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DISCRETE-CONTINUOUS MODELS IN THE ANALYSIS OF LOW STRUCTURES SUBJECT TO KINEMATIC EXCITATIONS CAUSED BY TRANSVERSAL WAVES

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The paper consists of two parts. In the first one the influence of shear forces on vibration frequencies of a short beam is investigated utilizing: Timoshenko equations, simplified Timoshenko equations and classical wave equation. In the second part, two discrete-continuous models are proposed for dynamic investigations of the low structures subject to kinematic excitation caused by transversal waves. The models consist of rigid bodies and elastic elements which undergo only shear deformations. In the second multi-body model an additional elastic segment is located between the lower rigid body and remaining elements. In the discussion a wave method is applied, which utilizes the wave solution of the equations of motion. Numerical calculations are made for the model consisting of three rigid bodies. They concern on determination of the amplitude-frequency curves and investigation of the effect of diversified mechanical and geometrical properties of the segment on resonant amplitudes.

Key words: dynamics, discrete-continuous models, waves

1. Introduction

In the technical literature, the formal description of problems of both the analysis and synthesis starts with the modelling of various engineering structures including also low structures subject to kinematic excitations caused by transversal waves. In the literature available one can find a lot of items concerning dynamic analysis of low structures subject to various excitations connected directly with these structures. However, there is a lack of papers, which the kinematic excitations of seismic type or caused by highway traffic, surface and subsurface railways and machinery in a nearby location are accounted for. There are, in some exceptions, i.e., papers concerning such engineering structures like buildings and nuclear powers. These structures are usually modelled by means of discrete multi-degree-of-freedom models taking into account or disregarding the stiffness of soil surrounding the structure foundation.

In the paper discrete-continuous models are proposed for the dynamic investigation of low structures. Continuous elastic elements in these models are assumed to be described by the classical wave equation representing a shear beam.

The paper consists of two parts. In the first part it is shown when the use of the classical wave equation is justifiable. The results obtained agree with the suggestion put forward by Humar (1990) that many engineering structures can be idealized by shear beams, e.g. low buildings and isotropic or horizontally layered soil deposit undergoing horizontal deformations. Next, two discrete-continuous multi-body models undergoing shear deformations are studied using the wave solution of equations of motion.

2. Influence of shear deformations on vibration frequencies of a short beam

Shear forces exert a significant effect on displacements and strains in numerous mechanical and engineering structures subject to various external excitations caused, e.g., by seismic loadings, highway traffic, surface and subsurface railways, and by machinery in a nearby location. It mainly concerns constructional elements of the transverse dimension, alongside of which shear forces act, being close to the element length. Among such structures partially low columns, bridge piers and different machine supports may be discussed. In the practical analysis of vibrations many elements of these structures are modelled by a short beam. The vibration of the short beam can be investigated utilizing of the Timoshenko equations and their special cases.

Application of the Timoshenko equations to beam problems was considered, e.g. by Huang (1961) and Wang (1970) while of simplified Timoshenko equations by Abramovich and Elishakoff (1987) and (1990). These papers deal with frequency equations of a slender beam under various simple boundary conditions for $r = \sqrt{I/A}/L \leq 0.1$, where L is the beam length, A stands for the cross-sectional area and I is the moment of inertia of a cross-section, r is the reciprocal of the slenderness ratio for a beam.

In the present paper the equations of frequency of free vibration for a beam under four simple boundary conditions are studied. In the discussion the Timoshenko equations, simplified Timoshenko equations and the classical wave equation for the shear beam are utilized. The last equation takes into account only pure shear deformations. In numerical calculations frequencies of free vibration in function of the parameter r are compared for two beam materials, i.e., for steel and timber.

2.1. Differential equations and boundary conditions

The coupled equations for the total deflection y and rotation ψ of the cross-section were given by Timoshenko (1955) as

$$\begin{aligned} &EI\psi_{,xx} + kAG(y_{,x} - \psi) - \rho I\psi_{,tt} = 0 \\ &\rho A y_{,tt} - kAG(y_{,xx} - \psi_{,x}) = 0 \end{aligned} \tag{2.1}$$

where

E – modulus of elasticity
G – shear modulus
ρ – mass density of the beam material
k – shear coefficient

and the comma denotes partial differentiation.

Eliminating ψ or y from Eqs (2.1) the following two differential equations in y and ψ are obtained

$$\begin{aligned} &EIy_{,xxxx} + my_{,tt} - mR^2\Big(1 + \frac{E}{kG}\Big)y_{,xxtt} + \frac{m^2R^2}{kAG}y_{,tttt} = 0 \\ &EI\psi_{,xxxx} + m\psi_{,tt} - mR^2\Big(1 + \frac{E}{kG}\Big)\psi_{,xxtt} + \frac{m^2R^2}{kAG}\psi_{,tttt} = 0 \end{aligned} \tag{2.2}$$

where $R^2 = I/A$ is the radius of gyration and $m = \rho A$.

Free harmonic oscillations at an angular frequency ω are expressed by

$$y(x,t) = Y(x)e^{i\omega t} \qquad \psi(x,t) = \Psi(x)e^{i\omega t} \qquad \xi = \frac{x}{L} \tag{2.3}$$

where

$Y(x)$ – transverse vibrational mode
$\Psi(x)$ – rotational mode
ξ – nondimensional spatial coordinate.

Substituting solutions (2.3) into Eqs (2.2) we obtain

$$
\begin{aligned}
&Y^{IV} + p^2(r^2 + b^2)Y'' - p^2(1 - p^2r^2b^2)Y = 0 \\
&\Psi^{IV} + p^2(r^2 + b^2)\Psi'' - p^2(1 - p^2r^2b^2)\Psi = 0
\end{aligned}
\tag{2.4}
$$

where

$$
r = \frac{R}{L} \qquad b^2 = \frac{E}{kG}r^2 \qquad p = \frac{\omega L}{cb} \qquad c^2 = \frac{kG}{\rho}
\tag{2.5}
$$

and the prime denotes differentiation with repect to ξ.

Below, four common types of beams will be identified by compound adjectives which describe the end conditions at $\xi = 0$ and $\xi = 1$. They are:

$$
\begin{array}{lll}
1) & \text{hinged-hinged:} & Y(0) = \Psi'(0) = Y(1) = \Psi'(1) = 0 \\
2) & \text{clamped-clamped:} & Y(0) = \Psi(0) = Y(1) = \Psi(1) = 0 \\
3) & \text{clamped-hinged:} & Y(0) = \Psi(0) = Y(1) = \Psi'(1) = 0 \\
4) & \text{clamped-free:} & Y(0) = \Psi(0) = Y'(1)/L - \Psi(1) = \Psi'(1) = 0.
\end{array}
\tag{2.6}
$$

The solutions of Eqs (2.4) can be found as

$$
\begin{aligned}
Y(\xi) &= B_1\cosh(ps_1\xi) + B_2\sinh(ps_1\xi) + B_3\cos(ps_2\xi) + B_4\sin(ps_2\xi) \\
\Psi(\xi) &= C_1\cosh(ps_1\xi) + C_2\sinh(ps_1\xi) + C_3\cos(ps_2\xi) + C_4\sin(ps_2\xi)
\end{aligned}
\tag{2.7}
$$

where

$$
s_1,\ s_2 = \frac{1}{\sqrt{2}}\sqrt{\mp(r^2 + b^2) + \sqrt{(r^2 - b^2) + \frac{4}{p^2}}}
\tag{2.8}
$$

Relations between the constants in Eqs (2.7) can be obtained by substituting Eqs (2.3), (2.7) into Eqs (2.1). They are as follows

$$
\begin{aligned}
\frac{C_1}{B_2} &= \frac{p(s_1^2 + b^2)}{Ls_1} & \qquad \frac{C_2}{B_1} &= \frac{p(s_1^2 + b^2)}{Ls_1} \\
\frac{C_3}{B_4} &= \frac{p(s_2^2 - b^2)}{Ls_2} & \qquad \frac{C_4}{B_3} &= -\frac{p(s_2^2 - b^2)}{Ls_2}
\end{aligned}
\tag{2.9}
$$

Eqs (2.2) with the last terms neglected, were discussed by Abramovich and Elishakoff (1990). Solutions of such simplified equations have the form (2.3) with (2.7), however the parameters s_1, s_2 now take the form

$$
s_1,\ s_2 = \frac{1}{\sqrt{2}}\sqrt{\mp(r^2 + b^2) + \sqrt{(r^2 + b^2) + \frac{4}{p^2}}}
\tag{2.10}
$$

When the cross-section rotation ψ may be assumed to be identically equal to zero, the classical wave equation is derived from Eq $(2.1)_2$

$$y_{,tt} - c^2 y_{,xx} = 0 \tag{2.11}$$

These two special cases of the Timoshenko equations (2.1) are considered in the present paper.

2.2. Frequency equations

Compliance with the appropriate boundary conditions (2.6) yields frequency equations. They are presented below for three types of governing equations.

A. Timoshenko equations (cf Huang (1961))

$$
\begin{array}{lll}
1) & \text{hinged-hinged beam} & \sin(ps_2) = 0 \\
2) & \text{clamped-clamped beam} & 2 - 2\cosh(ps_1)\cos(ps_2) + \\
 & & +A_1 \sinh(ps_1)\sin(ps_2) = 0 \\
3) & \text{clamped-hinged beam} & \lambda\zeta \tanh(ps_1) - \tan(ps_2) = 0 \\
4) & \text{clamped-free beam} & 2 + A_2\cosh(ps_1)\cos(ps_2) + \\
 & & +A_3 \sinh(ps_1)\sin(ps_2) = 0
\end{array}
\tag{2.12}
$$

where

$$
\begin{array}{ll}
A_1 = \dfrac{p}{\sqrt{1 - p^2r^2b^2}}\Big[p^2b^2(r^2 - b^2) + 3b^2 - r^2\Big] & \\
A_2 = p^2(r^2 - b^2)^2 + 2 & A_3 = -\dfrac{p(r^2 + b^2)}{\sqrt{1 - p^2r^2b^2}} \\
\lambda = \dfrac{s_1}{s_2} & \zeta = \dfrac{s_2^2 - b^2}{s_2^2 - r^2}
\end{array}
\tag{2.13}
$$

If s_1 becomes imaginary then Eqs (2.12) have to be suitably retransformed (cf Huang (1961))

B. Simplified Timoshenko equations (cf Abramovich and Elishakoff (1990))

1) $\sin(ps_2) = 0$

$$
\begin{aligned}
&2) \qquad 2 - 2\cosh(ps_1)\cos(ps_2) + \\
&\qquad + \frac{p[3b^2 - r^2 + p^2b^4(r^2+b^2)]}{1+p^2r^2b^2}\sinh(ps_1)\sin(ps_2) = 0 \\
&3) \qquad \lambda\zeta\tanh(ps_1) - \tan(ps_2) = 0 \\
&4) \qquad 2 + \frac{2+p^2(r^4+b^4)}{1+p^2r^2b^2}\cosh(ps_1)\cos(ps_2) + \\
&\qquad - p(r^2+b^2)\sinh(ps_1)\sin(ps_2) = 0
\end{aligned}
\tag{2.14}
$$

C. Classical wave equation

$$
\begin{aligned}
&1,2,3) \qquad \sin\frac{\omega L}{c} = 0 \qquad \frac{\omega_n L}{c} = (n-1)\pi \\
&4) \qquad \cos\frac{\omega L}{c} = 0 \qquad \frac{\omega_n L}{c} = (2n-1)\frac{\pi}{2}
\end{aligned}
\tag{2.15}
$$

2.3. Numerical results

Diagrams of the first three frequencies of free vibration versus the parameter r for steel $(E/kG = 4)$ and timber $(E/kG = 30)$ are plotted in Fig.1 and Fig.2 for the Timoshenko beam according to Eqs (2.12), and in Fig.3 and Fig.4 for the simplified Timoshenko equations using Eqs (2.14). They are marked by continuous lines. These results were obtained for four boundary conditions (2.6) for $0 \le r \le 1$. Additional broken lines in Fig.1 and Fig.2 represent the frequencies of free vibration (2.15) for the classical wave equation.

From Fig.1 it follows that the frequencies $\omega_1, \omega_2, \omega_3$ with the increase of parameter r approach constant values corresponding to the frequencies of free vibration for the classical wave equation; namely, the curves for the boundary conditions 1, 2, 3 approach π, 2π, 3π, and the curves for the boundary condition 4 approach $\pi/2, 3\pi/2, 5\pi/2$. The diagrams in Fig.1 show that differences between the frequencies of free vibration of the Timoshenko beam and suitable frequencies for the classical wave equation are already insignificant when $r > 0.35$, for example, in the case of clamped ends. Better conformability between the frequencies of free vibration for the Timoshenko equations and those for the classical wave equation has been obtained for timber. In this case insignificant differences occur already from $r = 0.15$. For instance, $r = 0.2$, corresponding to the slenderness ratio equal to 5, gives $h/L = 0.69$ where

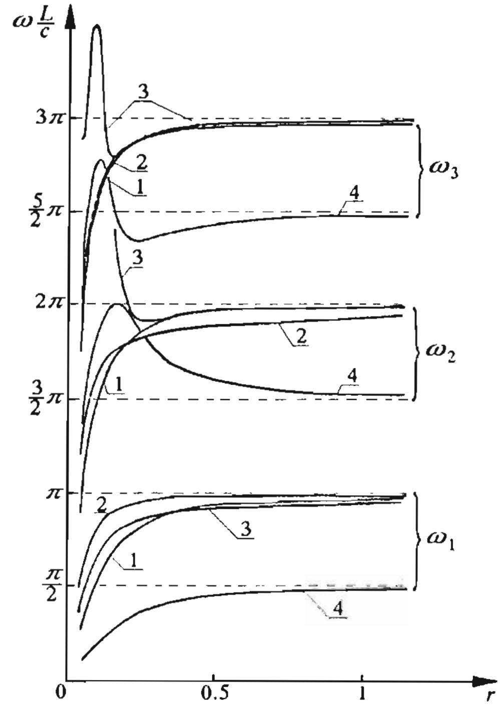

Fig. 1. Frequencies of free vibrations for the Timoshenko equations with E/kG=4

h is the beam height. For this reason, in dynamic investigations of structure elements having a transverse dimension, alongside of which shear forces act, being close to the element length and having a low slenderness ratio the approach using the classical wave equation is permissible.

From the diagrams plotted in Fig.1, Fig.2 and in Fig.3, Fig.4 it follows that the first three frequencies of free vibration of the Timoshenko equations differ insignificantly from the frequencies for the simplified Timoshenko equation only for $r \leq 0.1$, i.e., for a slender beam. When $r > 0.1$, then qualitative and quantitative differences occur between suitable curves. For this reason, it is inadvisable to apply the simplified Timoshenko equations proposed by Abramovich and Elishakoff (1987) and (1990) to investigations of short beams.

The next part of the present paper aims at the dynamic analysis of multibody discrete-continuous models for a low engineering structure in which shear forces dominate. From the considerations given above and from Fig.1 ÷ Fig.4 it follows that the motion of elastic elements in the discussed models can be described utilizing either the Timoshenko equations (2.1) or the classical wave

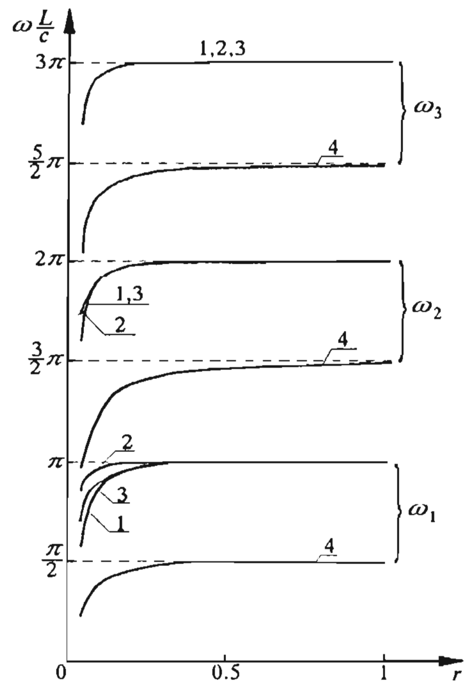

Fig. 2. Frequencies of free vibrations for the Timoshenko equations with E/kG=30

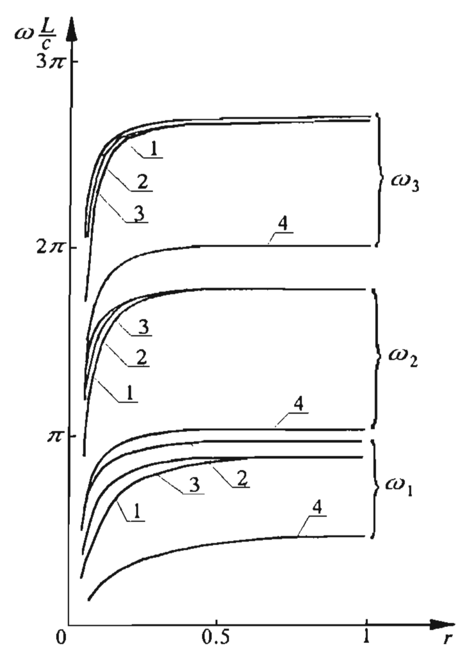

Fig. 3. Frequencies of free vibrations for the simplified Timoshenko equations with E/kG=4

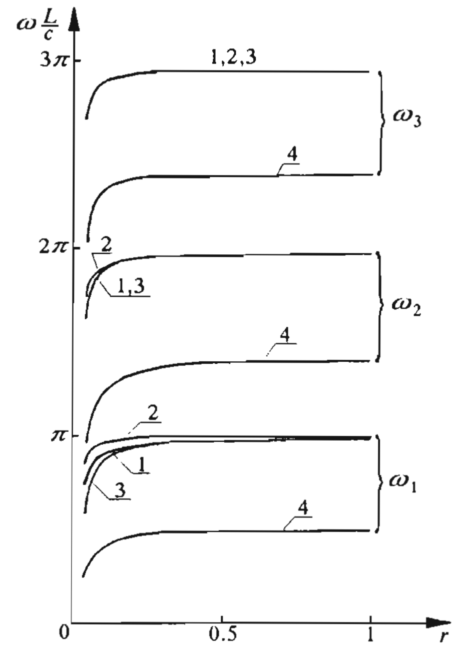

Fig. 4. Frequencies of free vibrations for the simplified Timoshenko equations with E/kG=30

equation (2.11). From the available literature it follows that the methods appropriate for the discussion of discrete-continuous systems consisting of several Timoshenko beams and rigid bodies have not been invented yet. However, the method used e.g. by Nadolski and Pielorz (1992) can be easily adopted for the investigation of multi-body discrete-continuous systems undergoing shear deformations described by Eq (2.11).

3. Discrete-continuous models in the analysis of low structures subject to kinematic excitation caused by transversal waves

3.1. Assumptions

The paper is devoted to dynamic investigations of low structures subject to kinematic excitation caused by transversal waves. Kinematic excitations can be of seismic type or can be caused by highway traffic, surface and subsurface railways, and by machinery in nearby location. In the literature, engineering structures subject to various kinematic excitations are discussed utilizing discrete as well as continuous models (cf Okamoto (1973); Sackman and Kelly (1979); Mengi and Dündar (1988)).

The elastic elements of the structures considered in the present paper have the transverse dimension, alongside of which shear forces act, close to the length of the element, i.e. they have a low slenderness ratio. These are, e.g., machine supports, bridge piers and low columns in buildings. Many structure elements subject to the transversal excitation can be modelled by means of the Timoshenko beam. In the first part of the paper it has been shown that in the case of short beams in which shear forces are predominant, the Timoshenko equations can be replaced by the classical wave equation. Some suggestion about applying classical wave equations, however with no discussion on frequencies included (see Section 2), were given by Humar (1990).

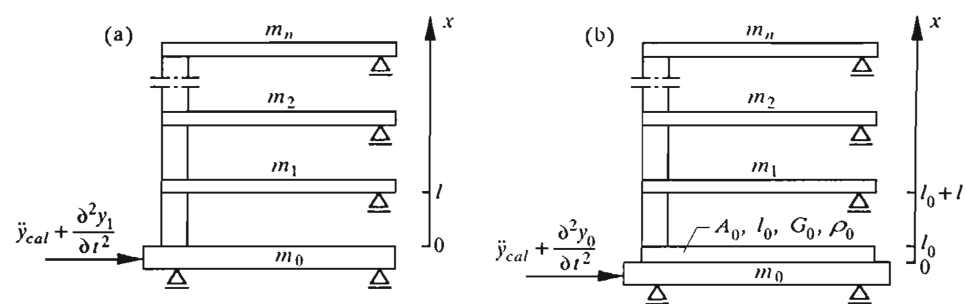

Fig. 5. Discrete-continuous models

The use of classical wave equation enables discussion of the models of engineering structures consisting of many elastic elements and rigid bodies. The approach applying the classical wave equation and its wave solution is employed in dynamic investigations of the discrete-continuous models shown in Fig.5. Special cases of these models can be employed when investigating of the structures mentioned above.

The studied models consist of n elastic elements connected by rigid bodies. All cross-sections of the elastic elements under external excitations remain flat and parallel to the cross-sections where rigid bodies are located. The elastic elements undergo only shear deformations. They may have different mechanical properties, however for the sake of simplicity it is assumed that all the elements are characterized by: shear modulus G, cross-sectional area A, shear coefficient k, density ρ and length l, Fig.5a. Between the rigid body

m_0 and the element (1) an elastic segment of an isolation type can be situated (cf Su et al. (1989)). It is characterized by the following parameters shear modulus G_0, density ρ_0, cross-sectional area A and length l_0, Fig.5b.

The rigid body m_0 is subject to the absolute acceleration $\partial^2[y_m(t) + y_{cal}(t)]/\partial t^2$ where y_m is the displacement of the rigid body m_0 with respect to the ground and $y_{cal}(t)$ is the ground displacement in the fixed spatial system. The function $y_m(t)$ is equal either to $y_1(0,t)$ when the model shown in Fig.5a is analyzed or to $y_0(0,t)$ for the model in Fig.5b.

Damping in the model is represented by equivalent external and internal damping expressed by

$$R_{di} = d_i y_{i,t} \qquad R_{Vi} = D_i y_{i,xt} \qquad i = 0,1,...,n \tag{3.1}$$

where $y_i(t)$ represents the displacement of the ith elastic element, d_i and D_i are the coefficients of external and internal damping, respectively, and the comma denotes partial differentiation. The equivalent damping is taken into account in the boundary conditions. It is assumed that the direction of x-axis is normal to the direction of displacements y_i, x-axis origin coincides with the position of the rigid body m_0 in the undisturbed state and that velocities and displacements of the cross-sections of all the elastic elements are equal to zero at the instant $t = 0$.

3.2. Governing equations

The problem of finding displacements, strains and velocities in the cross-sections of the elastic elements for the model shown in Fig.5b is reduced to solving the classical wave equations

$$\begin{aligned} & y_{0,tt} - c_0^2 y_{0,xx} = 0 \\ & y_{i,tt} - c^2 y_{i,xx} = 0 \qquad i = 1,2,...,n \end{aligned} \tag{3.2}$$

with the following initial conditions

$$y_i(x,0) = y_{i,t}(x,0) = 0 \qquad i = 0,1,2...,n \tag{3.3}$$

and the boundary conditions

$$\begin{aligned} & -m_0(\ddot{y}_{cal}(t) + y_{0,tt}) - d_0 y_{0,t} + AkG(D_0 y_{0,xt} + y_{0,x}) = 0 && \text{for} \quad x = 0 \\ & y_0 = y_1 && \text{for} \quad x = l_0 \end{aligned}$$

$$-AkG_0(D_0y_{0,xt}+y_{0,x})+AkG(D_1y_{1,xt}+y_{1,x})=0 \qquad \text{for} \quad x=l_0$$
$$y_i=y_{i+1} \qquad \text{for} \quad x=l_0+il \quad i=1,2,...,n-1 \tag{3.4}$$
$$-AkG(D_iy_{i,xt}+y_{i,x})+AkG(D_{i+1}y_{i+1,xt}+y_{i+1,x})-m_iy_{i+1,tt}-d_iy_{i+1,t}=0$$
$$\text{for} \quad x=l_0+il \quad i=1,2,...,n-1$$
$$-AkG(D_ny_{n,xt}+y_{n,x})-m_ny_{n,tt}-d_ny_{n,t}=0 \qquad \text{for} \quad x_0=l_0+nl$$

where $c_0^2 = kG_0/\rho_0$ and $c^2 = kG/\rho$. Eqs (3.4) represent conditions for displacements and forces acting in the adjacent cross-sections of adjacent elastic elements of the considered model, and $\ddot{y}_{cal}(t)$ is a given time function representing the external excitation, which can be irregular (cf Okamoto (1973); Sackman and Kelly (1979); Mengi and Dündar (1988)) or regular. If the elastic segment of the length l_0 is eliminated from the model presented in Fig.5b, i.e., $l_0 \equiv 0$ and $y_0(x,t) \equiv 0$, then the rigid body m_0 is subject to the acceleration $y_{1,tt}(0,t)+\ddot{y}_{cal}(t)$, Fig.5a, and the boundary conditions $(3.4)_{1,2,3}$ are reduced to the equation

$$-m_0(\ddot{y}_{cal}+y_{1,tt})-d_0y_{1,t}+AkG(D_1y_{1,xt}+y_{1,x})=0 \quad \text{for} \;\; x=0 \tag{3.5}$$

Although the equations of motion in which continuously distributed damping is taken into account would describe the problem more precisely, they do not have as effective solution methods as the wave method in the case of classical wave equations. Moreover, it was shown by Pielorz (1988) that appropriate equations with continuously distributed damping and with the equivalent damping gave practically the same results except for a short initial time interval.

Upon introduction of the nondimensional quantities

$$\begin{array}{llll} \bar{x}=\dfrac{x}{l} & \bar{t}=\dfrac{ct}{l} & K_r=\dfrac{A\rho l}{m_r} & \bar{D}_i=\dfrac{D_ic}{l} \\ \bar{d}_i=\dfrac{d_il}{m_rc} & \bar{y}_i=\dfrac{y_i}{y_r} & R_i=\dfrac{m_i}{m_r} & \bar{c}_0=\dfrac{c_0}{c} \\ \bar{l}_0=\dfrac{l_0}{l} & B_0=\dfrac{G_0}{G} & & \end{array} \tag{3.6}$$

where m_r and y_r represent the fixed mass and displacement, respectively, Eqs (3.2) ÷ (3.4) can be rewritten

$$\begin{aligned} &y_{0,tt}-c_0^2y_{0,xx}=0 \\ &y_{i,tt}-y_{i,xx}=0 \qquad \text{for} \;\; i=1,2,...,n \end{aligned} \tag{3.7}$$

$$y_i(x,0)=y_{i,t}(x,0)=0 \qquad \text{for} \;\; i=0,1,...,n \tag{3.8}$$

$$
\begin{aligned}
&R_0\ddot{y}_{cal}(t) + R_0 y_{0,tt} + d_0 y_{0,t} - B_0 K_r(D_0 y_{0,xt} + y_{0,x}) = 0 && \text{for} \quad x = 0\\
&y_0 = y_1 && \text{for} \quad x = l_0\\
&B_0(D_0 y_{0,xt} + y_{0,x}) - D_1 y_{1,xt} - y_{1,x} = 0 && \text{for} \quad x = l_0\\
&y_i = y_{i+1} \qquad \text{for} \quad x = l_0 + i \quad i = 1,2,...,n-1 && (3.9)\\
&K_r(D_i y_{i,xt} + y_{i,x}) - K_r(D_{i+1} y_{i+1,xt} + y_{i+1,x}) + R_i y_{i+1,tt} + d_i y_{i+1,t} = 0\\
&\qquad\qquad \text{for} \quad x = l_0 + i \quad i = 1,2,...,n-1\\
&K_r(D_n y_{n,xt} + y_{n,x}) + R_n y_{n,tt} + d_n y_{n,t} = 0 && \text{for} \quad x = l_0 + n
\end{aligned}
$$

For the sake of convenience in Eqs (3.7) $\div$ (3.9) bars denoting nondimensional quantities are omitted.

The solutions of Eqs (3.7) taking into account the initial conditions (3.8) are sought in the form

$$
\begin{aligned}
&y_0(x,t) = f_0\Big(c_0(t-x)\Big) + g_0\Big(c_0(t+x)\Big)\\
&y_i(x,t) = f_i(t - l_0c_0^{-1} - x + l_0) + g_i\Big(t - l_0c_0^{-1} + x - l_0 - 2(i-1)\Big) \qquad (3.10)\\
&\qquad\qquad i = 1,2,...,n
\end{aligned}
$$

where unknown functions f_i and g_i represent waves, caused by kinematic excitation, propagating in the ith elastic element of the discrete-continuous model in the directions coincident and opposite to the x-axis direction, respectively. In the sought solution (3.10) it is taken into account that the material of additional elastic segment can differ from the material of the remaining elements. Moreover, in the arguments of functions f_i and g_i it has been already taken into account that the first disturbance in the elastic segment of the length l_0 occurs at $t = 0$ in $x = 0$ while in the ith element at $t = l_0c_0^{-1} + i - 1$ in the cross-section $x = l_0 + i - 1$ for $i = 1,2,...,n$. The functions f_i and g_i are continuous and for negative arguments identical to zero.

Upon substituting the solution (3.10) into the boundary conditions (3.9) and denoting the largest argument in each equality by z_0 or z, respectively, the following equations are obtained for the functions f_i and g_i

$$
\begin{aligned}
&g_0(z_0) = f_1(z - 2l_0c_0^{-1}) + g_1(z - 2l_0c_0^{-1}) - f_0(z_0 - 2l_0)\\
&g_i(z) = f_{i+1}(z-2) + g_{i+1}(z-2) - f_i(z-2) \qquad i = 1,2,...,n-1\\
&r_{n+1,1}g_n''(z) + r_{n+1,2}g_n'(z) = r_{n+1,3}f_n''(z-2) + r_{n+1,4}f_n'(z-2)\\
&r_{01}f_0''(z_0) + r_{02}f_0'(z_0) = -R_0\ddot{y}_{cal}(z) + r_{03}g_0''(z_0) + r_{04}g_0'(z_0) \qquad (3.11)
\end{aligned}
$$

$$r_{11}f_1''(z) + r_{12}f_1'(z) = r_{13}g_1''(z) + r_{14}g_1'(z) + r_{15}f_0''(z_0) + r_{16}f_0'(z_0)$$
$$r_{i1}f_i''(z) + r_{i2}f_i'(z) = r_{i3}g_i''(z) + r_{i4}g_i'(z) + r_{i5}f_{i-1}''(z) + r_{i6}f_{i-1}'(z) \qquad i = 2,3,...,n$$

where

$$\begin{aligned}
&r_{01} = c_0(B_0K_rD_0 + R_0c_0) && r_{02} = B_0K_r + d_0c_0\\
&r_{03} = c_0(B_0K_rD_0 - R_0c_0) && r_{04} = B_0K_r - d_0c_0\\
&r_{11} = D_1 + B_0D_0c_0^{-1} && r_{12} = 1 + B_0c_0^{-1}\\
&r_{13} = D_1 - B_0D_0c_0^{-1} && r_{14} = 1 - B_0c_0^{-1}\\
&r_{15} = 2B_0D_0c_0 && r_{16} = 2B_0\\
&r_{i1} = K_rD_i + K_rD_{i-1} + R_{i-1} && r_{i2} = 2K_r + d_{i-1}\\
&r_{i3} = K_rD_i - K_rD_{i-1} - R_{i-1} && r_{i4} = -d_{i-1}\\
&r_{i5} = 2K_rD_{i-1} && r_{i6} = 2K_r \quad i = 2,3,...,n\\
&r_{n+1,1} = K_rD_n + R_n && r_{n+1,2} = K_r + d_n\\
&r_{n+1,3} = K_rD_n - R_n && r_{n+1,4} = K_r - d_n
\end{aligned} \tag{3.12}$$

If the elastic segment of the length l_0 is eliminated from the model presented in Fig.5b, then appropriate equations have to be derived replacing Eqs $(3.4)_{1,2,3}$ by Eq (3.5).

Eqs (3.11) are differential equations with a retarded argument. They can be solved analytically or numerically by means of the finite difference method similarly of Nadolski and Pielorz (1980) and (1992). The solution can be obtained in the transient as well as in steady states.

3.3. Numerical results

Numerical analysis is carried out for the models presented in Fig.5 when $n = 2$ with $l_0 = 0$ and for $l_0 \neq 0$. First, sample calculations are carried out for the model with $l_0 = 0$ with the following constant parameters

$$\begin{aligned}
&R_0 = 1 \qquad && R_1 = R_2 = 0.5 \qquad && d_0 = d_1 = D_1 = D_2 = 0.1\\
&m_r = m_0 && \Delta z = \frac{1}{20}
\end{aligned} \tag{3.13}$$

The function representing the external excitation $\ddot{y}_{cal}(t)$ can be taken arbitrary: irregular or regular, periodic or nonperiodic. In the paper it is assumed in the form

$$\ddot{y}_{cal}(t) = a_0\sin(pt) \qquad a_0 = 1.0 \tag{3.14}$$

and the considerations focus on the determination of displacements in the steady states. The function (3.14) can represent various direct and indirect external excitations, where p is the nondimensional frequency of external excitation.

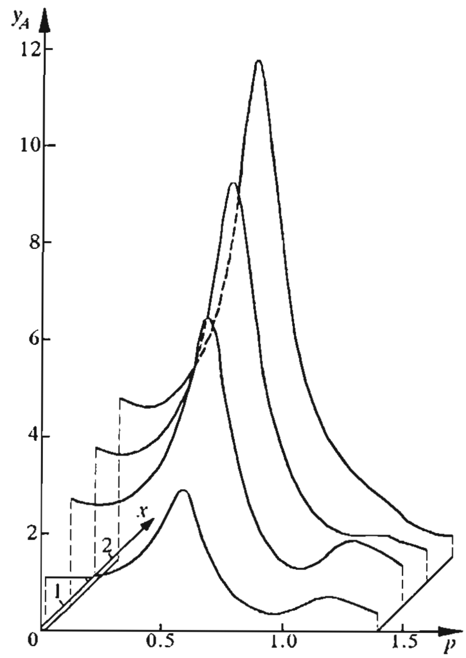

Fig. 6. Amplitude-frequency curves of the displacements $y(x,t) - y(0,t)$ for the model with $l_0 = 0$

In Fig.6 are plotted spatial diagrams of amplitude-frequency curves for $K_r = 0.3$ and the parameters (3.13) for relative displacements. From Fig.6 it follows that the curves for the function $y(x,t) - y(0,t)$ are regular and the amplitudes in the first resonance increase with the increase of x. Diagrams in Fig.7 concern amplitude-frequency curves for $K_r = 0.1$, 0.3, 0.5, for the displacements of the cross-sections $x = 1$ and $x = 2$ with respect to the displacements of the cross-section $x = 0$. These curves are regular, similarly as corresponding curves in Fig.6, the maximum amplitudes decrease with the increase of the parameter K_r and are higher in $x = 2$. Moreover, from Fig.6 and Fig.7 it follows that all the curves obtained are irregular in the region of second resonance, and the amplitudes in this region are many times lower than those in the region of first resonance.

Further numerical analysis concerns the three-body model with the addi-

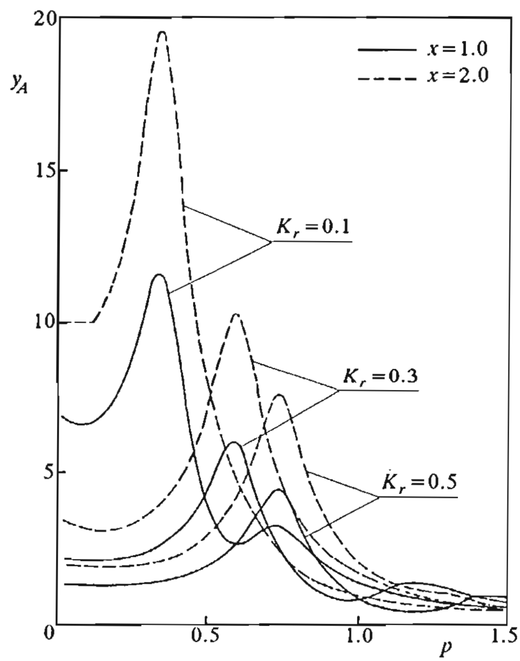

Fig. 7. Amplitude-frequency curves of the displacements $y(x,t) - y(0,t)$ in the cross-sections $x = 1$, 2 for $K_r = 0.1$, 0.3, 0.5 and $l_0 = 0$

tional elastic segment characterized by l_0, ρ_0, G_0 and the coefficient of internal damping D_0.

At first, the amplitude-frequency curves analogous to those plotted in Fig.6 were obtained using Eqs (3.11) with the nondimensional parameters (3.13) and with

$$c_0 = 1.0 \qquad B_0 = 1.0 \qquad l_0 = 0.1 \qquad D_0 = 0.1 \tag{3.15}$$

It has appeared that the displacements of the cross-sections $x = 0$ and $x = l_0$ were practically equal, and therefore the amplitude-frequency curves for the function $y(x,t) - y(l_0,t)$ were determined in the cross-sections $x = l_0 + 0.5, l_0 + 1, l_0 + 1.5, l_0 + 2$. These cross-sections correspond to those for which the diagrams plotted in Fig.6 and Fig.7 were determined. The amplitude-frequency curves obtained for the parameters (3.15) are not far from the appropriate curves presented in Fig.6 and Fig.7. For this reason they are not presented in this paper.

The following numerical calculations are performed for $B_0 \in <0.1, 10>$ and for nondimensional wave speed $c_0 = 0.5$, 1, 2. The parameters B_0 and

c_0 are not independent. It can be noted that $c_0^2 = B_0/\gamma_0$ where $\gamma_0 = \rho_0/\rho$, and that a rather slow change of γ_0 occurs for $c_0 \in< 0.5, 2 >$.

Further numerical calculations concern the investigation of the effect of the parameters c_0 and l_0, which characterize the additional elastic segment in the considered model, on the amplitudes of the displacements in selected cross-sections. From Fig.6 and Fig.7 it follows that maximum amplitudes occur in the region of first resonance, so the effect of these parameters is investigated for the frequency of external excitation p equal to the first frequency of free vibration of the system. The frequency equation is derived seeking the solution of Eqs (3.7) using the method of separation of variables and neglecting damping and external excitation in the boundary conditions (3.9).

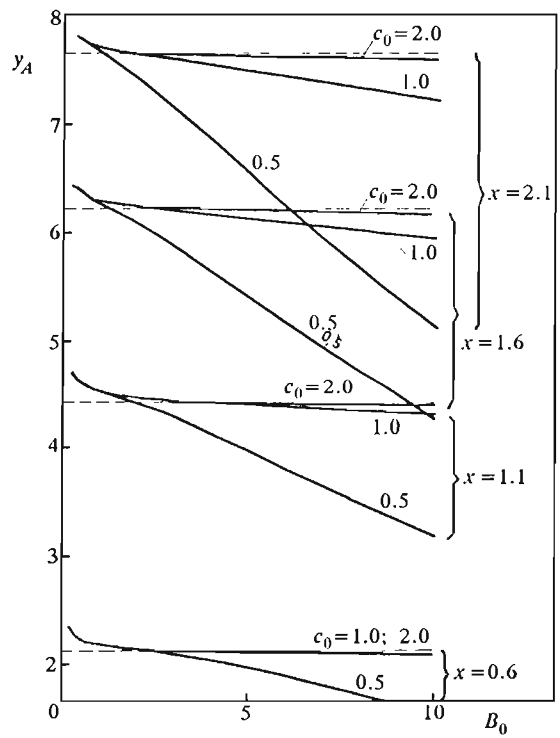

Fig. 8. Effect of the wave speed c_0 on resonant amplitudes of relative displacements for the model with $l_0 = 0.1$

The effect of the wave speed c_0 is investigated for $B_0 \in< 0.2, 10 >$, $D_0 = 0.1$, $K_r = 0.1,\ 0.3,\ 0.5$ in the cross-sections $x = 0.6,\ 1.1,\ 1.6,\ 2.1$ also for the relative displacements. It is assumed $c_0 = 0.5,\ 1,\ 2$. In Fig.8 the resonant amplitudes for $K_r = 0.3$ are plotted. From Fig.8 it follows that

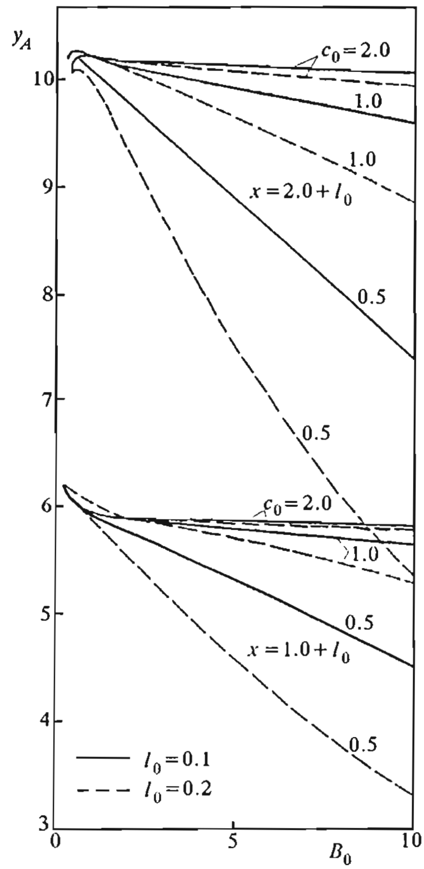

Fig. 9. Effect of the segment length l_0 on resonant amplitudes of relative displacements

the effect of c_0 is very significant, specially for $B_0 > 1$. The sensitivity to c_0 increases with the increase of x. When $B_0 < 0.8$ the dependence of the resonant amplitudes on B_0 is nonlinear while for $B_0 > 0.8$ it is rather linear, and the amplitudes increase with the increase of x. Comparing the continuous curves obtained for $l_0 = 0.1$ with broken lines corresponding to $l_0 = 0$, it can be seen that there exists a value of B_0 below which the results for $l_0 = 0.1$ are higher and above which are smaller than the appropriate amplitudes for $l_0 = 0$. Diagrams in Fig.8 give also the explanation to the fact that amplitude-frequency curves determined for $l_0 = 0.1$ and $c_0 = B_0 = 1$ were practically identical to those presented in Fig.6 and Fig.7 for $l_0 = 0$. The same conclusions of the effect of the wave speed c_0 can be drawn from the diagrams of the resonant amplitudes for $K_r = 0.1$ and $K_r = 0.5$, they are therefore not given in the paper. However, the amplitudes decrease with the increase of K_r, what can also be observed in Fig.7.

The curves presented in Fig.8 are determined for $l_0 = 0.1$. The diagrams in Fig.9 concern the resonant amplitudes of the function $y(x,t) - y(l_0,t)$ for $l_0 = 0.1$ and $l_0 = 0.2$ with the parameters (3.13), $K_r = 0.3$, $D_0 = 0.1$, $c_0 = 0.5$, 1, 2 in the cross-sections $x = l_0 + 1$ and $x = l_0 + 2$. From Fig.9 it follows that resonant amplitudes for $l_0 = 0.2$ are lower than for $l_0 = 0.1$ except when $B_0 < 1.7$ and $c_0 = 1$. For the fixed c_0 the differences between corresponding amplitudes for $l_0 = 0.1$ and $l_0 = 0.2$ increase with the increase of B_0. The highest differences occur for $c_0 = 0.5$ and the least ones for $c_0 = 2$.

4. Final remarks

The considerations presented in this paper have a theoretical character. The comparison between the frequencies of free vibration for the Timoshenko equations for a beam and their specific cases indicates that in dynamic investigations of the systems subject to shear loadings, which can be modelled by means of short beams, it is permissible to employ the classical wave equation and its wave solution.

In the paper two discrete-continuous models, which can represent various low engineering structures subject to transversal external excitation are investigated. In the multi-body model the segment of isolation type is neglected or taken into account. From numerical calculations concerning the effect of the wave speed and the segment length on vibrations it follows that taking into account the segment in most cases results in the decrease of amplitudes.

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Modele dyskretno-ciągłe w analizie niskich obiektów poddanych wymuszeniom kinematycznym wywołanym falami poprzecznymi

Streszczenie

Praca składa się z dwóch części. W pierwszej zbadano wpływ sił poprzecznych na częstości drgań własnych krótkiej belki wykorzystując równania Timoshenki, uproszczone równania Timoshenki i klasyczne równanie falowe. Następnie zaproponowano dwa modele dyskretno-ciągłe do badań dynamicznych niskich obiektów poddanych wymuszeniu kinematycznemu wywołanemu falami poprzecznymi. Modele te składają się z bryl sztywnych i elementów sprężystych poddanych tylko odkształceniom ścinającym. Drugi model zawiera dodatkowy segment sprężysty usytuowany pomiędzy dolną bryłą a pozostałymi elementami modelu. W rozważaniach zastosowano metodę falową, w której wykorzystuje się rozwiązanie falowe równań ruchu. Obliczenia numeryczne wykonano dla modeli z trzema bryłami sztywnymi. Koncentrują się one na wyznaczaniu krzywych amplitudowo-częstościowych i badaniu wpływu własności mechanicznych i geometrycznych dodatkowego segmentu na amplitudy rezonansowe.

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