedom defined at nodes I and K referred to the αth component of the displacement vector.

4. Solution to cell problem by equilibrium model of FEM

After splitting into two 2-dimensional problems, the dual cell problem can be set as follows:

The anti-plane shear problem: Find $\sigma_{3\alpha} \in Y_1$ such that

$$\int_{\Omega} \frac{1+\nu}{E} \sigma_{3\alpha}\tau_{3\alpha}\, dx = \int_{\Omega} E_{3\alpha}\tau_{3\alpha}\, dx \qquad \forall \tau \in Y_1 \tag{4.1}$$

where

$$Y_1 = \Big\{ \tau \in [L^2(\Omega)]^2 : \quad \tau_{3\alpha,\alpha} = 0 \text{ on } \Omega \quad \tau_{3\alpha}n_\alpha \Omega\text{-anti-periodic} \Big\} \tag{4.2}$$

The plane strain problem: Find $\sigma_{\alpha\beta} \in Y_2$ such that

$$\int_{\Omega} \frac{1+\nu}{E} (\sigma_{\alpha\beta}\tau_{\alpha\beta} - \nu\sigma_{\alpha\alpha}\tau_{\beta\beta})\, dx = \int_{\Omega} (E_{\alpha\beta}\tau_{\alpha\beta} + \nu E_{33}\tau_{33})\, dx \qquad \forall \tau \in Y_2 \tag{4.3}$$

where

$$Y_2 = \Big\{ \tau \in [L^2(\Omega)]^4 : \ \tau_{\alpha\beta} = \tau_{\beta\alpha} \ \tau_{\beta\alpha,\beta} = 0 \text{ on } \Omega \quad \tau_{\beta\alpha}n_\beta \Omega\text{-anti-periodic} \Big\} \tag{4.4}$$

In Eqs (4.1) and (4.3) the considerations have been restrained to the case of isotropic material with the Young modulus E, and the Poisson ratio ν.

The crucial point in the equilibrium approach to the finite element method is the choice of representation of statically admissible fields of stresses. Generally, they can be represented by the stress functions of Maxwell and Morera (cf Fung (1965), Truesdell (1959/60)).

In the anti-plane shear problem, only two components of the stress tensor are non-zero

$$\sigma_{3\alpha} \neq 0 \qquad \alpha = 1,2$$

and the equilibrium conditions reduce to one equation

$$\sigma_{3\alpha,\alpha} = 0 \quad \text{on } \Omega \tag{4.5}$$

Eq (4.5) is satisfied if stresses are expressed by the Prandtl stress function $\psi \in H^1(\Omega)$ according to the formula

$$\sigma_{3\alpha} = e_{\alpha\beta}\psi_{,\beta}$$

where $e_{\alpha\beta}$ denotes the permutation symbol, $e_{11} = e_{22} = 0$, $e_{12} = -e_{21} = 1$.

The finite dimensional space of statically admissible stress fields Y_{1h} being a subspace of Y_1 can be defined for the anti-plane shear problem as follows

$$\begin{aligned} Y_{1h} \; = \; \Big\{\tau_{h3\alpha} : \quad & \tau_{h3\alpha}(x) = e_{\alpha\beta} \sum_{I=1}^{N} \varphi^I p^I_{,\beta}(x) \quad p^I(x) \in C^0(\overline{\Omega}) \\ & p^I(x) \in H^1(\omega^I) \quad \tau_{h3\alpha} n_\alpha \, \Omega\text{-anti-periodic}\Big\} \end{aligned} \tag{4.6}$$

In the plane strain problem, the following stress components are non-zero

$$\sigma_{\alpha\beta},\ \sigma_{33} \neq 0$$

and two equilibrium equations are to be satisfied

$$\sigma_{\beta\alpha,\beta} = 0 \qquad \text{on} \quad \Omega \tag{4.7}$$

Now the self-equilibrated stresses are expressed by the Airy stress function $F \in H^2(\Omega)$

$$\sigma_{\alpha\beta} = e_{\alpha\gamma} e_{\beta\delta} F_{,\gamma\delta}$$

For the plane strain problem, the finite dimensional set of statically admissible stress fields is defined as follows

$$\begin{aligned} Y_{2h} \; = \; \Big\{\tau_{h\alpha\beta} : \quad & \tau_{h\alpha\beta}(x) = e_{\alpha\gamma} e_{\beta\delta} \sum_{I=1}^{N} \varphi^I p^I_{,\gamma\delta}(x) \quad p^I(x) \in C^1(\overline{\Omega}) \\ & p^I(x) \in H^2(\omega^I) \quad \tau_{h\alpha\beta} n_\beta \, \Omega\text{-anti-periodic}\Big\} \end{aligned} \tag{4.8}$$

The discrete problems corresponding to Eqs (4.1) and (4.3) can now be written as follows:

— Find $\sigma_h \in Y_{Kh}$ such that

$$b(\sigma_h, \tau - \sigma_h) = g(\tau - \sigma_h) \qquad \forall \tau \in Y_{Kh} \qquad K = 1,2$$

To fulfill the conditions of Ω-anti-periodicity of the stress field, the method of Lagrange multipliers is used.

5. Estimation of lower and upper bounds for effective moduli

The value of strain energy

$$e(u) = \frac{1}{2}\int\limits_{\Omega} A_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(u)\, dy$$

stored up in the volume of an elastic body satisfies the following inequalities

$$\frac{1}{2}\int\limits_{\Omega} A^{-1}_{ijkl}\sigma^{s}_{ij}\sigma^{s}_{kl}\, dy \equiv e(\sigma^{s}) \leq e \leq e(u^{k}) \equiv \frac{1}{2}\int\limits_{\Omega} A_{ijkl}u^{k}_{i,j}u^{k}_{k,l}\, dy \tag{5.1}$$

where u^{k} and σ^{s} are the kinematically and statically admissible solutions, respectively. For the homogenized material, the strain energy accumulated in the basic cell can be expressed as

$$e = \frac{1}{2}A^{\rm eff}_{ijkl}E_{ij}E_{kl}\mathrm{meas}(\Omega)$$

It follows from Eq (5.1) that the effective moduli occurring in the diagonal of the 6×6 matrix of elastic constants, it means the components $A^{\rm eff}_{IJIJ}$, can be estimated using the inequalities

$$e\Big(\sigma^{s}(E_{IJ})\Big) \leq \frac{1}{2}A^{\rm eff}_{IJIJ}\mathrm{meas}(\Omega) \leq e\Big(Au^{k}(E_{IJ})\Big)$$

where it is assumed that only $E_{IJ} \neq 0$, and $E_{IJ} = 1$. It should be noticed that the summation convention does not stand for capital indices. For other components of effective moduli, we can write the following inequalities

$$e\Big(\sigma^{s}(E_{IJ}, E_{KL})\Big) \leq \frac{1}{2}\Big(A^{\rm eff}_{IJIJ} + A^{\rm eff}_{KLKL} + 2A^{\rm eff}_{IJKL}\Big)\mathrm{meas}(\Omega) \leq$$
$$\leq e\Big(Au^{k}(E_{IJ}, E_{KL})\Big)$$

which imply

$$e\Big(\sigma^{s}(E_{IJ}, E_{KL})\Big) - \frac{1}{2}\Big(A^{\rm upp}_{IJIJ} + A^{\rm upp}_{KLKL}\Big)\mathrm{meas}(\Omega) \leq A^{\rm eff}_{IJKL}\mathrm{meas}(\Omega) \leq$$
$$\leq e\Big(Au^{k}(E_{IJ}, E_{KL})\Big) - \frac{1}{2}(A^{\rm low}_{IJIJ} + A^{\rm low}_{KLKL})\mathrm{meas}(\Omega)$$

where it is assumed that only $E_{IJ}, E_{KL} \neq 0$, and $E_{IJ} = E_{KL} = 1$.

6. Numerical results

6.1. Square cell

The composite material consisting of the epoxy matrix ($E = 2.5$ GPa, $\nu = 0.4$) and steel fibres ($E = 209$ GPa, $\nu = 0.3$) of circular cross-section is considered. The cell geometry is shown in Fig.4, where the following dimensions are applied

$$l_1 = l_2 = 1 \qquad\qquad R = 0.75$$

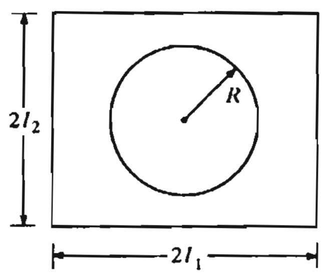

Fig. 4. Square basic cell

In case of the anti-plane shear problem the finite element mesh has consisted of 208 triangular elements and 117 nodes for the whole cell domain. For the solution to the plane strain problem, 100 square elements have been used to discretize the quarter of cell region.

Because of double symmetry of the problem, it is sufficient to consider the following states of mean strain

- the anti-plane shear problem

$$E_{31} = 1 \qquad\qquad E_{32} = 0$$

- the plane strain problem

$$\begin{array}{ll} E_{11} = 1 & E_{22} = E_{33} = E_{12} = 0 \\ E_{33} = 1 & E_{11} = E_{22} = E_{12} = 0 \\ E_{12} = 1 & E_{11} = E_{22} = E_{33} = 0 \end{array}$$

The results concerning the anti-plane shear problem are given in Fig.5 and Fig.6, while Fig.7 ÷ Fig.10 correspond to the plane strain problem. The components of the homogenization function χ, obtained by means of the displacement model of the finite element method is shown in Fig.5 and Fig.7. In Fig.6 and Fig.8 ÷ Fig.10, the stress distributions, obtained by the displacement (the left diagrams), and equilibrium (the right diagrams) models of the finite element method, are compared. In Fig.6, the continuous and dashed lines represent the stress components σ_{31} and σ_{32}, respectively.

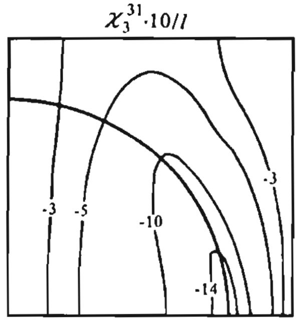

Fig. 5. Homogenization function – the anti-plane shear problem, $E_{31} = 1$

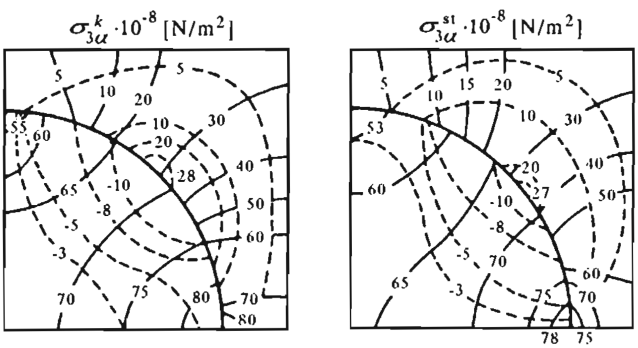

Fig. 6. Stress distribution – the anti-plane shear problem, $E_{31} = 1$

Some of effective moduli (e.g. A_{1122}) can be considered to be evaluated inaccurately. Using the mesh consisting of 1600 square elements and 1681 nodes for the plane strain problem, we obtain the results shown in Table 2.

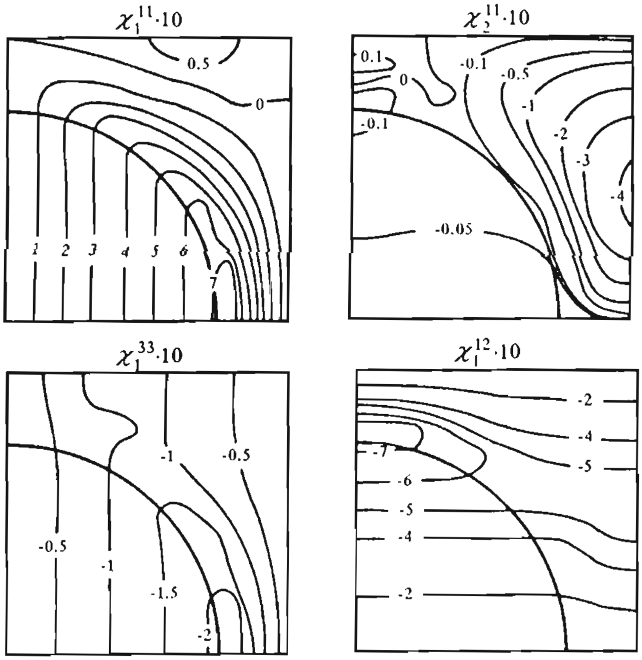

Fig. 7. Homogenization function – the plane strain problem

Table 1. Effective moduli

$A^{(eff)}_{ijkl}$	$A^{(low)}$ [GPa]	$A^{(up)}$ [GPa]	$A^{(mean)}$ [GPa]	ΔA [%]
$A_{1111} = A_{2222}$	11.500	13.700	12.600	8.73
A_{1122}	4.488	6.686	5.587	19.67
$A_{1133} = A_{2233}$	5.867	6.793	6.330	7.31
A_{3333}	97.879	98.270	98.074	0.20
A_{1212}	1.7413	2.0013	1.8713	6.95
$A_{1313} = A_{2323}$	2.39	2.48	2.43	1.93

Table 2. Effective moduli – more accurate results

$A^{(eff)}_{ijkl}$	$A^{(low)}$ [GPa]	$A^{(up)}$ [GPa]	$A^{(mean)}$ [GPa]	ΔA [%]
$A_{1111} = A_{2222}$	11.509	11.891	11.700	1.63
A_{1122}	5.1940	5.9430	5.5685	6.73
$A_{1133} = A_{2233}$	5.7755	6.3345	6.0550	4.62
A_{3333}	97.992	98.057	98.024	0.000332
A_{1212}	1.7378	1.7805	1.7592	1.21

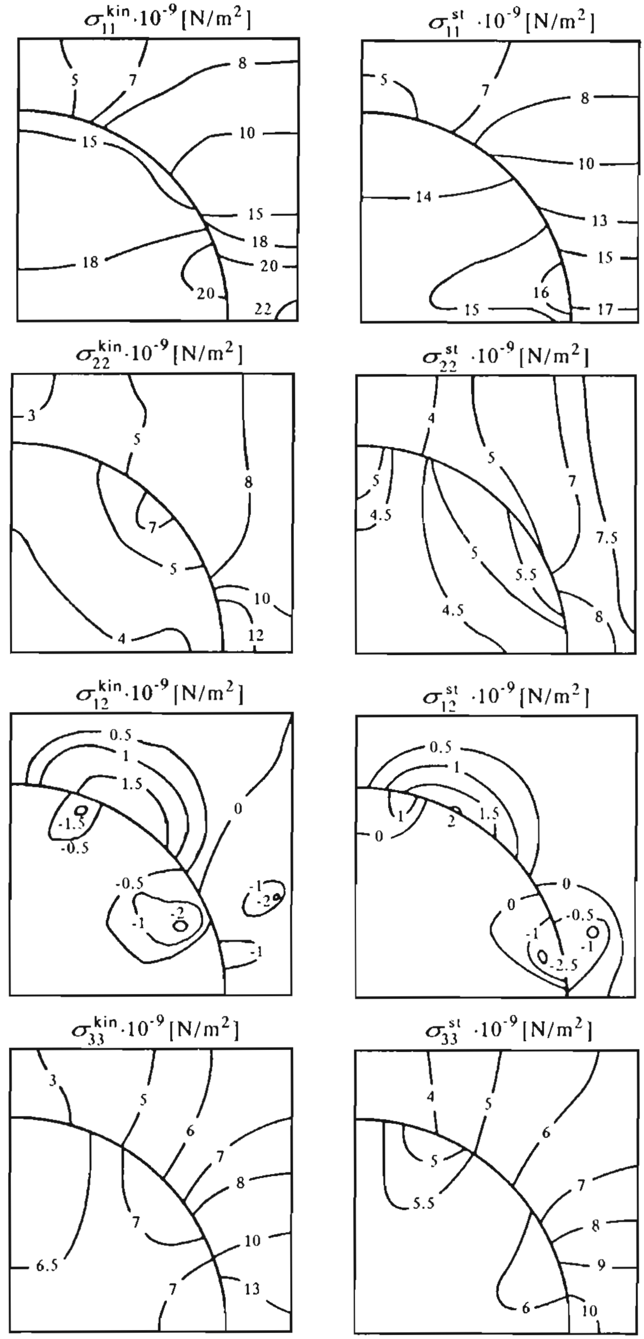

Fig. 8. Stress distribution – the plane strain problem, $E_{11} = 1$

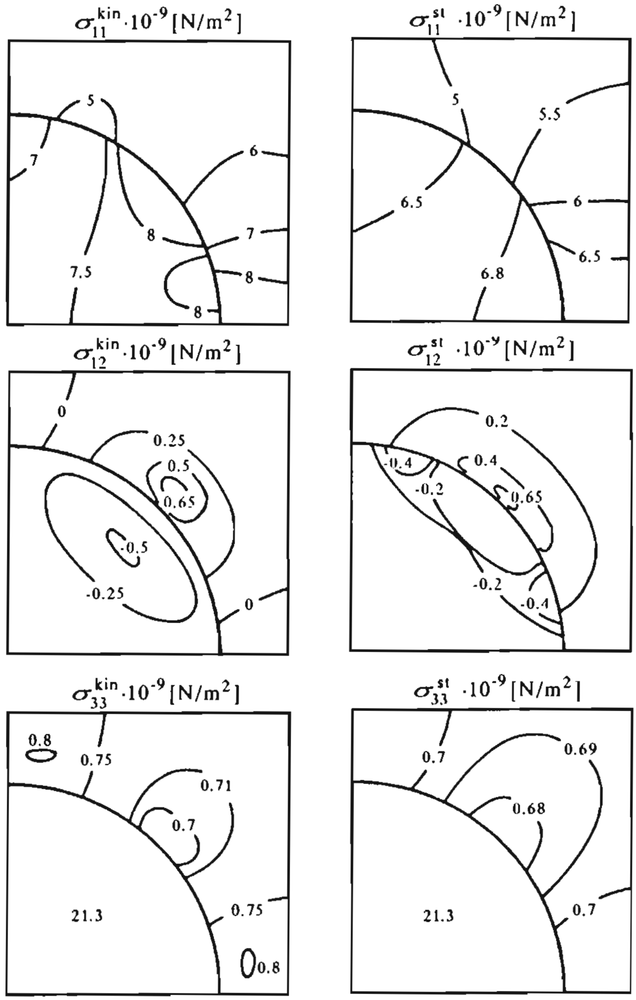

Fig. 9. Stress distribution – the plane strain problem, $E_{33} = 1$

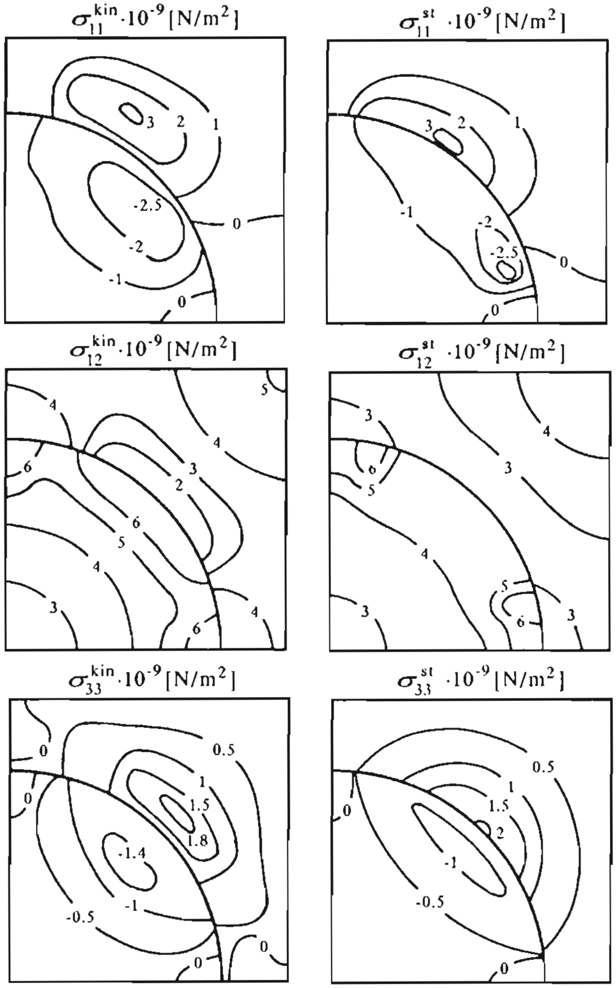

Fig. 10. Stress distribution – the plane strain problem, $E_{12} = 1$

7. Hexagonal cell

Let us consider the anti-plane shear problem of a composite material with the hexagonal cell, the quarter of which is shown in Fig.11. Calculations have been made for the following values of shear moduli μ: 1.3 GPa for the matrix (epoxy resin), and 29 GPa for fibres (glass). The problem has been solved for

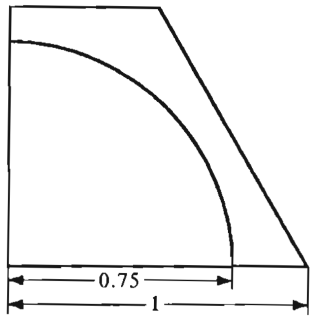

Fig. 11. Hexagonal basic cell

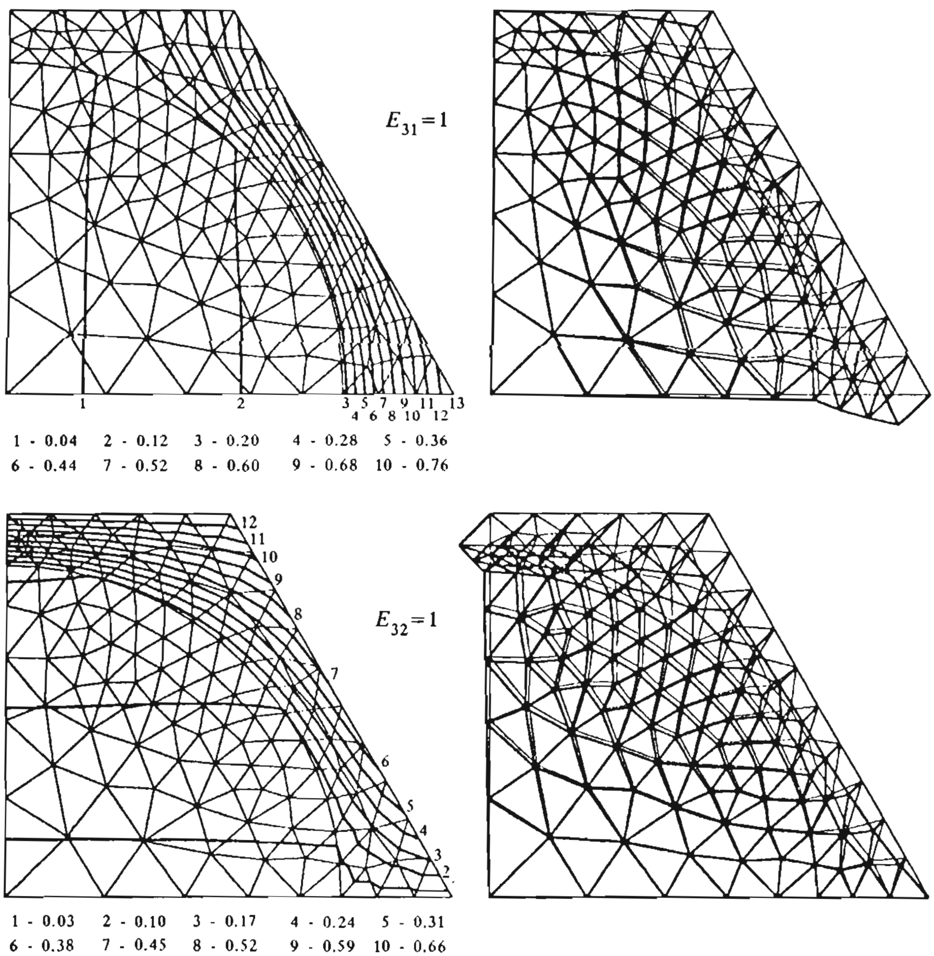

Fig. 12. Deformation of a hexagonal cell

the following two states of mean strain:

$$E_{31} = 1 \qquad E_{32} = 0$$
$$E_{31} = 0 \qquad E_{32} = 1$$

The diagrams of displacement fields u_3 are shown in the form of isolines and 3-dimensional plots in Fig.12. Fig.13 presents the solutions to the problem obtained by the equilibrium model of the finite element method; the isolines of the Prandtl stress function, and the distribution of the principal stresses are shown in the figure.

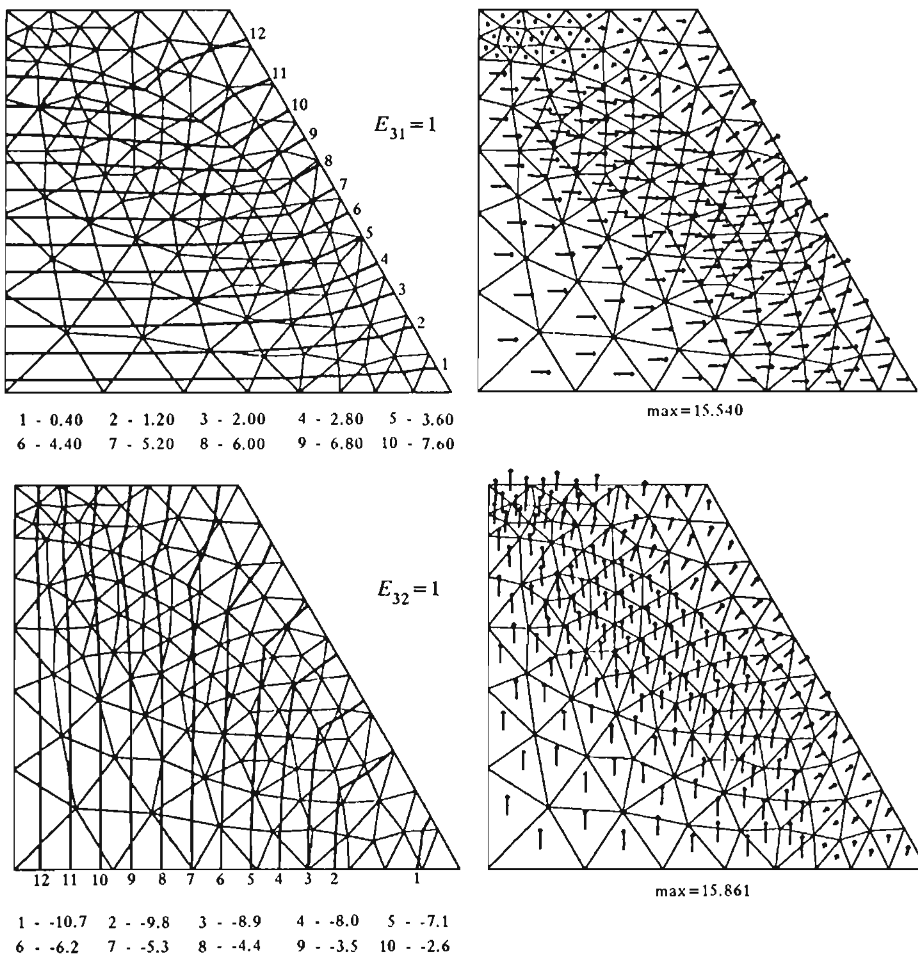

Fig. 13. Stress distribution – the statically admissible solution

*The lower and upper bounds of effective moduli, their mean values and errors are given in Table 3. It can be noticed that the difference between the values of both moduli is very small. It is natural because the composite material considered in the example reveals the material symmetry of the hexagonal type.

Table 3. Effective moduli – hexagonal cell

$A_{ijkl}^{(\mathrm{eff})}$	$A^{(\mathrm{low})}$ [GPa]	$A^{(\mathrm{up})}$ [GPa]	$A^{(\mathrm{mean})}$ [GPa]	ΔA [%]
A_{1313}	5.607	5.658	5.638	0.453
A_{2323}	5.609	5.656	5.632	0.408

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Modele przemieszczeniowy i naprężeniowy metody elementów skończonych w mechanice kompozytów

Streszczenie

W pracy rozważono zastosowanie metody elementów skończonych w zagadnieniu homogenizacji periodycznego, liniowo sprężystego materiału kompozytowego o strukturze włóknistej. Metodę elementów skończonych sformułowano w dwóch ujęciach: przemieszczeniowym i naprężeniowym, wykorzystując dualne zasady wariacyjne. Otrzymano dwa rozwiązania przybliżone: kinematycznie i statycznie dopuszczalne. Na podstawie tych rozwiązań oszacowano dolne i górne granice modułów sprężystości materiału efektywnego. Praca zawiera sformułowanie metody rozwiązania rozważanego zagadnienia i przyklady obliczeń.

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