JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
51, 1, pp. 171-181, Warsaw 2013

ON THE TORQUE AS A MEASURE OF MECHANICAL INTERACTION IN THE PRINCIPLES OF DYNAMICS

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In most discussions, the *Principles of Dynamics* are expressed using the force as a measure of mechanical interaction between the bodies. The intention of the paper is to extend the usual discussion on basic theorems, laws and principles in Dynamics of rigid bodies including the torque as another independent measure of mechanical interaction between the bodies. In *D'Alembert's principle of Dynamics*, beside the forces, the active and reaction torques are also included. The torque is introduced in the Euler-Newton equations for general motion of the rigid body. The *General Equation of Dynamics* is reformulated by including the virtual work of the torques on the virtual rotation. An additional view to Newton's Laws is also given.

Key words: Newton's Laws, active and reaction torques, d'Alembert's principle of dynamics

1. Introduction

In 1687, Sir Isaac Newton published his three laws in *Philosophie Naturalis Principia Mathematica: Axiomata sive leges motus* for which is stated that represent the basis of the classical mechanics. Let us adduce the Laws, as were given by Newton (1687), with the translation from Latin to English done by Prof. Johns in late 2005 (the first translation from Latin to English was published in 1727):

Lex I: *Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus illud a viribus impressis cogitur statum suum mutare.* (Law I: Every body preserves in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.)

Lex II: *Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur.* (Law II: The alteration of motion is ever proportional to the motive force impressed, and is made in the direction of the right line in which that force is impressed.)

Lex III: *Actioni contrariam semper et eaqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse eaquales et in partes contrarias dirigi.* (Law III: To every Action there is always opposed an equal Reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.)

Analyzing the text of the Laws, it is evident that they refer to the general behavior of a "body". This conclusion is proved by the examples given by Newton in the same section of his book where the bodies are considered as macroscopic, ordinary objects.

The three Newton's Laws are axiomatic to studies in Mechanics as they are based on experimental observation without analytical description and proof. It gives the chance to be translated and interpreted in a variety of ways from their original state. For example, the most often used version of the Law II is as follows (see the textbooks of Starzhunskii (1982), Ginsburg (2008), *etc.*): The resultant force $\mathbf{F}$ (that is, the sum of all forces) acting on a particle is proportional to the acceleration $\mathbf{a}$ of the particle

$$m\mathbf{a} = \mathbf{F} \tag{1.1}$$

where the factor of proportionality is the mass m. The remarkable feature of this interpretation of the Law is that it addresses only the object that can be modeled as a particle.

Remark: The resultant force includes all active and passive forces (constraint forces) which act on the particle. This assumption will be used in the whole paper.

An extension to Newton's Law II was done by multiplying Eq. (1.1) with the position vector $\mathbf{r}$ of the particle respectively to the fixed point O. It follows

$$\mathbf{r} \times m\mathbf{a} = \mathbf{M}_0^{\mathbf{F}} \tag{1.2}$$

where $\mathbf{M}_0^{\mathbf{F}} = \mathbf{r} \times \mathbf{F}$ is the moment about the fixed point O of the force $\mathbf{F}$. This feature of motion is not described with the mentioned Newton's Laws, explicitly.

Let us consider the body as a system of particles and form equations of motion (1.1) for each of the particles with mass m_i and acceleration $\mathbf{a}_i$ on which also the inner forces between the particles act. Summarizing these equations and using Newton's Law III, which eliminates the inner forces, it follows

$$\sum_i m_i\mathbf{a}_i = \mathbf{F} \tag{1.3}$$

where $\mathbf{F}$ is the resultant force of the system. Multiplying each of Eq. (1.2) with the position vector $\mathbf{r}_i$ and summarizing the so obtained relations, we have

$$\sum_i \mathbf{r}_i \times m_i\mathbf{a}_i = \mathbf{M}_0^{\mathbf{F}} \tag{1.4}$$

where the moment $\mathbf{M}_0^{\mathbf{F}}$ is the sum of moments of each force according to the fixed point O, i.e.,

$$\mathbf{M}_0^{\mathbf{F}} = \sum_i \mathbf{M}_0^{\mathbf{F}_i} = \sum_i \mathbf{r}_i \times \mathbf{F}_i$$

If the body is assumed to be with continual mass distribution, the sums on the left-hand side of (1.3) and (1.4) transform into the integrals $\int_{(m)} \mathbf{a}\, dm$ and $\int_{(m)} \mathbf{r} \times \mathbf{a}\, dm$. Using the property of mass centre of the body, we have $\int_{(m)} \mathbf{a}\, dm = \mathbf{a}_C m$, where m is the mass of the body and $\mathbf{a}_C$ is the acceleration of the mass centre C. Using the aforementioned, Eq. (1.3) is rewritten as

$$m\mathbf{a}_C = \mathbf{F} \tag{1.5}$$

where $\mathbf{F}$ is the resultant force. If the position of the particle of the body is expressed as a function of the position of the mass centre $\mathbf{r}_C$ and of the position of the particle to the mass centre ρ, i.e., $\mathbf{r} = \mathbf{r}_C + \rho$ and the properties of the mass centre are applied ($\int_{(m)} \boldsymbol{\rho}\, dm = 0$), Eq. (1.4) with the corresponding integral gives

$$\int\limits_{(m)} \boldsymbol{\rho} \times \mathbf{a}_\rho\, dm = \mathbf{M}_C^{\mathbf{F}} \tag{1.6}$$

where

$$\mathbf{a}_\rho = \boldsymbol{\varepsilon} \times \boldsymbol{\rho} + \boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) \tag{1.7}$$

$\boldsymbol{\varepsilon}$ is the angular acceleration of the body, $\boldsymbol{\omega}$ is the angular velocity of the body, $\mathbf{M}_C^{\mathbf{F}} = \boldsymbol{\rho} \times \mathbf{F}$ is the moment of the force $\mathbf{F}$ with respect to the mass centre C and

$$\mathbf{r}_C \times m\mathbf{a}_C = \mathbf{r}_C \times \mathbf{F} \tag{1.7}$$

Expressions (1.5) and (1.6) describe the general motion of the body and are usually called the Newton-Euler equations. As the rigid body is treated as an indefinite number of particles, differential equations of motion (1.5) and (1.6) are obtained by summaring the equations of motion for each particle, separately. Due to this procedure, we obtain only the moment caused by the forces which are introduced on the right side of the equations. Namely, as the only measure of interaction between two particles is the force, only the action of the forces on the rigid body are treated. The forces produce translation but also rotation due to the moment. As it can be seen, the force is assumed as a basic measure of mechanical interaction between the bodies. The question is: is the force the only basic measure of mechanical interaction between the bodies, and is relation (1.1) the only possible mathematical interpretation of second Newton's law?

In this paper, the torque, as another basic measure of mechanical interaction between the bodies, is considered. The paper is arranged as follows: in Section 2, torques resulting from contact between the bodies and also as a measure of the action-at-a-distance are discussed. Various types of torques, active and passive, are analyzed. The rheological models of the torques are considered. In Section 3, we introduce active and passive torques (resulting from constraints) into d'Alembert's balancing equations. The Euler-Newton equations are rewritten. For the special case when only a torque acts, Newton's Laws are reformulated defining the relation between the torque and the rotation of a body about an axis. In Section 4, applying the virtual work of the torques on the virtual rotation, the Lagrange-D'Alamber principle (the General Equation of Dynamics) is reformulated. The inclusion of the torque into the generalized force is presented. Finally, this type of generalized force, due to the torque, is considered in Lagrange's second kind differential equations of motion.

2. About torques in general

The answer to the aforementioned question is: undoubtedly, the force is not the only measure of mechanical interaction between bodies. Still in 1754, Immanuel Kant (see Efroimsky and Williams, 2009) in his short work, known as "Spin-Cycle Essey", mentioned the so called "retarding torque" which slows down the Earth's rotation. He believed that this torque is a quite another measure of mechanical interaction between the bodies than is the force. Johannes Kepler, the key figure in the 17th century scientific revolutions, remarked that the body deformation may be caused not only by a force, but also by another unique physical impression called the torque. Since that time, various torque models have been formed in celestial mechanics, quantum and classical mechanics, electromagnetism, radiation, ... For example, in the planetary astronomy, Karato (2007) considered the tidal torque, emerging from the bodily tides. Various rheological models called MacDonald's torque (Darwin, 1879), Darwin's torque (Darwin, 1880), ... have been developed and discussed in celestial mechanics. Nowadays, when nanotechnology is developed and nanosystems are investigated, the importance of the torque is increased. In electromagnetics, the spin torques, induced by current in ferromagnetic materials, attract increased attention (see Gambardella and Miron, 2011). Ralph and Stiles (2008) considered the effect of flow of an electric current in a crystalline structure lacking inversion symmetry, which transfers orbital angular momentum from the lattice to the spin system, and named it the spin-orbit torque. It gives an opportunity to integrate magnetic functionality into electronic circuits. The spin-orbit torque enabled reduction of dimensions of write heads and the extension of the stray field. Very recently, the spin-orbit torque, relying on the presence of a strong spin orbit coupling intrinsic to the nuclear composition and atomic structure of a material, opens a promising new avenue to manipulate the magnetization of spintronic devices by means of electric currents as it is discussed by Miron *et al.* (2010). Dresselhaus (1955) concluded that the spin-orbit torques originate from either bulk inversion asymmetry in noncentrosymmetric crystals with a zinc blende structure or from a wurtzite-type crystal with a single high symmetry axis, which presents that structure inversion asymmetry allow the transfer of orbital angular momentum from the crystal lattice to local spin magnetization. The discovery of the Rashba-type torque (1960) in a ferromagnetic metal at room temperature opens very promising perspectives in spintronics, namely for fabrication of magnetic storage and logic gates operating through intrinsic current-induced spin-orbit torques. Spin-orbit torques are equivalent to an effective magnetic field because of the intrinsic coupling between charge and spin, and can be induced in a uniformly magnetized layer without the need of non-collinear polarization layers. In the paper of Sluka *et al.* (2011), the spin-transfer torque induced vortex dynamics in metallic nanopillars within in-plane magnetized layers is considered. Spin-transfer devices are promising candidates for future information technology. As spin-polarized currents can propel steady spin precession, spin-transfer torque devices are also envisaged to be used as integrated microwave sources. Therefore, finding highly tunable spin-transfer torque nano oscillators is a matter of great current interest. It was found that the spin-transfer torque can drive oscillatory motion of a magnet vortex and of gyrotropic vortex motion. Experimentally is shown that the spin-transfer torque can excite vortex dynamics in Fe/Ag/Fe pillars. The spin-transfer torque effect which appears in tunnel ferromagnetic junctions can be used in the random access memories and is connected with the charge transport in systems composed of ferromagnetic materials. Wilczynski (2011) investigated the spin-transfer torque generated by the temperature gradient in the planar tunnel junction consisting of ferromagnetic layers and a nonmagnetic tunnel barrier in the free-electron-like spin one-band model.

Bohren (2011) proved that, beside radiation forces, the radiation torques are exerted by radiation and are the consequence of electric and magnetic fields acting on charges and currents that the fields induce within illuminated objects. The importance of the radiation forces and torques is that the treatment of them on illuminated objects usually invokes photon linear and angular momentum transfer. The radiation forces and torques have physical origins.

The review given by Junge *et al.* (2009) attempts to describe the mechanisms of torque generation which powers the electrical rotary nanomotor of the enzyme, which drives the chemical nanomotor by elastic mechanical-power transmission producing adenosine triphosphate, the universal fuel of the cell. Finally, new approaches are developed which extend the description from the nanosecond time domain of molecular mechanics to the level of milliseconds. This provides fresh mechanics insights and gives us the way for new experiment approaches. Only when we solve the problem, we will come close to full understanding of this remarkable piece of cellular machinery. It requires inclusion of the torque into consideration.

In technique, the effect of torque is very widely applied, too. The most exploited is the effect of gyroscopic torques as mentioned by Birtae *et al.* (2011), for example. The gyrostatic torques, affine gyroscopic torques, nonlinear torques studied by Yehia (2003), gyroscopic torques along one axis of inertia investigated in the paper of Yehia and Elmandouh (2011), are already in use. All of these torques are generated by the axisymmetric force field. The effects of these torques are applied in micro/nanocoordinate measuring machines (see Liang *et al.*, 2012). Shi *et al.* (2011) emphasized the cutterhead torque as an important parameter in the design and operation of the earth pressure balance shields of tunneling machines. Tunneling plays a very important role in the underground engineering. The earth pressure balance shield tunneling machine is the most applied one to the tunnel construction for subway, highway, metro tunnel, etc., due to its ability to adapt to a variety of geological conditions and discharge control. The torque model is based on the experimentation proves. The model takes into account the cutterhead structure, cutting principle and the interaction between the cutterhead and soil. It is concluded that the cutterhead torque varies with geological conditions apparently, and the opening ratio of the cutterhead as well as the earth pressure are the two most important factors in determining the cutterhead torque. Rotational speed and torque are two critical parameters of the cutterhead drive, and they are directly related to drive power.

Nowadays, Fujiya *et al.* (2011) stated that very little is known about the effect of applied torque about the long axis of the bone in combination with muscle load, specially in legs and arms, and generally in biomechanics. The same is the case for tibia in combination with muscle loads on anterior cruciate ligament properties. It is of special interest to study the effect of the torque applied about the long axis of the bone on the anterior cruciate ligament strain behavior. The torques that are generated internally and externally to the knee or hands are thought to produce anterior cruciate ligament injuries by internal or external rotation about the axis of these bones. The experimental investigation gives us the opportunity to form the rheological model of the torque.

For all of the aforementioned torques, it is common that they represent a unique vector which is the measure of mechanical interaction between bodies. The most of the torques are the consequence of the contact between the bodies.

Based on the aforementioned, it is evident that beside the force, the torque is also a measure of body interaction. Djukic and Jones (1997) suggests that two independent measures of mechanical interaction exist: force, which causes translation and torque, which causes rotation around an axis of a body. In the paper of Djukic and Jones (1997), the reformulation of the *Axioms of Statics* are suggested where forces and torques are treated as the independent measures of the mechanical action.

The formulation given by Djukic and Jones (1997) includes the statement of the *Theorem in Kinematics* given by Chasles (1830) about the two independent motions: translatory and rotory with respect to an axis. Namely, Chales proves that the general motion of a rigid body can be represented as a superposition of two independent motions: translation following any point in the body and pure rotation about that point.

Based on the previous consideration, this paper suggests, beside the force, the inclusion of the torque vector as an independent measure of mechanical interaction into the classical Dynamics. Namely, some extension of the existing theorems in quantum mechanics, like the Virial theorem, which is usually expressed through the coordinate and the force, is extended to the case of angular displacement-torque variables, as has already been done by Jiang *et al.* (2011). The Virial theorem applied in celestial and galactic mechanics is about the balance between the kinetic and potential contribution to the total energy, which although originally deduced from classical mechanics (see the textbook of Johns, 2005) has a quantum mechanical counterpart, and – as given by Hellmann (1937) – is related to the Hellmann-Feynman theorem. The result of these investigation gives us an idea to include the torque as a unique vector into dynamic equations of motion and to give an additional interpretation of the Newton's laws.

The aim of this communication is to introduce the torque into the Euler-Lagrange differential equations of motion, but not only by adding them to the already written equations but including the physical sense of the torque. The torque, as a vector, is described using the following information: the intensity of the rotational effect, the axis and/or plane of rotation and the direction of rotation. The vector is aligned along the axis of rotation and directed toward the side from which this rotation will be seen in the counterclockwise direction. The rotation effect of the torque is given with the moment. The torque has the mechanical equivalent, and it is the couple of forces, which is often mentioned in the textbooks in Mechanics.

Remarks *on the couple of forces*: The couple of forces is a system of two parallel forces with the same intensity but opposite direction settled in the action plane which give the rotation around the axis orthogonal to the plane of the forces and with the moment which is the product of the force intensity and the normal distance between the forces. The couple of forces is characterized with the plane of action, magnitude of its moment and direction of rotation. The couple of forces and the torque with the parallel rotation axis, with the same rotation direction and with the same absolute values of their moments are said to be equivalent. However, the couple can be transferred in its plane as a 'rigid construction' without changing its action on the body, and the action of the couple does not change when the magnitudes of the forces forming the couple and the distance between the forces are varied so that the absolute value of the moment and the direction of rotation of the couple remain unchanged. Due to this property of the couple of forces, the transformation of the torque into the couple of forces is not straightforward: an infinite number of solutions exist. In spite of the fact that the effect of the couple of forces is formally equivalent to that of the torque, the physical sense is quite another. This lack seems to be only of formal character, but it is not true, as it is mentioned in previously. To eliminate this formal uncertainty and knowing the physical existence of the independent value of the torque, we have to include the torque with its rheological model directly into mechanical expressions.

3. Torque in Euler-Newton equations

Based on the experience of the investigators shown in the previous Section, it is obvious that the torque, as the independent measure of bodies interaction, exists. These torques do not appear automatically in the equations as dynamical equations are obtained treating the rigid body as a system of indefinite number of particles (see Sect. 2). Namely, the dynamic laws for the system of particles are generalized for the body, and as the torque does not act on the particle it is impossible for the torque to appear in the differential equation of motion.

To eliminate this disadvantage, let us consider d'Alembert's principle of equilibrium for a rigid body. Due to d'Alembert's principle for a system of forces and torques applied to a rigid body to be balanced, it is necessary and sufficient that the resultant force vector $\mathbf{F}_r$ and the resultant moment vector $\mathfrak{M}_{rC}$, with respect to the mass centre C, should be equal to zero

$$\mathbf{F}_r = 0 \qquad\qquad \mathfrak{M}_{rC} = 0 \tag{3.1}$$

Let us discuss two balance equations (3.1):

1) The balance of forces for a rigid body exists if the sum of the resultants of the inertial forces $\mathbf{I} = \sum_i \mathbf{I}_i$, active forces $\mathbf{F} = \sum_i \mathbf{F}_i$ and passive forces $\mathbf{N} = \sum_i \mathbf{N}_i$ (constraint reactions) of the body, is zero

$$\mathbf{I} + \mathbf{F} + \mathbf{N} = \mathbf{0} \tag{3.2}$$

According to (1.5), the inertial force is $\mathbf{I} = -\mathbf{a}_C m$. If the resultant of passive forces is zero, relation (3.2) transforms into (1.5).

2) The balance of moments of forces and torques for a rigid body exists if the sum of the resultant moment of inertial forces ($\mathbf{M}_C^{\mathbf{I}} = \sum_i \mathbf{M}_C^{\mathbf{I}_i}$), active and passive forces with respect to the mass centre ($\mathbf{M}_C^{\mathbf{F}} = \sum_i \mathbf{M}_C^{\mathbf{F}_i}$ and $\mathbf{M}_C^{\mathbf{N}} = \sum_i \mathbf{M}_C^{\mathbf{N}_i}$), and of the resultant active and passive torques ($\mathfrak{M}$ and $\mathfrak{R}$) is equal to zero

$$\mathbf{M}_C^{\mathbf{I}} + \mathbf{M}_C^{\mathbf{F}} + \mathbf{M}_C^{\mathbf{N}} + \mathfrak{M} + \mathfrak{R} = \mathbf{0} \tag{3.3}$$

Substituting the relation for the moment of the inertial force into (3.3)

$$\mathbf{M}_C^{\mathbf{I}} = -\int\limits_{(m)} \boldsymbol{\rho} \times \mathbf{a}_\rho \, dm = -\int\limits_{(m)} \frac{d}{dt}(\boldsymbol{\rho} \times \mathbf{v}) \, dm \tag{3.4}$$

and assuming that no constraints exist ($\mathbf{M}_C^{\mathbf{N}} = 0$, $\mathfrak{R} = 0$), a simplified version of (3.3) is

$$\int\limits_{(m)} \frac{d}{dt}(\boldsymbol{\rho} \times \mathbf{v})\, dm = \mathbf{M}_C^{\mathbf{F}} + \mathfrak{M} \tag{3.5}$$

where $\mathbf{v} = \boldsymbol{\omega} \times \boldsymbol{\rho}$ and $\boldsymbol{\omega}$ is the angular velocity of the body. If no active torque acts, Eq. (3.5) is equivalent to (1.6).

Analyzing relation (3.5), it is obvious that the rotation of the body may be affected not only by the moment of the force but also by the moment generated by the torque $\mathfrak{M}$. Even, if the moment of forces is zero, Eq. (3.5) is

$$\int\limits_{(m)} \frac{d}{dt}(\boldsymbol{\rho} \times \mathbf{v})\, dm = \mathfrak{M} \tag{3.6}$$

and the rotation around the mass centre C is forced by the torque. This fact has already been known for a long time, but it is not explicitly shown.

Using relation (3.6) and the assumption that

$$\boldsymbol{\rho} = x\mathbf{i} + y\mathbf{j} \qquad\qquad \boldsymbol{\omega} = \dot{\varphi}\mathbf{k} \tag{3.7}$$

where x and y are coordinates of an arbitrary particle dm, $\dot{\varphi}$ is the intensity of the angular velocity of the body and $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit vectors of the coordinate system $Oxyz$, the rotation of the body around the axis z under influence of the torque $\mathfrak{M}$ is expressed as

$$J_z\ddot{\varphi} = \mathfrak{M} \tag{3.8}$$

where $\int_{(m)}(x^2 + y^2)\; dm$ is the so called moment of inertia J_z for the z axis. The moment of inertia J_z can also be determined experimentally, if the mass and the position of the mass center C are known.

The obtained relation can be discussed in the sense of the first and the second Newton law:

Law I: Every body preserves in its state of rest, or uniform rotation around an axis, unless it is compelled to change that state by a torque impressed thereon.

Law II: The alteration of rotation around an axis is ever proportional to the motive torque impressed, and is made around and in the direction in which that torque is impressed. The coefficient of proportionality is the moment of inertia of the body for the corresponding rotation axis.

4. Torque in the general equation of dynamics

Let us multiply Eq. (3.2) with the virtual displacement of the mass centre $\delta\mathbf{r}_C$ and Eq. (3.3) with the virtual rotation angle $\delta\mathbf{\Phi}_k$ around an arbitrary axis in C, respectively

$$\begin{aligned}
&\sum_i(\mathbf{I}_i + \mathbf{F}_i + \mathbf{N}_i)\delta\mathbf{r}_C = 0\\
&\sum_i(\mathbf{M}_C^{\mathbf{I}_i} + \mathbf{M}_C^{\mathbf{F}_i} + \mathbf{M}_C^{\mathbf{N}_i})\delta\mathbf{\Phi}_k + (\mathfrak{M} + \mathfrak{R})\delta\mathbf{\Phi}_k = 0
\end{aligned} \tag{4.1}$$

where the moments of the inertial force $\mathbf{I}_i$, active force $\mathbf{F}_i$ and passive force $\mathbf{N}_i$ are

$$\mathbf{M}_C^{\mathbf{I}_i} = \boldsymbol{\rho}_i \times \mathbf{I}_i \qquad\qquad \mathbf{M}_C^{\mathbf{F}_i} = \boldsymbol{\rho}_i \times \mathbf{F}_i \qquad\qquad \mathbf{M}_C^{\mathbf{N}_i} = \boldsymbol{\rho}_i \times \mathbf{N}_i \tag{4.2}$$

and $\boldsymbol{\rho}_i$ is the position vector of the particle to the mass centre C. Substituting (4.2) into $(4.1)_2$, it follows

$$\sum_i [\boldsymbol{\rho}_i \times (\mathbf{I}_i + \mathbf{F}_i + \mathbf{N}_i)]\delta\boldsymbol{\Phi}_k + (\mathfrak{M} + \mathfrak{R})\delta\boldsymbol{\Phi}_k = 0 \tag{4.3}$$

Using the fact that the position of the i-th particle due to the position of the mass centre C is

$$\mathbf{r}_i = \mathbf{r}_C + \boldsymbol{\rho}_i \tag{4.4}$$

we obtain the virtual displacement of the mass centre

$$\delta\mathbf{r}_C = \delta\mathbf{r}_i - \delta\boldsymbol{\rho}_i \tag{4.5}$$

and the virtual displacement due to body rotation for the angle $\delta\boldsymbol{\Phi}_k$

$$\delta\boldsymbol{\rho}_i = \delta\boldsymbol{\Phi}_k \times \boldsymbol{\rho}_i \tag{4.6}$$

Substituting (4.5) and (4.6) into $(4.1)_1$, we obtain

$$\sum_i (\mathbf{I}_i + \mathbf{F}_i + \mathbf{N}_i)\delta\mathbf{r}_i = \sum_i (\mathbf{I}_i + \mathbf{F}_i + \mathbf{N}_i)(\delta\boldsymbol{\Phi}_k \times \boldsymbol{\rho}_i) \tag{4.7}$$

Using the property of vector multiplication

$$\sum_i (\boldsymbol{\rho}_i \times (\mathbf{I}_i + \mathbf{F}_i + \mathbf{N}_i)\delta\boldsymbol{\Phi}_k = \sum_i (\mathbf{I}_i + \mathbf{F}_i + \mathbf{N}_i)(\delta\boldsymbol{\Phi}_k \times \boldsymbol{\rho}_i) \tag{4.8}$$

and relation (4.7), Eq. (4.3) is rewritten as

$$\sum_i (\mathbf{I}_i + \mathbf{F}_i + \mathbf{N}_i)\delta\mathbf{r}_i + (\mathfrak{M} + \mathfrak{R})\delta\boldsymbol{\Phi}_k = 0 \tag{4.9}$$

Introducing into Eq. (4.9) the virtual works of the inertial forces δA^i, active forces and torques δA^a, and also passive forces and torques δA^p on the virtual displacement and rotation

$$\begin{aligned} &\delta A^i = \sum_i \mathbf{I}_i\delta\mathbf{r}_i \qquad \delta A^a = \sum_i \mathbf{F}_i\delta\mathbf{r}_i + \sum_k \mathfrak{M}\delta\boldsymbol{\Phi}_k \\ &\delta A^p = \sum_i \mathbf{N}_i\delta\mathbf{r}_i + \sum_k \mathfrak{R}\delta\boldsymbol{\Phi}_k \end{aligned} \tag{4.10}$$

yields

$$\delta A^i + \delta A^a + \delta A^p = 0 \tag{4.11}$$

Equation (4.12) expresses the *Lagrange-D'Alambert principle* (*The General Equation of Dynamics*): the total virtual work on the virtual displacement and rotation of all inertial forces, all active forces and torques, and for all reactive forces and torques of nonideal constraints during the motion is zero. It is worth to say that the active torques $\mathfrak{M}$ and reactive torques $\mathfrak{R}$ of nonideal constraints give the virtual work on the virtual rotation for the virtual rotation angle $\delta\boldsymbol{\Phi}_k$. Namely, expression (4.12) includes not only the virtual works of the forces (as it is usual – see for example Starzhunskii (1982)), but also the virtual works of the active and reactive torques.

For the ideal geometric constraints, the virtual work of the reaction forces and reaction torques on the virtual displacement and rotation is zero, and relation (4.11) simplifies into

$$\delta A^i + \delta A^a = 0 \tag{4.12}$$

where the virtual work of the active torques on the virtual rotation is included into consideration.

If the system of bodies has n degrees of freedom and n generalized coordinates q_α, where $\alpha = 1, 2, \ldots, n$, the virtual displacement and rotation angle are defined up to the first order of the virtual coordinate variations as

$$\delta\mathbf{r}_i = \sum_\alpha \frac{\partial \mathbf{r}_i}{\partial q_\alpha}\delta q_\alpha \qquad \delta\mathbf{\Phi}_k = \sum_\alpha \frac{\partial \mathbf{\Phi}_k}{\partial q_\alpha}\delta q_\alpha \tag{4.13}$$

Substituting (4.12) into (4.9), gives

$$-\sum_\alpha \Big(\sum_i \mathbf{I}_i \frac{\partial \mathbf{r}_i}{\partial q_\alpha}\Big)\delta q_\alpha = \sum_\alpha \Big(\sum_i (\mathbf{F}_i + \mathbf{N}_i)\frac{\partial \mathbf{r}_i}{\partial q_\alpha} + (\mathfrak{M} + \mathfrak{R})\frac{\partial \mathbf{\Phi}_k}{\partial q_\alpha}\Big)\delta q_\alpha \tag{4.14}$$

Let us define the generalized force Q_α and the generalized inertial force Z_α

$$Z_\alpha = \sum_i \mathbf{I}_i \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \qquad Q_\alpha = \sum_i (\mathbf{F}_i + \mathbf{N}_i)\frac{\partial \mathbf{r}_i}{\partial q_\alpha} + (\mathfrak{M} + \mathfrak{R})\frac{\partial \mathbf{\Phi}_k}{\partial q_\alpha} \tag{4.15}$$

Equation (4.14) is rewritten as

$$\sum_\alpha (Z_\alpha + Q_\alpha)\delta q_\alpha = 0 \tag{4.16}$$

Relation (4.16) is the *Lagrange-d'Alembert principle in generalized coordinates*: during motion, the sum of the products of generalized forces and generalized inertial forces with the generalized coordinates, is zero. The generalized forces are generated not only from the forces, as it is usual, but also from the torques.

Remark: Introducing the well known formulation of the generalized inertial force based on the kinetic energy T

$$Z_\alpha = -\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha} + \frac{\partial T}{\partial q_\alpha} \tag{4.17}$$

into (4.16) and separating the relations for every independent variation of the generalized coordinate δq_α, the well known system of n second kind Lagrange's differential equations is obtained

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} = Q_\alpha \qquad \alpha = 1, 2, \ldots, n \tag{4.18}$$

The generalized force Q_α in (4.18) includes the action of torques, too.

Equations (4.1) are written for one rigid body. For the case of N rigid bodies, all active forces and torques and those of the nonideal constraints have to be included into consideration, and the corresponding generalized force transforms into

$$Q_\alpha = \sum_i (\mathbf{F}_i + \mathbf{N}_i)\frac{\partial \mathbf{r}_i}{\partial q_\alpha} + \sum_{k=1}^{N} (\mathfrak{M} + \mathfrak{R})\frac{\partial \mathbf{\Phi}_k}{\partial q_\alpha} \tag{4.19}$$

All of the statements and definitions given in the paper remain invariant.

Acknowledgement

This research has been supported by the Provincial Secretariat for Science and Technological Development, Autonomous Province of Vojvodina (Proj. No. 114-451-2094/2011) and Ministry of Science of Serbia (Proj. No. ON 174028 and IT 41007).

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O momencie, jako mierze mechanicznego oddziaływania w zasadach dynamiki

Streszczenie

W większości rozważań, zasady dynamiki są wyrażane poprzez siły rozumiane jako miary mechanicznych oddziaływań pomiędzy ciałami. Celem tej pracy jest rozszerzenie zwyczajowego podejścia do aksjomatów, praw i zasad dynamiki o pojęcie momentu jako niezależnej miary mechanicznego oddziaływania. Wielkość tę wstawiono do zasady d'Alemberta w postaci momentu czynnego i biernego reakcyjnego. Przedstawiono również momentowe równania Eulera-Newtona dla ogólnego przypadku ruchu bryły sztywnej. Na nowo sformułowano ogólne równanie dynamiki poprzez wstawienie pracy przygotowanej momentu na przemieszczeniu kątowym. Dodatkową dyskusją objęto trzy zasady dynamiki Newtona.

Manuscript received January 28, 2012; accepted for print April 23, 2012