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TWO AXI-SYMMETRICAL CONTACT PROBLEMS WITH THE STEADY-STATE FRICTIONAL HEATING

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We study two axi-symmetrical contact problems for a sliding/rotating sphere on an elastic half space. The effect of frictional heating is considered. The problems are reduced to Fredholm integral equations of the second kind which are solved numerically. Distributions of the contact pressure and temperature are shown in terms of input parameters.

Key words: thermoelasticity, frictional heating, contact mechanics

1. Introduction

Many investigators have presented methods for prediction surface temperature. We can mention only some basic monographs by Carslaw and Jaeger (1959), Parkus (1959), Boley and Weiner (1960), Ling (1973), Özişik (1980), Nowacki (1986), Rożnowski (1988). Since the frictional heat is proportional to the contact pressure in order to apply these theoretical approaches we must know the pressure distribution and contact area dimensions in advance. The usual approach is to assume that the pressure distribution and contact area are independent of the thermal deformation and friction force. Then the Hertz contact solution can be used to determine the contact area and pressure field (cf Korovchinski (1968); Tiang and Winer (1989); Yevtushenko and Ukhanska (1992)). However, owing, to some undefined factors in the real sliding process, there appear discrepancies between the calculated and measured values (cf Lingard (1984)).

One possible factor contributing to such inconsistent results is the uncertainty of the real contact area. Also, the assumption that the thermal deformation does not change the pressure distribution is questionable.

Barber (1975) and (1976), Generalov et al. (1976), Grilitskij and Kultchytsky-Zhyhailo (1991), Aleksandrov (1992) studied the axi-symmetrical contact problem involving the steady-state frictional heating. They found that the effect of thermal deformation on the contact pressure and temperature distributions is very essential.

The difference methods are employed for the solution of these problems. So, Barber (1975) obtained an approximate solution representing a general pressure distribution in terms of a finite series analogical to the form of Hertzian distribution but in which the radius of the loaded circle varies from 0 to a, where a is unknown a priori radius of the contact area. Another approximate solution is obtained by Barber (1976) in terms of trigonometrical series. The essence of this paper is formulation of considered problems in terms of the governing Fredholm integral equation for unknown pressure at the interface. The equation is solved numerically, using the approximate method proposed by Barber (1975).

2. Formulation of the problem

Let an elastic sphere of the radius R (body 1) be pressed by the force P into an elastic isotropic half space (body 2). Two cases of the uniform motion of the sphere are considered:

1. Sliding at the velocity V (problem 1)

2. Rotation about the symmetry axis at the angular speed ω (problem 2).

Friction effect involves heat sources in the contact region. We suppose that the total amount of heat, generated in the contact region, in the case of problem 1 is absorbed by one sliding body 1 but in the case of problem 2 by two contacting bodies (with the condition of temperature equality in over the contact area imposed). Remaining parts of the half space and sphere are assumed to be insulated and unloaded. We assume that there is no coupling between tangent and normal tractions however the tangent traction on the surface is not neglected. Indeed, the work done against these tractions is the source of the heat generation. However, the elastic displacements normal to the surface, caused by the tangencial tractions, are much smaller than those produced by the normal tractions, and the coupling effect is negligible. This approximation becomes more accurate if the elastic properties of two solids are similar.

We introduce now the cylindrical coordinate axes r, z, rigidly connected to the sphere; in these coordinate axes the contact region $0 \le r \le a$, $z = 0$ is motionless and thermomechanical processes are steady.

In such statement both problems will be axi-symmetrical and reduced to the solution of equations for the thermoelastic half spaces $z \ge 0$ and $z \le 0$, respectively, (cf Nowacki (1986))

$$\begin{aligned} &\nabla^2 u_r^{(i)} - \frac{1}{r^2} u_r^{(i)} + \frac{1}{1-2\nu_i} \frac{\partial \Theta^{(i)}}{\partial r} = \frac{2(1+\nu_i)\alpha_i}{1-2\nu_i} \frac{\partial T^{(i)}}{\partial r} \\ &\nabla^2 u_z^{(i)} + \frac{1}{1-2\nu_i} \frac{\partial \Theta^{(i)}}{\partial z} = \frac{2(1+\nu_i)\alpha_i}{1-2\nu_i} \frac{\partial T^{(i)}}{\partial z} \qquad i = 1,2 \end{aligned} \tag{2.1}$$

where

$$\Theta^{(i)} = \frac{\partial u_r^{(i)}}{\partial r} + \frac{u_r^{(i)}}{r} + \frac{\partial u_z^{(i)}}{\partial z} \qquad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

and the heat conduction equations

$$\nabla^2 T^{(i)} = 0 \qquad i = 1,2 \tag{2.2}$$

under the following boundary conditions on the surface $z = 0$

— mechanical

$$\sigma_z^{(i)} = -p(r) \qquad r \le a \tag{2.3}$$

$$\sigma_z^{(i)} = 0 \qquad r > a \tag{2.4}$$

$$\sigma_{rz}^{(i)} = 0 \qquad r \ge 0 \tag{2.5}$$

$$\frac{\partial}{\partial r}\left(u_z^{(2)} - u_z^{(1)}\right) = \frac{r}{R} \qquad r \le a \tag{2.6}$$

— thermal

$$-K_1 \frac{\partial T^{(1)}}{\partial z} = fVp(r) \qquad r \le a; \quad \text{(problem 1)} \tag{2.7}$$

$$-K_1 \frac{\partial T^{(1)}}{\partial z} + K_2 \frac{\partial T^{(2)}}{\partial z} = f\omega r p(r) \qquad r \le a; \quad \text{(problem 2)} \tag{2.8}$$

$$\frac{\partial T^{(i)}}{\partial z} = 0 \qquad r > a \tag{2.9}$$

$$T^{(1)} = T^{(2)} \qquad r \le a; \quad \text{(problem 2)} \tag{2.10}$$

$$u_r^{(i)}\ , u_z^{(i)}\ , T^{(i)} \to 0 \qquad \text{as} \qquad r^2 + z^2 \to \infty$$

where
- f – coefficient of friction
- K_i – thermal conductivity
- R – radius of the sphere
- p – contact pressure.

The values, referring to the sphere and half space, here and hereinafter are denoted by the indices $i = 1, 2$, respectively.

3. Reduction to the integral equation

The general solution of Eqs (2.1) and (2.2) which satisfies Eq (2.11), obtained by applying the Hankel integral transformation of zero order with respect to variable r has the form

$$\begin{aligned} u_z^{(i)}(r,z) &= \eta_i \int\limits_0^\infty \Big[-A_i(\xi) + B_i(\xi)(3 - 4\nu_i - \eta_i\xi z)\Big] e^{\eta_i\xi z} J_0(\xi r)\, d\xi\, + \\ &+ \alpha_i(1+\nu_i)\eta_i \int\limits_0^\infty \xi^{-1} C_i(\xi) e^{\eta_i \xi z} J_0(\xi r)\, d\xi \end{aligned} \tag{3.1}$$

$$T^{(i)}(r,z) = \int\limits_0^\infty C_i(\xi) e^{\eta_i\xi z} J_0(\xi r)\, d\xi \qquad \eta_i = (-1)^i \quad i = 1,2 \tag{3.2}$$

where
- ν_i – Poison ratios
- α_i – coeficients of thermal expansion
- $J_0(\cdot)$ – Bessel function of the first kind.

From the stress-strain relation we have

$$\sigma_z^{(i)}(r,z) = 2\mu_i \int\limits_0^\infty \Big[-A_i(\xi) + B_i(\xi)(2 - 2\nu_i - \eta_i\xi z)\Big] e^{\eta_i\xi z} J_0(\xi r)\xi\, d\xi \tag{3.3}$$

$$\sigma_{rz}^{(i)}(r,z) = 2\mu_i\nu_i \int_0^\infty \Big[A_i(\xi) - B_i(\xi)(1 - 2\nu_i - \eta_i\xi z)\Big] e^{\eta_i \xi z} J_1(\xi r)\xi \, d\xi \quad (3.4)$$

where μ_i are the shear moduli.

Substituting for $\sigma_z^{(i)}$, $\sigma_{rz}^{(i)}$ from Eqs (3.3), (3.4) into the boundary conditions (2.3) ÷ (2.5), we find the following relations between $A_i(\xi)$ and $B_i(\xi)$

$$A_i(\xi) = (1 - 2\nu_i)B_i(\xi) \quad (3.5)$$

$$2\mu_1 B_1(\xi) = 2\mu_2 B_2(\xi) = B(\xi) = -\int_0^a sp(s)J_0(\xi s)\, ds \quad (3.6)$$

By substituting Eq (3.2) into the conditions (2.7) ÷ (2.9) we obtain

$$C_1(\xi) = \frac{fV}{K_1}\int_0^a sp(s)J_0(\xi s)\, ds \qquad C_2(\xi) = 0 \quad \text{(problem 1)} \quad (3.7)$$

$$C_1(\xi) = C_2(\xi) = \frac{f\omega}{K}\int_0^a s^2 p(s)J_0(\xi s)\, ds \qquad K = K_1 + K_2 \quad \text{(problem 2)} \quad (3.8)$$

After substituting Eqs (3.1) into the boundary condition (2.6) and taking into account Eqs (3.5) ÷ (3.8) we obtain the integral equation

$$\int_0^\infty \xi J_1(\xi r)\, d\xi \int_0^a sp(s)J_0(\xi s)\, ds - \gamma_j r^{-1}\int_0^r s^j p(s)\, ds = \frac{r\mu}{R} \qquad r \le a \quad j = 1,2 \quad (3.9)$$

Here

$$\gamma_1 = \delta_1 fV\mu \qquad \gamma_2 = \delta f\omega\mu$$

$$\frac{1}{\mu} = \frac{1-\nu_1}{\mu_1} + \frac{1-\nu_2}{\mu_2} \quad (3.10)$$

$$\delta = \frac{\delta_1 K_1 + \delta_2 K_2}{K} \qquad \delta_i = \frac{(1+\nu_i)\alpha_i}{K_i} \qquad i = 1,2$$

The superscript $j = 1$ here and herein after denotes the problem 1, while $j = 2$ denotes the problem 2.

The solution of the Fredholm integral equation (3.9) must satisfy the equation of equilibrium

$$2\pi \int_0^a r p(r)\, dr = P \tag{3.11}$$

4. Numerical solution and analysis

An approximate solution of the integral equation (3.9) we obtain by the numerical method of Barber (1975). We divide the contact region $0 \le r \le a$ into N rings by introducing points $r_n = n\Delta r$ $(n = 0, 1, ..., N)$, $\Delta r = a/N$, and the contact pressure $p(r)$ can be writtenin the form

$$p(r) = \sum_{n=m}^{N} x_n \sqrt{r_n^2 - r^2} \qquad r \le a \tag{4.1}$$

where m is the smallest integer greater than $r/\Delta r$.

Finally, substituting Eq (4.1) into the integral equation (3.9) and calculating integrals at the points r_m $(m = 1, 2, ..., N)$, we obtain the system of linear algebraic equations

$$\sum_{n=1}^{N} X_n (a_{nm} - \beta_j b_{nm}^j) = m \qquad m = 1, 2, ..., N \quad j = 1, 2 \tag{4.2}$$

Here

$$X_n = \frac{\pi R x_n}{4\mu} \tag{4.3}$$

$$\beta_j = \gamma_j a^j \qquad j = 1, 2 \tag{4.4}$$

$$a_{nm} = \begin{cases} m & m \le n \\ \frac{2}{\pi}\Big(-\frac{n}{m}\sqrt{m^2 - n^2} + m \arcsin\frac{n}{m}\Big) & m > n \end{cases} \tag{4.5}$$

$$b_{nm}^1 = \begin{cases} \frac{4}{3\pi m N}\Big[n^3 - \sqrt{(n^2 - m^2)^3}\Big] & m \le n \\ \frac{4n^3}{3\pi m N} & m > n \end{cases} \tag{4.6}$$

$$b_{nm}^2 = \begin{cases} \frac{1}{2\pi m N^2}\Big[n^4 \arcsin\frac{m}{n} - m(n^2 - 2m^2)\sqrt{(n^2 - m^2)}\Big] & m \le n \\ \frac{n^4}{4mN^2} & m > n \end{cases} \tag{4.7}$$

From Eq (3.11) we find the total load as

$$P = P_H \sum_{n=1}^{N} X_n \Big(\frac{n}{N}\Big)^3 \tag{4.8}$$

where

$$P_H = \frac{8a^3\mu}{3R} \tag{4.9}$$

is the force, necessary for attainment the contact radius a in the isothermal case (Hertz contact problem, cf Johnson (1985)).

In the case of absence of heat generation ($\beta_j = 0,\ j = 1,2$), Eqs (4.2), have the trivial solution

$$X_n = \delta_{nN} \tag{4.10}$$

δ_{nN} is the Kronecker delta and Eq (4.8) gives $P = P_H$.

To solve the problem numerically it is necessary to select the parameters N, β_j ($j = 1,2$). The method (cf Yevtushenko and Kultchtsky-Zhyhailo (1995)) was employed for obtaining results for $N = 20$ and a range of values β_j between 0 and β_j^*

$$\beta_1^* = 2 \qquad \text{(problem 1)} \tag{4.11}$$

$$\beta_2^* = \frac{12}{\pi} \approx 3.82 \qquad \text{(problem 2)} \tag{4.12}$$

The constants β_j^* ($j = 1,2$) appearing in Eqs (4.11), (4.12) are critical values of parameters β_j, for which the solutions of the considered contact problems exist and are unique.

Relationships between the dimensionless load P_H/P and the parameters β_j (4.4) are linear, what corresponds to the equations (cf Barber (1975); Barber (1976); Yevtushenko and Kultchtsky-Zhyhailo (1995))

$$\frac{P_H}{P} = 1 - \frac{\beta_j}{\beta_j^*} \qquad j = 1,2 \tag{4.13}$$

Taking into account Eqs (4.11), (4.12) and Eqs (4.4), we obtain the critical values of contact radius

$$a_1^* = \frac{\beta_1^*}{\delta_1 f V \mu} \qquad \text{(problem 1)} \tag{4.14}$$

$$a_2^* = \sqrt{\frac{\beta_2^*}{\delta f \omega \mu}} \qquad \text{(problem 2)} \tag{4.15}$$

At a constant speed Eqs (4.14), (4.15) yield the limited value of the contact region radius for increasing load P. We note that in the isothermal case in the absence of frictional heating such limit does not exist.

Denote

$$b = \frac{a_H}{a} \tag{4.16}$$

$$b_j = \frac{a_H}{a_j^*} \qquad j = 1,2 \tag{4.17}$$

where

$$a_H = \sqrt[3]{\frac{3PR}{8\mu}} \tag{4.18}$$

is the radius of contact circle in the isothermal Hertz problem under fixed load P (cf Johnson (1985)). The non-dimensional parameter b characterizes decreasing of the radius of contact area due to frictional heating. We note that b^2 is the ratio of the mean contact pressure in the present problem to the mean pressure in the Hertz contact problem. The input parameter b_j (see Eq (4.17)) depends on mechanical, thermal and geometrical properties, respectively, of the contacting pair of materials and on the assumed conditions in the contact region. Taking into account Eqs (4.16) and (4.17), Eq (4.13) can be rewritten in the following form

$$b^3 - b_j^j b^{3-j} = 1 \qquad j = 1,2 \tag{4.19}$$

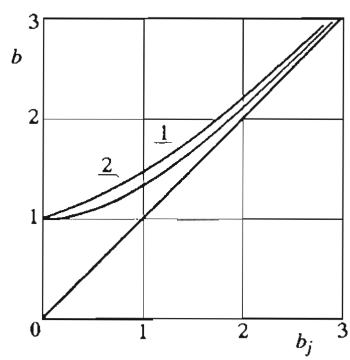

Fig. 1. Dependence of a_H/a from a_H/a_j^*; $j = 1$ – sliding, $j = 2$ – rotating

The relation of the non-dimensional parameter b (4.16) on the input parameter b_j (4.17) is given in Fig.1. In the absence of frictional heating we have

$b_j = 0$ ($a_j^* = \infty$) and $b = 1$ ($a = a_H$). As b_j increases (e.g. by increasing a sliding/rotational speed), the radius a of the contact circle falls, approaching at $b_j > 2.5$ the critical value a_j^*, ($j = 1, 2$) (see Eqs (4.14) and (4.15)).

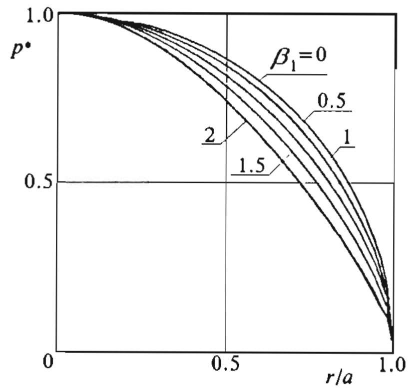

Fig. 2. Distribution of the dimensionless contact pressure $p^*(r) = p(r)/p(0)$ for the sliding sphere

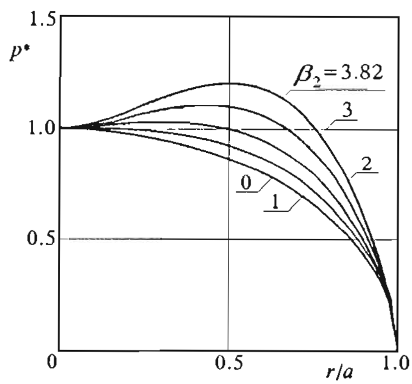

Fig. 3. Distribution of the dimensionless contact pressure $p^*(r) = p(r)/p(0)$ for the rotation sphere

The non-dimensional contact pressure distribution $p^*(r) = p(r)/p(0)$ in the case of problem 1 (sliding) is shown in Fig.2. The maximum value of contact pressure in sliding is reached at the centrum of the contact circle. In the case of problem 2 (rotation) the distribution $p^*(r)$ is given in Fig.3. The contact pressure distribution in rotation (Fig.3) essentially differs from the case of sliding of the bodies (Fig.2). The maximum value of the contact pressure in rotation is reached at a point $r \approx 0.5a$. Thus, at $\beta_2^* = 3.82$, $p^*_{max} = 1.202$ at $r = 0.52a$.

5. Themperature field

Taking into account Eqs (3.2), (3.7), (3.8) and (4.1), we have the temperature of the sphere

$$T_j^{(1)}(\rho_m,\zeta_k) = t_j \sum_{n=1}^{N} X_n\Big(\frac{n}{N}\Big)^{j+1} I_{nmk}^j \qquad j=1,2 \tag{5.1}$$

Here

$$I_{nmk}^j = \frac{2}{\pi}\int\limits_0^1 \frac{s^j\sqrt{1-s^2}}{\sqrt{\Big(\frac{k}{n}\Big)^2+\Big(s+\frac{m}{n}\Big)^2}} K\Bigg[\frac{2\sqrt{s\Big(\frac{m}{n}\Big)}}{\sqrt{\Big(\frac{k}{n}\Big)^2+\Big(s+\frac{m}{n}\Big)^2}}\Bigg]\,ds \qquad j=1,2 \tag{5.2}$$

$$t_1 = \frac{3fVP_H}{2\pi aK_1} \qquad t_2 = \frac{3f\omega P_H}{2\pi K} \tag{5.3}$$

$$\rho_m = \frac{m}{N} \qquad \zeta_k = \frac{k}{N} \qquad m,k = 0,1,2,... \tag{5.4}$$

where $K(\cdot)$ is a complete elliptic integral of the first kind.

The total heat flux through the contact region is

$$Q_j = q_j \sum_{n=1}^{N} X_n\Big(\frac{n}{N}\Big)^{2+j} \qquad j=1,2 \tag{5.5}$$

where

$$q_1 = fVP_H \qquad q_2 = \frac{3}{16}\pi f\omega aP_H \tag{5.6}$$

The distribution of dimensionless temperature

$$T_j^* = \frac{\Lambda_j T_j^{(1)}}{Q_j} \qquad j=1,2 \tag{5.7}$$

where

$$\Lambda_1 = \frac{2}{3}\pi aK_1 \qquad \Lambda_2 = \frac{2}{3}\pi aK \tag{5.8}$$

for $\beta_1^* = 2$ (sliding) and $\beta_2^* = 3.82$ (rotation) is shown in Fig.4 ÷ Fig.7. The surface $z=0$ temperature T_j^* $(j=1,2)$ of the sliding/rotation sphere is given in Fig.4 and its change along the symmetry axis $r=0$ in Fig.5. We can see that the maximum values of the contact temperature are reached at the points of maximum values of the contact pressure.

The isotherms of temperature fields T_j^* (5.9) in the case of sliding $j=1$ and rotation $j=2$ sphere are shown in Fig.6 and Fig.7, respectively.

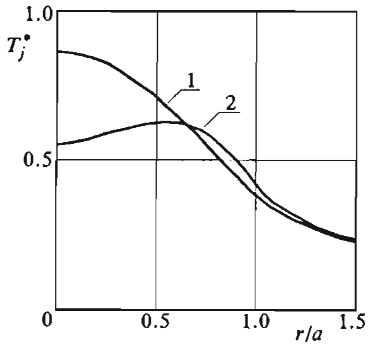

Fig. 4. Distribution of the dimensionless temperature T_j^* over the surface $z = 0$ at: $j = 1$ – sliding for $\beta_1 = 2$, $j = 2$ – rotation for $\beta_2 = 3.82$

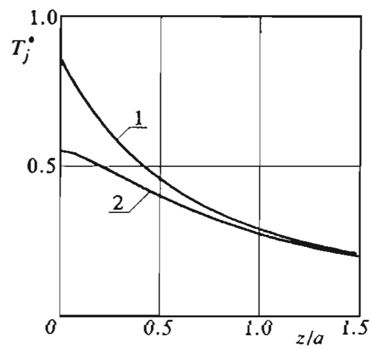

Fig. 5. Change of the dimensionless temperature T_j^* along the axis of symmetry $r = 0$: $j = 1$ – sliding for $\beta_1 = 2$, $j = 2$ – rotation for $\beta_2 = 3.82$

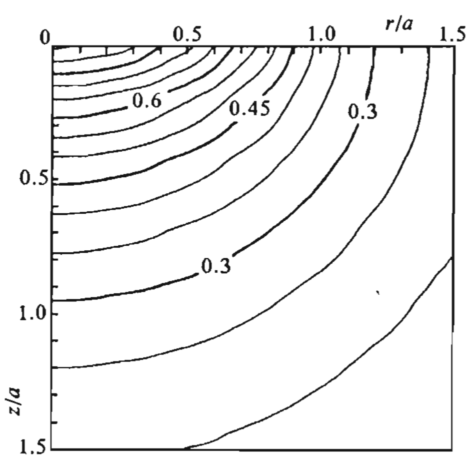

Fig. 6. The isotherms of the temperature field T_1^* in case of sliding sphere at $\beta_1 = 2$

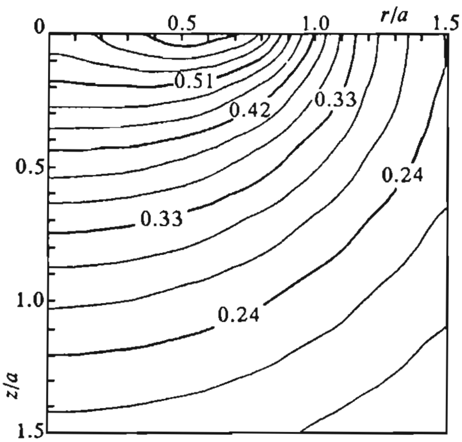

Fig. 7. The isotherms of the temperature field T_2^* in case of rotation sphere at $\beta_2 = 3.82$

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O dwóch osiowosymetrycznych zagadnieniach kontaktowych w ustalonym stanie ogrzania wywołanego tarciem

Streszczenie

W pracy rozpatrzono dwa zagadnienia kontaktowe dla ślizgającej się lub obracającej się sfery na sprężystej półprzestrzeni. Zbadano efekt ogrzania wywołanego tarciem. Zagadnienia zostały zredukowane do całkowych równań Fredholma drugiego rodzaju, które następnie rozwiązano numerycznie. Rozklady ciśnienia kontaktowego i temperatury zostały przedstawione na wykresach.

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