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HOW TO SPEED UP FAST FOURIER TRANSFORM COMPUTATION – A RECURRENCE METHOD

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There is a strong need for new approaches to non-stationary signal analysis. These signals call for advanced time-frequency analysis techniques (see References). The recurrence procedure for calculating the Fast Fourier Transform, which enables calculations to be made several times faster, especially in the case of precise time-frequency approach, has been establish in the present contribution. This method was invented in the course of transient signal analysis carried out in the Institute of Aviation upon the aircraft impulse response in flight.

Key words: Fourier Transform, non-stationary signal, identification

1. Introduction

One faces significant limitations when applying an ordinary spectral analysis to highly non-stationary signal. These are: speech, music, noise of significant variation with time, rapidly decaying signal of the aircraft impulse response in flight (Fig.1), etc. Information about the transients cannot be extracted from traditional spectral analysis techniques based on the Fast Fourier Transform; i.e., one cannot determine time-varying properties of a spectrum. The aforementioned information refers to e.g., flutter properties of the aircraft in flight, concert hall quality or noise sources detection possibilities.

Lenort (1989) applied the signal decay in time to determination of the system damping coefficient, using the Fourier Transform (Fig.3).

For proper graphical presentation of the time-frequency analysis results an additional, third coordinate is to be introduced. Lenort (1989) used the τ-axis representing the shift of analysis window of time duration T_a (Fig.3). The diagram presented was fairly clear since the shift $\Delta\tau$ of analysis window

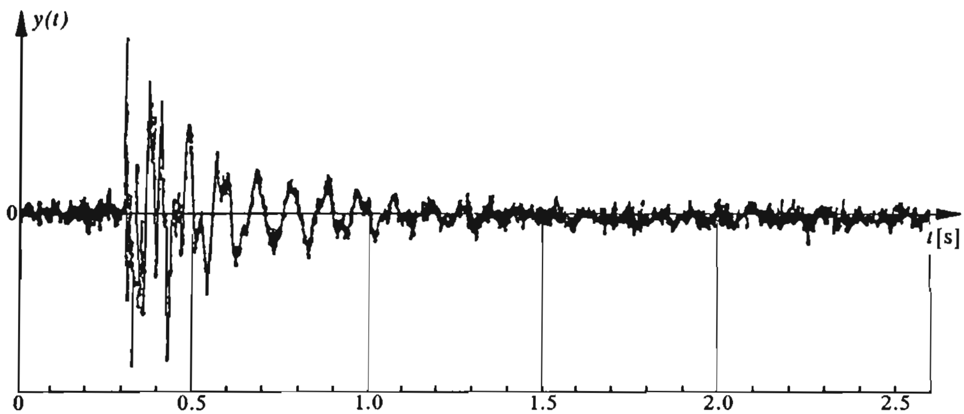

Fig. 1. Sample impulse aircraft wing response in flight

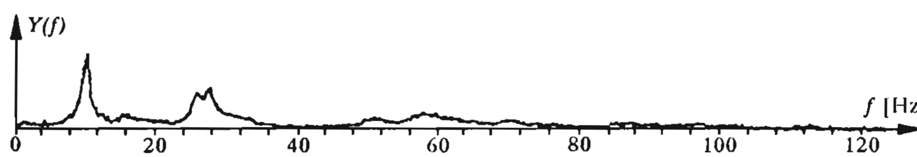

Fig. 2. The response of Fig.1 after traditional spectral analysis

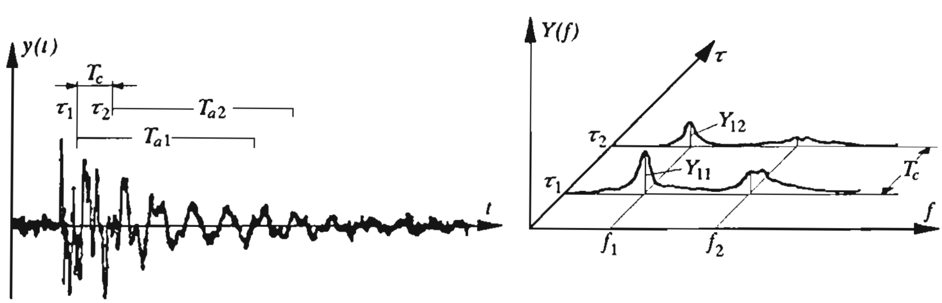

Fig. 3. Time-frequency analysis applied to determination of a damping coefficient. For the first vibration component of the frequency f_1 we have $\lambda_1 = \frac{1}{T_c} \ln \frac{|Y_{11}|}{|Y_{12}|}$

was equal to T_c, with T_c representing the period of the analysed component vibration.

The analysis window shift should be as small as possible i.e., equal to Δt (period of the signal sampling) when analysing the signal of speech-type or other highly non-stationary acoustic ones. Such compact presentation of spectral patterns makes the diagram nearly illegible. Brüel & Kjær put therefore forward [5,6,7] the concept of contour plots (Fig.4).

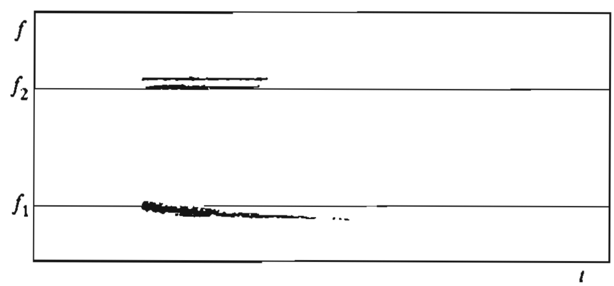

Fig. 4. Simplified contour plot of the results obtained after the time-frequency analysis of the signal of Fig.1. The frequency variation with time may prove the object non-linearity

As in the music score, vertical axis represents the vibration frequency, while the horizontal one is time axis. The intensity of black colour reflects respective spectral component magnitude. The computer set supplied with colour monitor and printer enables the spectral characteristics to be presented in the form similar to that used in geographical maps. The resolution of analysis method both in time and frequency domain should be as high as possible since it affects the precision of these contour plots. The following product enables the analysis method resolution quality to be assessed

$$r = T_a \Delta f$$

where

T_a – analysis window time duration

Δf – resolution in frequency domain.

The smaller the value of r is, the more precise analysis is being performed.

In general, the following three method [5,6,7] are applied when performing the time-frequency analysis: Short Time Fourier Transform (STFT), Wavelet Transform (WT) and Wigner-Ville distribution.

The Author put forward the concept of introducing the Discrete Fourier Transform (DFT), as being already tested in practice and revealing high resolution for close sampling of time-dependent signals, into solution to the aforementioned problems (cf Lenort (1995)).

When calculating the STFT, usually the FFT algorithm is to be used with a relatively small number of samples, i.e., $N = 256$ or $N = 128$ [5,7].

A comparison between characteristics of the STFT and the DFT being proposed is given in Table 1 for the signal sampling frequency $f_s = 25\,600$ samples/s.

When using the DFT approach it was assumed that $T_a = T_c$.

Table 1

f [Hz]		100	500	1000	5000
STFT (FFT)	Δf [Hz]	100	100	100	100
$N = 256$	$\Delta f/f$ [%]	100	20	10	2
	$T_a\Delta f$	1	1	1	1
DFT	Δf [Hz]	0.39	9.76	39.0	976
$T_a = T_c$	$\Delta f/f$ [%]	0.39	1.95	3.9	19.5
	$T_a\Delta f$	0.0039	0.0195	0.039	0.195

A comparison between the analysis window time duration imposed in SFTF and DFT methods, respectively is shown in Fig.5.

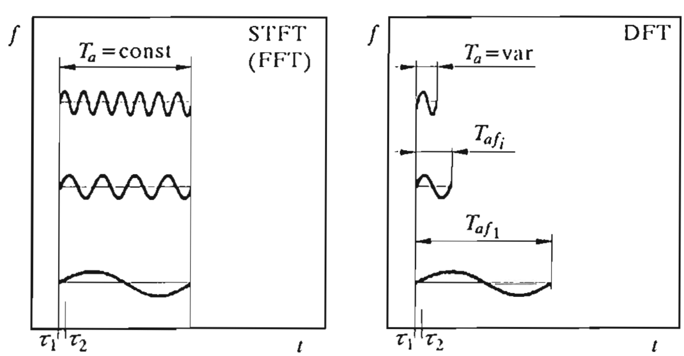

Fig. 5. Demonstration of the signal time duration. In contrast to the FFT properties in the DFT method the analysis window time duration is frequency-dependent and it can be assumed that $T_{af} = nT_{cf}$, $n = 1, 2, \ldots$

The FFT procedure deals with the vibration segment of T_{cb} time duration, encompassing one cycle of vibration at the first frequency, i.e., $T_a = T_{cb}$, despite the vibration frequency range.

On the contrary, when employing the DFT method it is possible to analyse the vibration segment duration of which is frequency-dependent. The best possible resolution in time domain is achieved for the vibration segment of time duration equal to the vibration period at a given frequency f, i.e., $T_{af} = T_{cf}$.

For almost stationary signals it can be, of course, assumed that

$$T_{af} = 3T_{cf}$$

putting aside the resolution aspect.

It follows from Table 1 that the FFT method suits better when analysing signals of higher frequency, while the DFT method being put forward works better for low frequency signal analysis purposes.

The analysis window of time duration T_a, originating from the point τ_1, after the analysis has been performed is shifted to the point τ_2. For the reasonable results to be obtained this shift should be equal to Δt, with Δt representing the signal sampling period. With this window shift there arises the possibility for speeding up the Fourier Transform computing since the results of signal processing made at a previous calculation stage can be exploited, at least partially, at the successive ones.

2. Recurrence method for the Fourier Transform calculation

The continuous Fourier Transform of a time-dependent signal of definite time duration T_a can be represented as follows

$$Y(f) = \frac{2}{T_a} \int_0^{T_a} y(t) \exp(-\mathrm{j}2\pi f t)\, dt \tag{2.1}$$

where $\mathrm{j} = \sqrt{-1}$ and

$y(t)$ – signal being analysed

$Y(f)$ – complex function in frequency domain.

The Fast Fourier Transform can be computed after introducing the following time-discretization of the function $f(t)$

$$t = n\Delta t \qquad n = 0, 1, ..., N-1 \tag{2.2}$$

In result of which the discrete values of the function $Y(f)$ calculated at the points

$$f = k\Delta f \qquad k = 0, 1, ..., N-1 \tag{2.3}$$

are obtained according to the following formula

$$Y(k\Delta f) = \frac{2}{N\Delta t}\sum_{n=0}^{N-1} y(n\Delta t)\exp(-\mathrm{j}2\pi k\Delta f n\Delta t)\Delta t \qquad k = 0,1,...,N-1 \tag{2.4}$$

where

Δt – sampling period of the time-dependent signal $y(t)$

Δf – resolution in frequency domain, being equal for the FFT

$$\Delta f = \frac{1}{T_a} = \frac{1}{N\Delta t} \tag{2.5}$$

where N represents the number of samples in the signal segment under investigation.

Generally, the function $y(t)$ may take complex values, but for the sake of transformations legibility let us assume that this function has only real values, which is really true in the case of the vibration signal under investigation (Fig.1).

Upon substituting Eq (2.5) into Eq (2.4) we have

$$Y(k\Delta f) = \frac{2}{N}\sum_{n=0}^{N-1} y(n\Delta t)\exp\Big(-\mathrm{j}\frac{2\pi}{N}kn\Big) \qquad k = 0,1,...,N-1 \tag{2.6}$$

Let us focus our attention on the problem of finding the way of fastest possible calculation of the above sum, utilising the results obtained for the previous analysis window position. Assume we have obtained $Y_0(k\Delta f)$ and now, after shifting the analysis window by Δt we have to calculate the value of $Y_1(k\Delta f)$ for the same value of k.

Applying some simplifying transformations to Eq (2.6) yields the following clear form

$$\sum_{n=0}^{N-1} y_n \exp(-\mathrm{j}\alpha_n) = \sum_{n=0}^{N-1} y_n \cos\alpha_n - \mathrm{j}\sum_{n=0}^{N-1} y_n \sin\alpha_n = Y_{0_R} + \mathrm{j}Y_{0_{Im}} \tag{2.7}$$

where

$$y_n = y(n\Delta t) \qquad \alpha_n = \Big(\frac{2\pi}{N}k\Big)n \tag{2.8}$$

For the initial and first positions, respectively, of the analysis window we can write the following formulae for the real parts of transformations

$$
\begin{aligned}
Y0_R &= \sum_{n=0}^{N-1} y_n \cos\alpha_n = y_0\cos\alpha_0 + \\
&+ (y_1\cos\alpha_1 + y_2\cos\alpha_2 + ... + y_{N-1}\cos\alpha_{N-1})
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
Y1_R &= \sum_{n=1}^{N} y_n \cos\alpha_{n-1} = (y_1\cos\alpha_0 + y_2\cos\alpha_1 + ... + y_{N-1}\cos\alpha_{N-2}) + \\
&+ y_N\cos\alpha_{N-1}
\end{aligned}
\tag{2.10}
$$

And similarly for the imaginary parts we have

$$
\begin{aligned}
-Y0_{Im} &= \sum_{n=0}^{N-1} y_n \sin\alpha_n = y_0\sin\alpha_0 + \\
&+ (y_1\sin\alpha_1 + y_2\sin\alpha_2 + ... + y_{N-1}\sin\alpha_{N-1})
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
-Y1_{Im} &= \sum_{n=1}^{N} y_n \sin\alpha_{n-1} = (y_1\sin\alpha_0 + y_2\sin\alpha_1 + ... + y_{N-1}\sin\alpha_{N-2}) + \\
&+ y_N\sin\alpha_{N-1}
\end{aligned}
\tag{2.12}
$$

By virtue of Eq $(2.8)_2$ we can write

$$
\begin{array}{ll}
\cos\alpha_0 = \cos 0 = 1 & \sin\alpha_0 = 0 \\
\cos\alpha_{N-1} = \cos\alpha_1 & \sin\alpha_{N-1} = -\sin\alpha_1
\end{array}
$$

Substituting the above formulae into Eqs (2.9)÷(2.12) yields

$$
Y0_R = y_0\cos\alpha_0 + \sum_{n=1}^{N-1} y_n\cos\alpha_n = y_0 + W0_R \tag{2.13}
$$

$$
Y1_R = \sum_{n=1}^{N-1} y_n\cos\alpha_{n-1} + y_N\cos\alpha_{N-1} = W1_R + y_N\cos\alpha_1 \tag{2.14}
$$

$$
-Y0_{Im} = y_0\sin\alpha_0 + \sum_{n=1}^{N-1} y_n\sin\alpha_n = W0_{Im} \tag{2.15}
$$

$$
-Y1_{Im} = \sum_{n=1}^{N-1} y_n\sin\alpha_{n-1} + y_N\sin\alpha_{N-1} = W1_{Im} - y_N\sin\alpha_1 \tag{2.16}
$$

Upon introducing the above formulae into the equations given above we have

$$W_{0_R} + \mathrm{j}W_{0_{Im}} = \sum_{n=1}^{N-1} y_n \cos\alpha_n + \mathrm{j}\sum_{n=1}^{N-1} y_n \sin\alpha_n = \sum_{n=1}^{N-1} y_n \exp(\mathrm{j}\alpha_n) =$$

$$= \sum_{n=1}^{N-1} y_n \exp\Big[\mathrm{j}\Big(\frac{2\pi}{N}k\Big)n\Big] = \exp(\mathrm{j}\alpha_1)\sum_{n=1}^{N-1} y_n \exp\Big[\mathrm{j}\Big(\frac{2\pi}{N}k\Big)(n-1)\Big] = \tag{2.17}$$

$$= \exp(\mathrm{j}\alpha_1)\sum_{n=1}^{N-1} y_n \exp(\mathrm{j}\alpha_{n-1}) =$$

$$= \exp(\mathrm{j}\alpha_1)\Big(\sum_{n=1}^{N-1} y_n \cos\alpha_{n-1} + \mathrm{j}\sum_{n=1}^{N-1} y_n \sin\alpha_{n-1}\Big) = \exp(\mathrm{j}\alpha_1)(W_{1_R} + \mathrm{j}W_{1_{Im}})$$

And then

$$W_{1_R} + \mathrm{j}W_{1_{Im}} = (W_{0_R} + \mathrm{j}W_{0_{Im}})\exp(-\mathrm{j}\alpha_1) = (W_{0_R} + \mathrm{j}W_{0_{Im}})(\cos\alpha_1 - \mathrm{j}\sin\alpha_1) \tag{2.18}$$

Therefore one obtains

$$W_{1_R} = W_{0_R}\cos\alpha_1 + W_{0_{Im}}\sin\alpha_1 \tag{2.19}$$

$$W_{1_{Im}} = W_{0_{Im}}\cos\alpha_1 - W_{0_R}\sin\alpha_1 \tag{2.20}$$

Instead of calculating the values of W_{1_R} and $W_{1_{Im}}$ from Eqs (2.10) and (2.12) (sums of relevant products in brackets) one may use the values of W_{0_R} and $W_{0_{Im}}$ calculated at the previous step.

By virtue of Eqs (2.13) and (2.15) we have

$$W_{0_R} = Y_{0_R} - y_0 \qquad W_{0_{Im}} = -Y_{0_{Im}}$$

Substituting the above formulae into Eqs (2.19) and (2.20) yields

$$W_{1_R} = (Y_{0_R} - y_0)\cos\alpha_1 - Y_{0_{Im}}\sin\alpha_1 \tag{2.21}$$

$$W_{1_{Im}} = -Y_{0_{Im}}\cos\alpha_1 - (Y_{0_R} - y_0)\sin\alpha_1 \tag{2.22}$$

While from Eqs (2.14) and (2.16) it follows

$$W_{1_R} = Y_{1_R} - y_N\cos\alpha_1 \tag{2.23}$$

$$W_{1_{Im}} = -Y_{1_{Im}} + y_N\sin\alpha_1 \tag{2.24}$$

After substituting Eqs(2.23) and (2.24) into Eqs (2.21) and (2.22) we have

$$Y_{1_R} - y_N\cos\alpha_1 = (Y_{0_R} - y_0)\cos\alpha_1 - Y_{0_{Im}}\sin\alpha_1$$

$$-Y_{1_{Im}} + y_N\sin\alpha_1 = -Y_{0_{Im}}\cos\alpha_1 - (Y_{0_R} - y_0)\sin\alpha_1$$

And finally we have

$$Y_{1_R} = (Y_{0_R} - y_0 + y_N)\cos\alpha_1 - Y_{0_{Im}}\sin\alpha_1 \tag{2.25}$$

$$Y_{1_{Im}} = (Y_{0_R} - y_0 + y_N)\sin\alpha_1 + Y_{0_{Im}}\cos\alpha_1 \tag{2.26}$$

In general, for an arbitrary taken analysis window position $l+1$ it is possible to utilise the results of calculations made for the previous position l, i.e.

$$Y_{(l+1)_R} = (Y_{l_R} - y_l + y_{N+l})\cos\alpha_1 - Y_{l_{Im}}\sin\alpha_1 \tag{2.27}$$

$$Y_{(l+1)_{Im}} = (Y_{l_R} - y_l + y_{N+l})\sin\alpha_1 + Y_{l_{Im}}\cos\alpha_1 \tag{2.28}$$

Therefore it can be easily seen that when using the recurrence method being put forward four products should be calculated. The number of multiplications made is assumed to represent an approximated measure of the transform calculation time (cf Brigham (1974)). When applying the FFT method to calculation of the N transform values (N points of a spectrum) one should, for the real signal $y(t)$ calculate $2(N/2)\log_2 N$ products, respectively (cf Brigham (1974)). It is well known that the number of $N/2.56$ points of spectrum obtained is assumed to be calculated accurately. So for one point of spectrum to be calculated one should make

$$\frac{2\frac{N}{2}\log_2 N}{\frac{N}{2.56}} = 2.56\log_2 N \tag{2.29}$$

multiplications on the average.

For $N = 256$ (usually taken in the STFT with the use of FFT) we have 20.48 products, while when applying the recurrence method the time of calculations is approximately 5 times shorter.

The difference between respective calculation times becomes even more apparent for calculations of higher accuracy, e.g., for closer sampling, i.e., at greater values of N. It can be easily shown that for $N = 2048$ the results can be obtained 7 times faster when using the proposed method. It can be achieved, however only for the time-frequency analysis with the analysis window shifted by Δt.

Shifting the analysis window by $2\Delta t$ calls for 8 multiplications to be made, while the $3\Delta t$ shift involves 12 multiplications, respectively, etc. It is therefore obvious that the application of recurrence method is reasonable only to calculations with the Δt window shift. The initial spectrum is obtained using the FFT and the recurrence approach is taken for further time-frequency spectrum calculation.

Table 1 proves that better resolution of time-frequency analysis is to be obtained when using the DFT method being proposed, especially for the frequency range below 10 kHz.

The initial spectrum can be found with the use of DFT method, and then in result of the recurrence approach, Eqs (2.27) and (2.28), the successive spectra can be obtained. In this case, however, N is frequency-dependent (Fig.5), i.e., the higher frequency is analysed the smaller number of sampled can be accepted in calculations.

In the presented approach the signal segment being analysed has not been pre-processed (e.g., using the Hann or Gaussian window). Therefore no signal distortion is introduced within the analysis window time duration.

3. Conclusions

- The recurrence method being put forward enables the time-frequency analysis of non-stationary signals to be performed several times faster in comparison with the other methods.
- The recurrence method supplied with the schemes of the DFT being proposed enables shorter calculation times and better resolutions to be obtained in comparison with the STFT method.

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Jak liczyć szybciej szybką transformatę Fouriera – metoda rekurencyjna

Streszczenie

Analiza sygnałów niestacjonarnych wymaga nowych metod postępowania. Do tych celów opracowano metody analizy czasowo-częstotliwościowej (patrz Literatura). W pracy wyprowadzono wzór do rekurencyjnego obliczana transformaty Fouriera, który kilkakrotnie przyspiesza obliczenia w przypadku dokładnej analizy czasowo-częstotliwościowej. Metoda powstała w związku z prowadzonymi w Instytucie Lotnictwa analizami sygnałów odpowiedzi impulsowych samolotu w locie.

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