MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
3, 34, 1996

ANALYSIS OF TORSIONAL VIBRATIONS OF DISCRETE-CONTINUOUS SYSTEMS IN A CLASS OF GENERALIZED FUNCTIONS

STANISŁAW KASPRZYK

Faculty of Mechanical Engineering and Robotics, Academy of Mining and Metallurgy
e-mail: kasprzyk@galaxy.uci.agh.edu.pl

The note presents a method of solving the mixed problem for the system of differential equations with distributional coefficients modelling small torsional vibrations of an elastic shaft carrying a number of rigid masses. The solution is searched in a class of generalized functions.

Key words: vibrations, discrete-continuous system, generalized functions

1. Introduction

The paper is devoted to the study of small torsional vibrations of an elastic shaft with variable stiffness carrying a number of rigid masses (the discrete-continuous system) using the model represented by equations with distributional coefficients.

The well known classical approach to the analysis of compound mechanical systems (cf e.g., Beer and Johnson (1977); Inman (1994)) consists either in discretization of the system, i.e. replacing it by a number of concentrated masses connected by springs and dampers, or in dividing it into typical elements with continuously distributed masses, each having constant mass density and stiffness, which are next analyzed separately.

For more complicated systems the first approach may lead to significant discrepancies between computed results based on a numerical model and the observed behaviour of the system.

The second one gives more accurate results but computations are more complicated since each element has to be analyzed separately, which requires solving a number of initial boundary value problems, and next the obtained solutions must be fitted together which involves additional computations.

The finite element techniques based on the Galerkin approach (cf White (1985)), very powerful and popular in applications, give approximate numerical models, the accuracy of which depends on the number of elements used. In the case when eigenfrequencies of the system are to be determined the method may give solutions suffering from big numerical errors.

For the recent works using a classical approach to the treatment of discrete-continuous systems see e.g., papers by Nadolski (1994), Pielorz (1992), (1995). Pielorz (1992) gives an extensive bibliography of the subject.

In this paper we propose a different approach to the modelling of mechanical systems, consisting in the use the discrete-continuous models, i.e. applying representation by equations with distributional coefficients. Solutions of such equations will be called *the generalized functions*, since they satisfy the equation in generalized (distributional) sense.

The use of distribution equations makes it dispensible dividing the analyzed structure into separate parts. Note that in the case of constant coefficients it is possible to obtain the explicit formula for the solution and in consequence the equation in natural frequencies, which in comparison with the finite element method, considerably simplifies the problem of finding eigenfrequencies of the system.

Application of distribution equations to modelling the mechanical systems has been used by various authors (cf e.g., Pan and Hohenstein (1981); Persson (1990)) and references therein. Kasprzyk (1984) and (1994) this approach have applied to investigation of transverse or longitudinal vibrations of the discrete-continuous systems.

In the present paper it is shown that the method of Kasprzyk (1984) and (1994), can be also applied to the situation when in the equation the distribution appears as a coefficient at the first derivative of the unknown function. Such a situation appears when modelling torsional damped vibrations of elastic shafts with concentrated loads under external moments.

2. Mathematical model

To be specific, we will carry out the analysis for the discrete-continuous system as depicted in Fig.1.

The shaft $0A$ of length l [m] with the mass moment of inertia J_0 [kgm] per unit lenght is subjected to external torques [Nm] acting at points x_i, varying

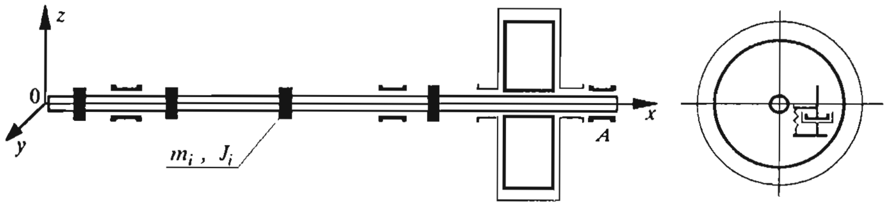

Fig. 1. The elastic shaft with concentrated rigid masses and a dynamic damper

in time, described by the functions $f_i(t)$ $i = 1,2,\dots,n$ and to a damper located at the point x_{n+1}. A rigid body with the mass m_i [kg] and the axial moment of inertia J_i [kgm^2] is attached to the shaft at a point x_i. It is assumed that a cross-section of the shaft at x_i does not undergo torsion. The shaft is connected with a damper at the point x_{n+1}. The (linear) dissipative and elastic elements of the damper are attached at a distance R [m] from the x-axis, the damper has the axial mass moment of inertia J_p [kgm^2].

Denote by $\widetilde{J}_{0i}$ [m^4] the polar moments of inertia, of the cross-sections of the shaft perpendicular to $0x$ axis taken at point x_i. We assume that at the remaining points the shaft has the constant moment of inertia $\widetilde{J}_0$.

We assume that the shaft has the coefficient of internal damping β [Ns/m^2], the shear modulus G [N/m^2] and the damper has a coefficient of damping $h > 0$ [Ns/m] and a coefficient of elasticity $k \geq 0$ [N/m].

Using the assumptions made above the distribution $J(x)$ of the mass moment of inertia per unit length and the geometric moment of inertia $\widetilde{J}(x,\varepsilon)$ of the shaft cross section can be written in the form

$$J(x) = J_0 + \sum_{i=1}^{n+1} J_i\delta_i \tag{2.1}$$

$$\widetilde{J}(x,\varepsilon) = \widetilde{J}_0 + \sum_{i=1}^{n} \Delta\widetilde{J}_i\Big(H(x_i^-) - H(x_i^+)\Big) \qquad \widetilde{J}_0 \geq 0 \tag{2.2}$$

where

$$\begin{array}{lll} \Delta\widetilde{J}_i = \widetilde{J}_{0i} - \widetilde{J}_0 & \delta_i = \delta(x - x_i) & \\ H(x_i^-) = H(x - x_i^-) & H(x_i^+) = H(x - x_i^+) & \\ x_i^+ = x_i + \varepsilon & x_i^- = x_i - \varepsilon & \varepsilon > 0 \end{array}$$

$H(x)$ is the Heaviside function

$$H(x) = \begin{cases} 1 & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 0 \end{cases}$$

and δ is a Dirac function with a peak at 0, i.e. δ_i denotes the Dirac distribution with a peak at x_i.

Denoting by $\varphi(x,t)$ the angle of torsion of the shaft at a point x and moment t of time and by $\theta(t)$ the angle of twist of the damper relative to the shaft, torsional vibrations of the system presented in Fig.1 are described by a system of one partial and one ordinary differential equations

$$\begin{aligned}
J(x)\frac{\partial^2\varphi}{\partial t^2} &= \frac{\partial}{\partial x}\Big[\widetilde{J}(x,\varepsilon)\frac{\partial}{\partial x}\Big(G\varphi+\beta\frac{\partial\varphi}{\partial t}\Big)\Big] + \\
&\quad -\Big[\Big(\frac{\partial\varphi}{\partial t}-\theta'\Big)hR^2+(\varphi-\theta)kR^2\Big]\delta_{n+1}+f(t) \\
J_p\theta'' &= \Big[\frac{\partial\varphi(x_{n+1},t)}{\partial t}-\theta'(t)\Big]hR^2+\Big[\varphi(x_{n+1},t)-\theta(t)\Big]kR^2 \\
f(t) &= \sum_{i=1}^{n} f_i(t)\delta_i \qquad (\cdot)'=\frac{d}{dt} \qquad (\cdot)''=\frac{d^2}{dt^2}
\end{aligned} \tag{2.3}$$

It will be convenient for further considerations to rewrite (2.3) in the form

$$\begin{aligned}
J(x)\frac{\partial^2\varphi}{\partial t^2} &= \frac{\partial}{\partial x}\Big[\widetilde{J}(x,\varepsilon)\frac{\partial}{\partial x}\Big(G\varphi+\beta\frac{\partial\varphi}{\partial t}\Big)\Big] - \frac{hR^2}{\beta}\Big[\Big(G\varphi+\beta\frac{\partial\varphi}{\partial t}\Big) + \\
&\quad -(G\theta+\beta\theta')\Big]\delta_{n+1} - R^2k_2(\varphi-\theta)\delta_{n+1}+f(t) \\
J_p\theta'' &= \frac{hR^2}{\beta}\Big[\Big(G\varphi(x_{n+1},t)+\beta\frac{\partial\varphi(x_{n+1},t)}{\partial t}\Big)-[G\theta(t)+\beta\theta'(t)]\Big] + \\
&\quad +R^2k_2\Big[\varphi(x_{n+1},t)-\theta(t)\Big]
\end{aligned} \tag{2.4}$$

where $k=k_1+k_2$, $h/k_1=\beta/G$. The solution of system (2.3) will be searched in the class of functions φ, θ satisfying conditions $\varphi(\cdot,t)\in C^1([0,l])\cap C^2(\Omega)$, $\Omega=(0,l)\setminus\{x_1,\dots,x_{n+1}\}$, $\varphi(x,\cdot),\theta(\cdot)\in C^2(0,\infty)$.

Substituting Eq (2.2) into the expression $\frac{\partial}{\partial x}\Big[\widetilde{J}(x,\varepsilon)\frac{\partial}{\partial x}\Big(G\varphi+\beta\frac{\partial\varphi}{\partial t}\Big)\Big]$ and computing the distributional derivative (cf Schwartz (1965), Ch.II) after letting $\varepsilon\to 0$ ($i=1,\dots,n$) we obtain from Eqs (2.3)

$$\begin{aligned}
&\frac{\partial}{\partial x}\Big[\widetilde{J}(x,\varepsilon)\frac{\partial}{\partial x}\Big(G\varphi+\beta\frac{\partial\varphi}{\partial t}\Big)\Big] = \widetilde{J}_0\Big(G\frac{\partial^2\varphi}{\partial x^2}+\beta\frac{\partial^3\varphi}{\partial x^2\partial t}\Big)\Big|_{\Omega} + \\
&+\widetilde{J}_0\sum_{i=1}^{n}\Big[\Big(G\frac{\partial\varphi(x_{i+},t)}{\partial x}+\beta\frac{\partial^2\varphi(x_{i+},t)}{\partial x\partial t}\Big)-\Big(G\frac{\partial\varphi(x_{i-},t)}{\partial x}+\beta\frac{\partial^2\varphi(x_{i-},t)}{\partial x\partial t}\Big)\Big]\delta_i +
\end{aligned}$$

$$+\sum_{i=1}^{n}\Delta\tilde{J}_i\Big[\Big(G\frac{\partial\varphi(x_{i+},t)}{\partial x}+\beta\frac{\partial^2\varphi(x_{i+},t)}{\partial x\partial t}\Big)+$$
$$-\Big(G\frac{\partial\varphi(x_{i-},t)}{\partial x}+\beta\frac{\partial^2\varphi(x_{i-},t)}{\partial x\partial t}\Big)\Big]\delta_i == \tilde{J}_0\Big(G\frac{\partial^2\varphi}{\partial x^2}+\beta\frac{\partial^3\varphi}{\partial x^2\partial t}\Big)+$$
$$+\sum_{i=1}^{n}\tilde{J}_i\Big[G\Big(\frac{\partial\varphi(x_{i+},t)}{\partial x}-\frac{\partial\varphi(x_{i-},t)}{\partial x}\Big)+\beta\Big(\frac{\partial^2\varphi(x_{i+},t)}{\partial x\partial t}-\frac{\partial^2\varphi(x_{i-},t)}{\partial x\partial t}\Big)\Big]\delta_i$$

where

$$\frac{\partial\varphi(x_{i+},t)}{\partial x}=\lim_{\varepsilon\to 0}\frac{\partial\varphi(x_i^+,t)}{\partial x}\qquad\qquad \frac{\partial\varphi(x_{i-},t)}{\partial x}=\lim_{\varepsilon\to 0}\frac{\partial\varphi(x_i^-,t)}{\partial x}$$

From Eqs (2.1), (2.3) and the equation above it follows that for given initial conditions the torsional vibrations of the system presented in Fig.1 are described by the solution of the system of differential equations

$$\begin{aligned}J(x)\frac{\partial^2\varphi}{\partial t^2} &= \tilde{J}_0\Big(G\frac{\partial^2\varphi}{\partial x^2}+\beta\frac{\partial^3\varphi}{\partial x^2\partial t}\Big)+\\ &+\sum_{i=1}^{n}\tilde{J}_{0i}\Big[G\Big(\frac{\partial\varphi(x_{i+},t)}{\partial x}-\frac{\partial\varphi(x_{i-},t)}{\partial x}\Big)+\beta\Big(\frac{\partial^2\varphi(x_{i+},t)}{\partial x\partial t}-\frac{\partial^2\varphi(x_{i-},t)}{\partial x\partial t}\Big)\Big]\delta_i+\\ &-hR^2\Big[\Big(\frac{\partial\varphi(x,t)}{\partial t}-\theta'(t)\Big)+\frac{k}{h}[\varphi(x,t)-\theta(t)]\Big]\delta_{n+1}+f(t)\\ J_p\theta'' &= hR^2\Big[\frac{\partial\varphi(x_{n+1},t)}{\partial t}-\theta'+\frac{[\varphi(x_{n+1},t)-\theta(t)]k}{h}\Big]\end{aligned}\tag{2.5}$$

subjected to the boundary and geometric conditions

$$\begin{aligned}&\frac{\partial}{\partial x}\varphi(0,t)=\frac{\partial}{\partial x}\varphi(l,t)=0\\ &\varphi(x_{i-},t)-\varphi(x_{i+},t)=\alpha_i\varphi(x_i,t)\quad (i=1,2,\dots,n)\quad \text{for}\quad t>0\end{aligned}\tag{2.6}$$

and the initial conditions

$$\begin{aligned}&\varphi(x,0)=\psi_1(x) &&\frac{\partial}{\partial t}\varphi(x,0)=\psi_2(x)\quad\text{for}\quad x\in(0,l)\\ &\theta(0)=\theta_1 &&\theta'(0)=\theta_2\end{aligned}\tag{2.7}$$

where

$$\varphi(x_{i+},t)=\lim_{\varepsilon\to 0}\varphi(x_i^+,t)\qquad\qquad \varphi(x_{i-},t)=\lim_{\varepsilon\to 0}\varphi(x_i^-,t)$$

The second formula in Eqs (2.6) represents geometric conditions. Constants α_i will be defined later, after solving the eigenvalue problem associated with Eqs (2.5), and the first condition Eqs (2.6).

The initial-boundary problem Eqs (2.5), (2.6) and (2.7) will be solved by the generalized Fourier method. To simplify its presentation, introduce a new independent variable $u(x,t)$ and operators C, L, L_1 defined by

$$\boldsymbol{u}(x,t) = \begin{bmatrix} \varphi(x,t) \\ \theta(t) \end{bmatrix} \qquad \boldsymbol{F}(t) = \begin{bmatrix} f(t) \\ 0 \end{bmatrix} \qquad \mathsf{C} = \begin{bmatrix} J(x) & 0 \\ 0 & J_p \end{bmatrix}$$

$$\mathsf{L} = \tilde{J}_0 G \begin{bmatrix} \dfrac{\partial^2}{partial x^2} + \sum\limits_{i=1}^{n} \dfrac{\tilde{J}_{0i}}{\tilde{J}_0}(\partial^+ - \partial^-)_i\delta_i - \dfrac{kR^2}{\tilde{J}_0 G}\delta_{n+1} & \dfrac{kR^2}{\tilde{J}_0 G}\delta_{n+1} \\ \dfrac{kR^2}{\tilde{J}_0 G}\nu_{n+1} & -\dfrac{kR^2}{\tilde{J}_0 G} \end{bmatrix}$$

$$\mathsf{L}_1 = \begin{bmatrix} \dfrac{\partial^2}{\partial x^2} + \sum\limits_{i=1}^{n} \dfrac{\tilde{J}_{0i}}{\tilde{J}_0}(\partial^+ - \partial^-)_i\delta_i - \dfrac{hR^2}{\tilde{J}_0 \beta}\delta_{n+1} & \dfrac{hR^2}{\tilde{J}_0 \beta}\delta_{n+1} \\ \dfrac{hR^2}{\tilde{J}_0 \beta}\nu_{n+1} & -\dfrac{hR^2}{\tilde{J}_0 \beta} \end{bmatrix}$$

Here

$$(\partial^+)_i\varphi(x,t) = \frac{\partial\varphi(x_{i+},t)}{\partial x} \qquad (\partial^-)_i\varphi(x,t) = \frac{\partial\varphi(x_{i-},t)}{\partial x}$$

$$\nu_{n+1}\varphi(x,t) = \varphi(x_{n+1},t)$$

It can be proved that that operator C is positive definite and operators L, L_1 are selfconjugate, negative definite with a point spectra.

In notations Eqs (2.5) assumes the form

$$\mathsf{C}\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \mathsf{L}\boldsymbol{u} + \tilde{J}_0\beta\mathsf{L}_1\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{F}(t) \tag{2.8}$$

From the definition of L and L_1 it follows immediately that

$$\frac{1}{\tilde{J}_0 G}\mathsf{L} - \mathsf{L}_1 = \frac{R^2}{\tilde{J}_0}\Big(\frac{h}{\beta} - \frac{k}{G}\Big)\mathsf{L}_3 = -\frac{R^2 k_2}{G\tilde{J}_0}\mathsf{L}_3$$

(see Eqs (2.4)), where

$$\mathsf{L}_3 = \begin{bmatrix} -\delta_{n+1} & \delta_{n+1} \\ \nu_{n+1} & -1 \end{bmatrix} \qquad \frac{h}{\beta} = \frac{k_1}{G} \qquad k_2 = k - k_1$$

Setting $\mathsf{L}_2 = -R^2k_2/(G\tilde{J}_0)\mathsf{L}_3$ one can write L as $\mathsf{L} = \tilde{J}_0 G(\mathsf{L}_1 + \mathsf{L}_2)$, to obtain

$$\mathsf{C}\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \tilde{J}_0 G\mathsf{L}_1\boldsymbol{u} + \tilde{J}_0\beta\mathsf{L}_1\frac{\partial \boldsymbol{u}}{\partial t} - k_2R^2\mathsf{L}_3\boldsymbol{u} + \boldsymbol{F}(t) \tag{2.9}$$

We will consider firstly the case when $\boldsymbol{F}(t) \equiv \boldsymbol{0}$ and $k_2 = 0$, i.e. Eq (2.9) reduces to

$$\mathsf{C}\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \tilde{J}_0 \mathsf{L}_1 \Big(G\boldsymbol{u} + \beta \frac{\partial \boldsymbol{u}}{\partial t} \Big) \tag{2.10}$$

3. Generalized Fourier method

Passing to the exposition of the method, we will consider firstly the homogeneous initial-boundary problem Eqs (2.10), (2.6) and (2.7).

Following the classical method of separation of variables, assume that the solution $\boldsymbol{u}$ of the problem can be written in the form $\boldsymbol{u}(x,t) = \boldsymbol{U}(x)T(t)$ with $T(t)$ being a scalar function and $\boldsymbol{U}(x)$ given by

$$\boldsymbol{U}(x) = \begin{bmatrix} X(x) \\ A \end{bmatrix} \qquad\qquad A = \mathrm{const} \tag{3.1}$$

Taking into consideration Eqs (2.6), we obtain the boundary conditions for X

$$\begin{aligned} &X'(0) = 0 && X'(l) = 0 \\ &X'(x_{i-}) - X'(x_{i+}) = \alpha_i X(x_i) && i = 1, 2, \dots, n \end{aligned} \tag{3.2}$$

Substituting Eq (3.1) into Eq (2.10) with the term $\tilde{J}_0 \beta \mathsf{L}_2 \boldsymbol{u}$ neglected, we get

$$\mathsf{C}\boldsymbol{U}T'' = \tilde{J}_0 (GT + \beta T') \mathsf{L}_1 \boldsymbol{U} \tag{3.3}$$

Suppose $\boldsymbol{U}$ is chosen in the way that it satisfies Eqs (3.2) and

$$\mathsf{L}_1 \boldsymbol{U} = \lambda \mathsf{C} \boldsymbol{U} \tag{3.4}$$

then from Eqs (3.3) and (3.4) it follows that

$$\mathsf{C}\boldsymbol{U}T'' = \tilde{J}_0 (GT + \beta T') \lambda \mathsf{C} \boldsymbol{U}$$

which, in turn, implies that

$$T'' + \omega^2 T' + \omega_0^2 T = 0 \tag{3.5}$$

where $\omega^2 = -\tilde{J}_0 \beta \lambda$, $\omega_0^2 = -\tilde{J}_0 G \lambda$ $(\lambda < 0)$.

Since the problem Eqs (3.4), (3.2) has, as it will be shown below, an infinite set of eigenvalues $\{\lambda_n\}$ with corresponding eigenfunctions $\{U_n\}$, the general solution of Eqs (2.10), (2.6) has the form

$$\boldsymbol{u}(x,t) = \sum_{n=1}^{\infty} \boldsymbol{U}_n(x)[c_n T_{1n}(t) + d_n T_{2n}(t)] \tag{3.6}$$

where $T_{1n}(t)$, $T_{2n}(t)$ are linearly independent solutions of the differential equation

$$T'' + \omega_n^2 T' + \omega_{0n}^2 T = 0 \tag{3.7}$$

$\omega_n^2 = -\widetilde{J}_0\beta\lambda_n$, $\omega_{0n}^2 = -\widetilde{J}_0 G\lambda_n$ and c_n, d_n are arbitrary integration constants.

The initial conditions $\boldsymbol{u}(x,0) = \boldsymbol{u}_1(x)$, $\boldsymbol{u}_t(x,0) = \boldsymbol{u}_2(x)$, where

$$\boldsymbol{u}_1(x) = \begin{bmatrix} \psi_1(x) \\ \theta_1 \end{bmatrix} \qquad \boldsymbol{u}_2(x) = \begin{bmatrix} \psi_2(x) \\ \theta_2 \end{bmatrix}$$

give the formulae

$$\sum_{n=1}^{\infty} \boldsymbol{U}_n(x)[c_n T_{1n}(0) + d_n T_{2n}(0)] = \boldsymbol{u}_1(x)$$
$$\sum_{n=1}^{\infty} \boldsymbol{U}_n(x)[c_n T'_{1n}(0) + d_n T'_{2n}(0)] = \boldsymbol{u}_2(x)$$

from which and the Fourier expansions of initial data

$$\boldsymbol{u}_1(x) = \sum_{n=1}^{\infty} \alpha_n \boldsymbol{U}_n(x) \qquad \boldsymbol{u}_2(x) = \sum_{n=1}^{\infty} \beta_n \boldsymbol{U}_n(x)$$

we obtain the system of equations in c_n and d_n

$$\begin{aligned} \alpha_n &= c_n T_{1n}(0) + d_n T_{2n}(0) \\ \beta_n &= c_n T'_{1n}(0) + d_n T'_{2n}(0) \end{aligned} \tag{3.8}$$

The function (3.6) with c_n, d_n satisfying Eqs (3.8) is the solution of the problem represented by Eqs (2.10), (2.6) and (2.7).

4. Generalized orthogonality of normal modes

Now we are going to prove that the eigenproblem represented by Eqs (3.4), (3.2) has an infinite set of solutions $\{\lambda_n\}$, $\{U_n\}$ such that $\lambda_n < 0$

and $\boldsymbol{U}_n$ satisfy certain generalized orthogonality conditions (the so-called $\mathbf{C}$-orthogonality).

At first we will prove orthogonality of the eigenfunctions $\boldsymbol{U}_n$.

To this end, define the product of vectors $\boldsymbol{U} = [X(x), A]^\top$ and $\boldsymbol{W} = [Y(x), B]^\top$ by

$$(\boldsymbol{U}|\boldsymbol{W}) = \int_0^l X(x)Y(x)\,dx + AB$$

Suppose $\boldsymbol{U}_n = [X_n(x), A_n]^\top$ is the eigenfunction corresponding to λ_n satisfying Eq (3.4) and Eqs (3.2). Using the fact that Eq (2.10) is equivalent to Eqs (2.4) with $f(t) = 0$, $k_2 = 0$, it can be proved by straightforward calculations that

$$\lambda J(x)X_n(x) = \frac{d}{dx}\Big[\tilde{J}(x,\varepsilon)\frac{d}{dx}X_n(x)\Big] - \frac{hR^2}{\beta}[X_n(x_{n+1}) - A_n]\delta_{n+1}$$

$$\lambda J_p A_n = \frac{hR^2}{\beta}[X_n(x_{n+1}) - A_n]$$

Multiplying the first of the equations above by an arbitrary function $Y(x)$, integrating the result on the interval $[0, l]$ and then using the formula for the integration by parts, the condition (3.2) and having in mind that $\int_0^l \delta_{n+1}Y(x)\,dx = Y(x_{n+1})$ we get the relation

$$\begin{aligned}\lambda \int_0^l J(x)X_n(x)Y(x)\,dx &= -\int_0^l \tilde{J}(x,\varepsilon)X_n'(x)Y'(x)\,dx + \\ &- \frac{hR^2}{\beta}[X_n(x_{n+1}) - A_n]Y(x_{n+1})\end{aligned}$$

which implies that for any two solutions $\boldsymbol{U}_k$, $\boldsymbol{U}_m$ of Eqs (3.4) and (3.2)

$$\begin{aligned}\lambda_k(\mathbf{C}\boldsymbol{U}_k|\boldsymbol{U}_m) &= \lambda_k\Big(\int_0^l J(x)X_k(x)X_m(x)\,dx + J_pA_kA_m\Big) = \\ &= -\int_0^l \tilde{J}(x,\varepsilon)X_k'(x)X_m'(x)\,dx - \frac{hR^2}{\beta}[X_k(x_{n+1}) - A_k][X_m(x_{n+1}) - A_m]\end{aligned} \tag{4.1}$$

Setting $k = n$ in Eq (4.1), we get

$$\lambda_n(\mathbf{C}\boldsymbol{U}_n|\boldsymbol{U}_n) = -\int_0^l \tilde{J}(x,\varepsilon)\Big(X_n'(x)\Big)^2\,dx - \frac{hR^2}{\beta}[X_n(x_{n+1}) - A_n]^2 < 0$$

hence $\lambda_n < 0$. Interchanging in Eq (4.1) k and m and substracting the obtained formulae yields

$$(\lambda_k - \lambda_m)(\mathbf{C}\boldsymbol{U}_k|\boldsymbol{U}_m) = 0$$

which implies the $\mathbf{C}$-orthogonality of $\boldsymbol{U}_k$ and $\boldsymbol{U}_m$ for $\lambda_k \neq \lambda_m$

$$(\mathbf{C}\boldsymbol{U}_k|\boldsymbol{U}_m) = \int\limits_0^l J(x)X_k(x)X_m(x)\,dx + J_pA_kA_m = 0$$

To show the existence of solution of Eq (3.4), observe first that Eq (3.4) is equivalent to the system of differential-algebraic equations

$$\begin{aligned}
&X'' + \sum_{i=1}^{n}\frac{\widetilde{J}_{0i}}{\widetilde{J}_0}[X'(x_{i+}) - X'(x_{i-})]\delta_i - \frac{hR^2}{\widetilde{J}_0\beta}[X(x_{n+1}) + A]\delta_{n+1} = \lambda J(x)X\\
&\frac{hR^2}{\widetilde{J}_0\beta}[X(x_{n+1}) - A] = \lambda J_pA
\end{aligned}$$

From the last equation we obtain $A = R^2h/(R^2h + \lambda J_p\widetilde{J}_0\beta)X(x_{n+1})$. Substituting it into the first and taking into consideration Eq (2.1) we get a single differential equation

$$\begin{aligned}
X'' - \lambda J_0X &= -\sum_{i=1}^{n}\frac{\widetilde{J}_{0i}}{\widetilde{J}_0}\Big(X'(x_{i+}) - X'(x_{i-})\Big)\delta_i + \\
&+ \lambda\sum_{i=1}^{n}J_iX(x_i)\delta_i + \lambda\frac{hR^2J_p}{hR^2 + \lambda J_p\widetilde{J}_0\beta}X(x_{n+1})\delta_{n+1}
\end{aligned} \tag{4.2}$$

With the help of Eq (4.2) we determine constants α_i appearing in the geometric condition (3.2). To this end compute $X'(x_{j+})$ and $X'(x_{j-})$ from Eq (4.2) integrating both sides over the interval $[0,x]$ for $x < x_j$ and $x > x_j$ and then assuming $x \to x_j$

$$\begin{aligned}
X'(x_{j+}) &= \lambda J_0\int\limits_0^{x_j}X(s)\,ds - \sum_{i=1}^{j-1}\frac{\widetilde{J}_{0i}}{\widetilde{J}_0}[X'(x_{i+}) - X'(x_{i-})] + \\
&- \frac{\widetilde{J}_{0j}}{\widetilde{J}_0}[X'(x_{j+}) - X'(x_{j-})] + \lambda\sum_{i=1}^{j-1}J_iX(x_i) + \lambda J_jX(x_j)\\
X'(x_{j-}) &= \lambda J_0\int\limits_0^{x_j}X(s)\,ds - \sum_{i=1}^{j-1}\frac{\widetilde{J}_{0i}}{\widetilde{J}_0}[X'(x_{i+}) - X'(x_{i-})] + \lambda\sum_{i=1}^{j-1}J_iX(x_i)
\end{aligned}$$

to obtain

$$X'(x_{j+}) - X'(x_{j-}) = -\frac{\widetilde{J}_{0j}}{\widetilde{J}_0}[X'(x_{j+}) - X'(x_{j-})] + \lambda J_j X(x_j)$$

which gives

$$X'(x_{j+}) - X'(x_{j-}) = \lambda \frac{\widetilde{J}_0 J_j}{\widetilde{J}_0 + \widetilde{J}_{0j}} X(x_j)$$

hence

$$\alpha_j(\lambda) = \lambda \frac{\widetilde{J}_0 J_j}{\widetilde{J}_0 + \widetilde{J}_{0j}}$$

Replacing in Eq (4.2) $X'(x_{j+}) - X'(x_{j-})$ by $\alpha_j X(x_j)$ we get finally the differential equation of the form

$$X'' + a^2(\lambda)X = \sum_{i=1}^{n+1} b_i(\lambda)X(x_i)\delta_i \tag{4.3}$$

where

$$a^2(\lambda) = -\lambda J_0 \qquad b_i(\lambda) = \lambda \frac{\widetilde{J}_0 J_i}{\widetilde{J}_0 + \widetilde{J}_{0i}} \qquad (i = 1, \dots, n)$$

$$b_{n+1}(\lambda) = \lambda \frac{hR^2 J_p}{hR^2 + \lambda J_p \widetilde{J}_0 \beta}$$

From the formula of distributional derivatives of a piecewise regular function (cf Schwartz (1985), Ch.II (II.2.25)), it follows immediately that the function $X(x) = (b/a)H(x - x_0)\sin a(x - x_0)$ is a solution of the second order ordinary differential equation with the right hand side being a Dirac function with a peak at $x = x_0$

$$X'' + a^2 X = b\delta_0$$

hence, applying the superposition principle, we get the formula for the general solution of Eq (4.3)

$$X(x) = C_1 \cos ax + C_2 \sin ax + \sum_{i=1}^{n+1} \frac{b_i(\lambda)}{a(\lambda)} X(x_i)H(x - x_i)\sin a(x - x_i) \tag{4.4}$$

To determine eigenvalues of the problem, note that Eq (4.4) involves $n+3$ constants: C_1, C_2 and $X(x_i)$, $i = 1, \dots, n+1$ not all equal to zero. Since

$$X'(x) = -C_1 a \sin ax + C_2 a \cos ax + \sum_{i=1}^{n+1} b_i(\lambda)X(x_i)H(x - x_i)\cos a(x - x_i)$$

from two first conditions of Eqs (3.2) we conclude that $C_2 = 0$ and C_1 satisfies

$$-C_1 a \sin al + \sum_{i=1}^{n+1} b_i(\lambda)X(x_i)\cos\xi_{li} = 0 \tag{4.5}$$

From Eq (4.4) it follows that

$$X(x_j) = C_1 \cos ax_j + \sum_{i=1}^{j-1} \frac{b_i(\lambda)}{a(\lambda)} X(x_i)\sin\xi_{ji} \qquad j = 1, 2, \ldots, n+1 \tag{4.6}$$

where $\xi_{li} = a(\lambda)(l - x_i)$, $\xi_{ji} = a(\lambda)(x_j - x_i)$ giving equations for unknowns $X(x_j)$. Eqs (4.5) and (4.6) constitute the homogeneous system of linear equations

$$\mathbf{A}(\lambda)z = \mathbf{0} \tag{4.7}$$

where

$$\mathbf{A}(\lambda) = \begin{bmatrix} \alpha & \boldsymbol{p}^\top \\ \boldsymbol{q} & \mathbf{A}_1 \end{bmatrix} \qquad z = \begin{bmatrix} C_1 \\ X(x_1) \\ \vdots \\ X(x_{n+1}) \end{bmatrix}$$

with

$$\alpha = -a \sin al \qquad \boldsymbol{p}^\top = [b_1 \cos\xi_{l,1}, \ldots, b_{n+1}\cos\xi_{l,n+1}]$$

$$\boldsymbol{q} = \begin{bmatrix} a\cos ax_1 \\ a\cos ax_2 \\ \vdots \\ a\cos ax_{n+1} \end{bmatrix}$$

$$\mathbf{A}_1 = \begin{bmatrix} -a & 0 & 0 & \ldots & 0 \\ b_1 \sin\xi_{2,1} & -a & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_1 \sin\xi_{n+1,1} & \ldots & \ldots & b_n \sin\xi_{n+1,n} & -a \end{bmatrix}$$

The system (4.7) has a nontrivial (nonzero) solution provided its coefficient matrix $\mathbf{A}(\lambda)$ is singular which gives the equation in eigenvalues λ of the eigenproblem reresented by Eqs (3.4) and (3.2)

$$\det \mathbf{A}(\lambda) = 0 \tag{4.8}$$

Eq (4.8) has infinitely many solutions, which follows from the observation that elements of $\mathbf{A}(\lambda)$ are transcendental functions with respect to λ. Solutions

λ_n of Eq (4.8) form an infinite sequence satisfying

$$0 = \lambda_1 \geq \lambda_2 \geq \dots \qquad \lim_{n\to\infty} \lambda_n = -\infty$$

The proof of the last theorem follows from the results of Kasprzyk and Sędziwy (1983).

Remark. Observe that for any λ_n we have $hR^2+\lambda_n J_p \widetilde{J}_0\beta \neq 0$. The equality $hR^2 + \lambda_s J_p \widetilde{J}_0\beta = 0$ holding for a certain λ_s by condition

$$\frac{hR^2}{\widetilde{J}_0\beta} X_s(x_{n+1}) - \frac{hR^2}{\widetilde{J}_0\beta} A = \lambda_s J_p A$$

would imply that $X_s(x_{n+1}) = 0$ from which, in turn, it would follow that the cross-section of the shaft at x_{n+1} remains in rest which is impossible in view of the assumptions made.

Let λ be the solution of Eq (4.8). To determine the eigenvector $\boldsymbol{z}$ corresponding to λ, observe that the $(n+1)\times(n+1)$ matrix A_1 is nonsingular, hence the rank of A equals to $n+1$ which, by the classical result of linear algebra, implies that Eq (4.7) has a one-parameter family of solutions which can be written in the form $\boldsymbol{z} = \gamma \boldsymbol{w}$ with $\boldsymbol{w}^\top = [1, \boldsymbol{d}^\top]$, and $\boldsymbol{d}$ being the solution of the equation

$$\begin{bmatrix} \alpha & \boldsymbol{p}^\top \\ \boldsymbol{q} & \mathsf{A}_1 \end{bmatrix} \begin{bmatrix} 1 \\ \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{0} \end{bmatrix}$$

It is clear that $\boldsymbol{d} = -\mathsf{A}_1^{-1}\boldsymbol{q}$ satisfies the above equation. Since A_1 is triangular, the components d_i of a vector $\boldsymbol{d}$ are given by the recurrence formulae

$$d_1 = a\cos ax_1$$

$$d_i = \frac{1}{a}\Big(a\cos ax_i - \sum_{j=1}^{i-1} d_j b_j \sin\xi_{ij}\Big) \qquad i = 2,\dots,n+1$$

Remark. Using the formula $\det \mathsf{A}(\lambda) = \alpha - \boldsymbol{p}^\top \mathsf{A}_1^{-1}\boldsymbol{q}$ and expressions for d_i we get

$$\begin{aligned} \det \mathsf{A}(\lambda) &= a\sin al - b_1 a\cos\xi_{l1}\cos ax_1 + \\ &- \sum_{i=2}^{n+1} \frac{b_i}{a}\cos\xi_{li}\Big(a\cos ax_i - \sum_{j=1}^{i-1} d_j b_j \sin\xi_{ij}\Big) \end{aligned}$$

which can be useful for computation of eigenvalues.

5. Solution of the problem in the case of dumping

The approximate solution of Eqs (2.9), (2.6), (2.7) with $k_2 \neq 0$ and $\boldsymbol{F}(t) \equiv \mathbf{0}$ will be searched in the form

$$\boldsymbol{u}_r(x,t) = \sum_{m=1}^{r} \boldsymbol{U}_m(x) s_m(t) \tag{5.1}$$

where $\{\boldsymbol{U}_m(x)\}$ denotes the set of normalized eigenvectors of Eq (3.4) $((\mathbf{C}\boldsymbol{U}_m|\boldsymbol{U}_m) = 1 \quad m = 1,2,\dots)$ and $s_m(t)$ is chosen in the way that $\boldsymbol{u}_r(x,t)$ satisfies the system of equations

$$\Big(\boldsymbol{U}_m|\mathbf{C}\frac{\partial^2 \boldsymbol{u}_r}{\partial t^2}\Big) = \Big(\boldsymbol{U}_m|\widetilde{J}_0\mathbf{L}_1\Big(G\boldsymbol{u}_r + \beta\frac{\partial \boldsymbol{u}_r}{\partial t}\Big)\Big) - k_2\Big(\boldsymbol{U}_m|R^2\mathbf{L}_3\boldsymbol{u}_r\Big) \qquad m = 1,\dots,r$$

and the initial conditions

$$\Big(\boldsymbol{U}_m|\boldsymbol{u}_r(x,0)\Big) = \Big(\boldsymbol{U}_m|\boldsymbol{u}_1(x)\Big)$$

$$\Big(\boldsymbol{U}_m|\frac{\partial \boldsymbol{u}_r(x,0)}{\partial t}\Big) = \Big(\boldsymbol{U}_m|\boldsymbol{u}_2(x)\Big) \qquad m = 1,\dots,m$$

By virtue of Eq (3.4), $(\boldsymbol{U}_m|\mathbf{L}_1\boldsymbol{U}_k) = \lambda_m(\mathbf{C}\boldsymbol{U}_m|\boldsymbol{U}_k)\delta_{mk}$ (δ_{mk} is the Kronecker delta). Using this and the linearity of the scalar product we conclude from the equations above that s_m is the solution of the following initial problem

$$s_m'' = \lambda_m\widetilde{J}_0(\beta s_m' + Gs_m) - k_2R^2\sum_{k=1}^{r}(\boldsymbol{U}_m|\mathbf{L}_3\boldsymbol{U}_k)s_k$$

$$s_m(0) = \alpha_m \qquad s_m'(0) = \beta_m \qquad m = 1,\dots,r$$

or in the matrix form

$$\begin{gathered} \boldsymbol{s}'' + \mathbf{A}\boldsymbol{s}' + \mathbf{B}\boldsymbol{s} = k_2\mathbf{H}\boldsymbol{s} \\ \boldsymbol{s}(0) = \boldsymbol{\alpha} \qquad \boldsymbol{s}'(0) = \boldsymbol{\gamma} \end{gathered} \tag{5.2}$$

where

$$\boldsymbol{s}^\top = [s_1,\dots,s_r] \qquad \boldsymbol{\alpha}^\top = [\alpha_1,\dots,\alpha_r]$$

$$\boldsymbol{\gamma}^\top = [\gamma_1,\dots,\gamma_r] \qquad \mathbf{A} = -\widetilde{J}_0\beta\mathrm{diag}(\lambda_1,\dots,\lambda_r)$$

$$\mathbf{B} = -\widetilde{J}_0 G\mathrm{diag}(\lambda_1,\dots,\lambda_r) \qquad \mathbf{H} = [h_{mk}]$$

H is a symmetric matrix with the entries $h_{mk} = -R^2(U_m|\mathsf{L}_3U_k)$.

To solve Eqs (5.2) one can replace it by the equivalent first order system of $2r$ equations and then solve it by standard methods (see e.g., Arnold (1984), Ch.III). We will propose here an alternative approach, used less frequently but probably more convenient for computations. For the sake of simplicity we describe the method in the case of pairwise different eigenvalues.

Assuming that the solution of Eqs (5.2) has the form $s(t) = ge^{\mu t}$ with $\|g\| = 1$ ($\|\cdot\|$ denotes the Euclidean norm), we get from Eqs (5.2), as in a scalar case, the condition imposed a g and μ

$$(\mu^2\mathsf{I} + \mu\mathsf{A} + \mathsf{D})g = \mathbf{0} \qquad \|g\| = 1$$
$$\mathsf{D} = \mathsf{B} - k_2\mathsf{H} \tag{5.3}$$

The system (5.3) of linear algebraic equations has a nontrivial solution provided its coefficient matrix is singular, i.e.

$$\det(\mu^2\mathsf{I} + \mu\mathsf{A} + \mathsf{D}) = 0 \tag{5.4}$$

Suppose μ_j, $j = 1, 2, \dots, 2r$ are pairwise distinct roots of Eq (5.4) and denote by g_j the solution of Eqs (5.3) corresponding to $\mu = \mu_j$.

Let $\mathsf{G}_1 = [g_1, \cdots, g_r]$, $\mathsf{G}_2 = [g_{r+1}, \dots, g_{2r}]$ be $r \times r$ matrices formed by the columns g_j and let $\mathsf{M}_1 = \mathrm{diag}(\mu_1, \dots, \mu_r)$, $\mathsf{M}_2 = \mathrm{diag}(\mu_{r+1}, \dots, \mu_{2r})$.

It can be proved by a direct computation that the function $s(t) = \mathsf{G}_1 \exp(\mathsf{M}_1 t)c + \mathsf{G}_2 \exp(\mathsf{M}_2 t)d$, with $c, d \in \mathbb{R}^r$ being arbitrary vectors, is a general solution of Eqs (5.2).

The values of c_r and d_r corresponding to the solution of the initial problem represented by Eqs (2.9) and (2.6) are computed from conditions $\alpha = s(0)$, $\gamma = s'(0)$ leading to the system of equations

$$\alpha = \mathsf{G}_1 c + \mathsf{G}_2 d \qquad \gamma = \mathsf{G}_1\mathsf{M}_1 c + \mathsf{G}_2\mathsf{M}_2 d \tag{5.5}$$

The approximate solution (5.1) of the problem represented by Eqs (2.9), (2.6) and (2.7) has then the form

$$u_r(x,t) = \sum_{m=1}^{r} U_m(x)\Big[\mathsf{G}_1 \exp(\mathsf{M}_1 t)c_r + \mathsf{G}_2 \exp(\mathsf{M}_2 t)d_r\Big]^{\mathsf{T}} e_m$$

where $e_m^{\mathsf{T}} = [0, \dots, 0, 1, 0, \dots, 0]$ (1 being the mth term).

Discussion of the convergence of Eq (5.1) to the exact solution of problem represented by Eqs (2.9), (2.6), (2.7) will be postponed to the further paper.

Remark. Note that for $\mathsf{H} = \mathbf{0}$ the system (5.2) reduces to r equations of the form of Eq (3.7) with $n = 1, \dots, r$ and the system (5.5) to Eqs (3.8).

6. Conclusions

- The method of separation of variables applied to vibration analysis of discrete continuous systems with internal dumping does not give satisfactory results for an arbitrary choice of parameters k_1, β, h, G.

  For parameters satisfying the condition $k_1\beta = hG$ the explicit form of solution and the equation in eigenfrequencies can be obtained (see Eqs (3.6) and (4.8)).

  For remaining values of parameters the problem of finding frequencies becomes nonlinear (see Eq (5.4)). There is no general approach to solving such a problem. The formula (5.1) for the general solution uses the sequence $\{U_n\}$ determined for the case $k_1\beta = hG$.

- The mathematical model described by Eq (2.9), presented in the paper is general becouse for particular choices of parameters appearing in Eq (2.9) one gets models describing various mechanical phenomena, namely:
  - If one takes $\Delta\widetilde{J}_i = 0$ $(i = 1,\dots,n)$, then Eq (2.9) represents damping of torsional vibrations of a shaft with constant torsional stiffness.
  - If $\Delta\widetilde{J}_i \neq 0$ for a certain i, then Eq (2.9) describes the torsional vibrations of a shaft with constant stiffness between points x_i, x_{i+1} at which stiffness is infinitely large.
  - The case $k \to \infty$ corresponds to the lack of damping. In this situation the damper is stiffly attached to a shaft.

- The mechanical system depicted in Fig.1 represents a crankshaft of an internal combustion engine.

- An analysis of vibrations of a system (e.g. determining its natural frequencies) based on Eq (4.8) or Eqs (5.2) in comparison with the one using the standard approach (e.g. finite elements) is easier to carry out, because explicit expressions for solutions can be obtained.

- For parameters h, β, k, k_1, k_2, G satisfying the conditions $h/\beta = k_1/G$, $k_2 = k - k_1$ one gets the analytic expression (Eq (3.6)) for the solution of the initial boundary problem represented by Eqs (2.10) and (2.6) which permits a detailed analysis of the system under consideration to be carried out. It is worthy to notice that the series (3.6) is convergent in a classical sense and that its convergence is strong, i.e. a few terms of the series are required to attain good accuracy with the exact solution.

- For an arbitrary choice of h, β, k, G the approximate solution (5.1) is even stronger convergent than the partial sums of Eq (3.6), which is due to the fact that the damping coefficient in this case is greater.

Acknowledgment

A partial support the State Committee for Scientific Research (KBN) under the grant No. 9T12A 027 08 is greatfully acknowledged.

7. References

1. ARNOLD W.I., 1984, *Ordinary Differential Equations,* in Russian, Nauka, Moscow
2. BEER F.P., JOHNSON E.R. JR., 1977, *Vector Mechanics for Engineers,* McGraw-Hill, New York, London
3. INMAN D.J., 1994, *Engineering Vibration,* Prentice-Hall, Inc., Englewood Cliffs, New Yersey
4. KASPRZYK S., 1983, An Analysis of Vibrations of a Discrete-Continuous System of the Type $(\infty, 2)$, *Nonlinear Vibration Problems,* **22**, 199-214
5. KASPRZYK S., 1984, Transvese Vibrations of a Beam with a Hinge, *Journal of Theoretical and Applied Mechanics,* **32**, 921-929
6. KASPRZYK S., SĘDZIWY S., 1983, Boundary-Value Problem for Differential-Algebraic Systems, *Bull. Pol. Ac. Sci., Math.,* **31**, 9-12
7. NADOLSKI W., 1994, The Influence of Ponderable Shafts on Dynamic Loads between Gear Teeth in Single Gear Transmission, *Archiwum Budowy Maszyn,* **XLI**, 295-314
8. PAN H.H., HOHENSTEIN R.M., 1981, A Method of Solution for an Ordinary Differential Equation Containing Symbolic Functions, *Quart. of Appl. Mechanics,* **XXXVIV**, 131-136
9. PERSSON J., 1990, Distribution Differential Equations, Regularization and Bending of a Rod, *Mathematica Scandinavica,* **66**, 291-301
10. PIELORZ A., 1992, Elastic Waves in Discrete-Continuous Mechanical Systems, in Polish, *IFTR Reports,* **21**, Instytut Podstawowych Problemów Techniki PAN, Warszawa
11. PIELORZ A., 1995, Dynamic Analysis of a Nonlinear Discrete-Continuous Torsional System by Means of Wave Method, *Z. Angew. Math. Mech.,* **75**, 9, 691-698
12. SCHWARTZ L., 1965, *Méthodes mathématiques pour les sciences physiques,* Herman, Paris
13. WHITE R.E., 1985, *An Introduction to the Finite Element Method with Applications to Nonlinear Problems,* J. Wiley & Sons, New York, Chichester, Brisbane

Analiza drgań skrętnych układu dyskretno-ciągłego w klasie funkcji uogólnionych

Streszczenie

W pracy przedstawiono metodę rozwiązywania zagadnienia mieszanego dla układu równań różniczkowych o współczynnikach dystrybucyjnych opisującego małe drgania skrętne wałka sprężystego ze skupionymi masami sztywnymi. Rozwiązania poszukiwane są w klasie funkcji uogólnionych.

Manuscript received January 13, 1995; accepted for print November 13, 1995