Science and Technology Indonesia
e-ISSN:2580-4391 p-ISSN:2580-4405
Vol. 8, No. 3, July 2023

Research Paper

The Subclasses of Analytic Functions of Complex Order with Application of q-Derivative Operators

Aini Janteng[1*], Desmond Lee Ching Yiing[2], Yong Enn Lun[1*], Jaludin Janteng[3], Lee See Keong[4], Rashidah Omar[5], Andy Liew Pik Hern[1]

[1]Faculty of Science and Natural Resources, Universiti Malaysia Sabah, Kota Kinabalu, Sabah, 88400, Malaysia
[2]Faculty of Science, Universiti Teknologi Malaysia, Johor Bahru, Johor, 81310, Malaysia
[3]Labuan Faculty of International Finance, Universiti Malaysia Sabah, Federal Territory of Labuan, 87000, Malaysia
[4]School of Mathematical Sciences, Universiti Sains Malaysia, Penang, 11800, Malaysia
[5]Faculty of Computer and Mathematical Sciences, Universiti Teknologi Mara Cawangan Sabah, Kota Kinabalu, Sabah, 88997, Malaysia
*Corresponding author: aini_jg@ums.edu.my; yongel@ums.edu.my

Abstract
In this article, we represent A as the of analytic functions in the open unit disk. Further, new subclasses of analytic functions of complex order utilising q-derivative operator are generated. The subclasses are symbolised by $H_{q,b}(\varphi)$ and $I_{q,b}(\varphi)$. Additionally, we discover that these function classes are implicated with the Fekete-Szegö inequalities.

Keywords
Analytic, q-Derivative Operator, Fekete-Szegö Inequality

Received: 24 March 2023, Accepted: 10 June 2023
https://doi.org/10.26554/sti.2023.8.3.436-442

1. INTRODUCTION

In recent times, research on the field of quantum calculus is actively being done by mathematicians. This is demonstrated by the use of it in the study of complex analysis, particularly geometry functions theories. Quantum calculus referred to as calculus without limits, is a kind of standard infinitesimal calculus that does not include the concept of limits.

The terms q-calculus and h-calculus are defined, where q is quantum and h is Planck's constant. q-calculus bridges the gap between physics and mathematics. Its applications are often seen in numerous branches of mathematics such as discrete mathematics, complex numbers, fundamental hypergeometric polynomials, symmetric coefficients, as well as other fields like quantum physics, general relativity and mechanics. Jackson (1908) and Jackson (1910) started the implementation of q-calculus. Jackson pioneered the q-integral and q-derivative systematically. Motivated by the research by Aldweby and Darus (2016) and Aldweby and Darus (2017), this study focuses on the q-derivative in defining several subclasses of analytic functions.

In the current study, let A symbolise the class of analytic functions in the open unit disk $\mathbb{O} = \{\zeta : \zeta \in \mathbb{C}, |\zeta| < 1\}$. The analytic function is represented by function f and has a Maclaurin series expansion as below:

$$f(\zeta) = \zeta + \sum_{\tau=2}^{\infty} a_\tau \zeta^\tau, \quad a_\tau \in \mathbb{C},\ \zeta \in \mathbb{O} \tag{1}$$

By Jeyaraman and Suresh (2014), if f and g are represented by the analytic functions in $\mathbb{O}$, then the subordinate to g is f denoted as $f \prec g$ in $\mathbb{O}$ or $f(\zeta) \prec g(\zeta)$ for all $\zeta \in \mathbb{O}$ if the presence of the Schwarz function $w(\zeta)$ is analytic in $\mathbb{O}$ that has the properties $|w(\zeta)| < 1$ and $w(0) = 0$ for all $\zeta \in \mathbb{O}$ such that

$$f(\zeta) = g(w(\zeta)), \quad \zeta \in \mathbb{O}$$

In the case g is univalent, we can state that $f \prec g$, if and only if $f(0) = g(0)$ and $f(\mathbb{O}) \subseteq g(\mathbb{O})$.

Now, we provide the basic definition of q-derivative in this study. Let $D_q f$ symbolise the q-derivative operator of f with $0 < q < 1$. It is defined in Aral et al. (2013); Jackson (1908) as below:

$$D_q f(\zeta) = \begin{cases} \frac{f(q\zeta)-f(\zeta)}{(q-1)\zeta}, & \zeta \neq 0 \text{ and } q \neq 1 \\ f'(0), & \zeta = 0 \end{cases} \tag{2}$$

In view of (1) and (2), it can be shown that

$$D_q(f(\zeta)) = 1 + \sum_{\tau=2}^{\infty} [\tau]_q a_\tau \zeta^{\tau-1}$$

where the formulae of $[\tau]_q$ is given as below:

$$[\tau]_q = \frac{1-q^\tau}{1-q}$$

and noted that as $q \to 1^-$, $[\tau]_q \to \tau$.

In the studies of previous researchers, several new subclasses of analytic functions utilising q-derivative operator have been proposed. Aldweby and Darus (2016) have introduced the following subclasses:

$$S_q^*(\gamma) = \left\{ f \in A : Re\left(\frac{\zeta D_q(f(\zeta))}{f(\zeta)}\right) > \gamma, \zeta \in \mathbb{O} \right\}$$

and

$$C_q(\gamma) = \left\{ f \in A : Re\left(1 + \frac{\zeta q D_q(D_q(f(\zeta))}{D_q(f(\zeta))}\right) > \gamma, \zeta \in \mathbb{O} \right\}$$

where $0 \leq \gamma < 1$.

Further, if $q \to 1^-$, the class $S_q^*(\gamma)$ and $C_q(\gamma)$ reduces to the starlike functions of order γ, $S^*(\gamma)$, and convex functions of order γ, $C(\gamma)$, respectively. Furthermore, there are several new subclasses of A involving q-derivative that have been introduced by other mathematicians (see Aldweby and Darus, 2017; Alsoboh and Darus, 2019; Altinkaya and Yalçin, 2017; Bulut, 2017; Janteng and Halim, 2009b; Karahuseyin, 2017; Lashin et al., 2021; Hern et al., 2022; Hern et al., 2020; Olatunji and Dutta, 2018; Piejko and Sokół, 2020; Ramachandran et al., 2017; Shamsan et al., 2021; Shilpa, 2022). In fact Seoudy and Aouf (2016) have introduced how classes of q-convex and q-starlike of complex order can be obtained using the principle of subordination and q-derivative. This study has inspired other mathematicians to study subclasses of analytic functions of complex order utilising q-derivative (see Ali and El Ashwah, 2021; Ibrahim et al., 2020; Purohit and Raina, 2014; Selvaraj et al., 2017; Srivastava and El Deeb, 2020; Srivastava and Zayed, 2019).

The research from Seoudy and Aouf (2016) and the research from Janteng and Halim (2009a); Janteng and Halim (2020) serves as our inspiration as we utilise the q-derivative of $f \in A$ and the subordination principle to propose new subclasses of analytic functions of complex order.

Definition 1.
Let P represents the class of analytic and univalent functions $\varphi(\zeta)$ in $\mathbb{O}$. $\varphi(\zeta)$ is convex with the properties $\varphi(0) = 1$ and $Re(\varphi(\zeta)) > 0$ for all $\zeta \in \mathbb{O}$. $f \in A$ is considered as belonging to class $H_{q,b}(\varphi)$ if f fulfils the subordination criteria as below:

$$1 + \frac{1}{b}\left[\frac{\zeta D_q(f(\zeta))}{f(\zeta)} + \frac{\epsilon q \zeta^2 D_q\left(D_q(f(\zeta))\right)}{f(\zeta)} - 1\right] \prec \varphi(\zeta) \quad (3)$$

with $\epsilon \geq 0,\ b \in \mathbb{C}\backslash\{0\}$, *and* $\varphi(\zeta) \in P$.

Definition 2.
$f \in A$ is considered as belonging to class $I_{q,b}(\varphi)$ if f fulfils the subordination criteria as below:

$$1 + \frac{1}{b}\left[\left(\frac{\zeta D_q(f(\zeta))}{f(\zeta)}\right)^{\epsilon}\left(1 + \frac{\zeta q D_q(D_q(f(\zeta)))}{D_q(f(\zeta))}\right)^{1-\epsilon} - 1\right] \prec \varphi(\zeta)$$

with $\epsilon \geq 0$, $b \in \mathbb{C} \setminus \{0\}$,and $\varphi(\zeta) \in P$.

The demonstration of our key findings requires the use of the preceding lemmas.

Lemma 1.
Ma and Minda (1992) If υ is a complex number and $p(\zeta) = 1 + c_1\zeta + c_2\zeta^2 + \cdots$ is a function that has a positive real part in $\mathbb{O}$, then

$$|c_2 - \upsilon c_1^2| \leq 2\max\{1, |2\upsilon - 1|\} \quad (4)$$

The Equality (4) holds for functions provided by

$$p(\zeta) = \frac{1+\zeta}{1-\zeta}$$

and

$$p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2}$$

Lemma 2.
Ma and Minda (1992) If $p(\zeta) = 1 + c_1\zeta + c_2\zeta^2 + \cdots$ is a function that has a positive real part in $\mathbb{O}$, then

$$|c_2 - \upsilon c_1^2| \leq \begin{cases} -4\upsilon + 2 & \text{if } \upsilon \leq 0 \\ 2 & \text{if } 0 \leq \upsilon \leq 1 \\ 4\upsilon - 2 & \text{if } \upsilon \geq 1 \end{cases}$$

When $\upsilon < 0$ or $\upsilon > 1$, the result is sharp if and only if $p(\zeta) = \dfrac{1+\zeta}{1-\zeta}$ or a rotation of itself. If $0 < \upsilon < 1$, the result is sharp if and only if $p(\zeta) = \dfrac{1+\zeta^2}{1-\zeta^2}$ or a rotation of itself. If $\upsilon = 0$, the result is sharp if and only if

$$p(\zeta) = \left(\frac{1}{2} + \frac{1}{2}\delta\right) \cdot \frac{1+\zeta}{1-\zeta} + \left(\frac{1}{2} - \frac{1}{2}\delta\right) \cdot \frac{1-\zeta}{1+\zeta} \quad (0 \leq \delta \leq 1)$$

or a rotation of itself. If $\upsilon = 1$, the result is sharp if and only if $p(\zeta)$ is the reciprocal of a function of itself such that the result is sharp when $\upsilon = 0$.

It is a sharp upper bound, with the possible improvements when $0 < \upsilon < 1$:

$$|c_2 - \upsilon c_1^2| + \upsilon |c_1|^2 \le 2 \quad (0 \le \upsilon \le 1/2)$$

and

$$|c_2 - \upsilon c_1^2| + (1-\upsilon)|c_1|^2 \le 2 \quad (1/2 \le \upsilon \le 1)$$

2. RESULTS AND DISCUSSION

Throughout this study, the Fekete-Szegö inequalities for classes $H_{q,b}(\varphi)$ and $I_{q,b}(\varphi)$ provided $0 < q < 1$, $b \in \mathbb{C} \setminus \{0\}$ and $\varphi(\zeta) \in P$ and $\zeta \in \mathbb{O}$ are obtained.

Theorem 1.
Let $\varphi(\zeta) = 1 + B_1\zeta + B_2\zeta^2 + \cdots$ with $B_1 \neq 0$. If f is provided by Equation (1) in class $\mathscr{H}_{q,b}(\varphi)$ with $\varepsilon \ge 0$, then

$$|a_3 - \mu a_2^2| \le \frac{|B_1 b|}{q[2]_q(\varepsilon[3]_q + 1)} \max\left\{1; \left|\frac{B_2}{B_1} + \frac{B_1 b}{q(\varepsilon[2]_q + 1)}\left(1 - \frac{[2]_q(\varepsilon[3]_q+1)}{\varepsilon[2]_q + 1}\mu\right)\right|\right\}$$

Proof. Let $f \in H_{q,b}(\varphi)$, then f fulfils subordination (3). According to the subordination principle, the Schwarz function $w(\zeta)$ is present, and it is analytic in $\mathbb{O}$, which has the properties $w(0) = 0$ and $|w(\zeta)| < 1$ such that

$$1 + \frac{1}{b}\left[\frac{\zeta D_q(f(\zeta))}{f(\zeta)} + \frac{\varepsilon q \zeta^2 D_q(D_q(f(\zeta)))}{f(\zeta)} - 1\right] = \varphi(w(\zeta)) \quad (5)$$

Let $p(\zeta)$ be a function which is defined by

$$p(\zeta) = \frac{1 + w(\zeta)}{1 - w(\zeta)} = 1 + c_1\zeta + c_2\zeta^2 + \cdots \quad (6)$$

Since $w(\zeta)$ is a Schwarz function, it is obvious for $\mathrm{Re}(p(\zeta)) > 0$ and $p(0) = 1$.

In view of Equation (6), we obtain

$$w(\zeta) = \frac{p(\zeta) - 1}{p(\zeta) + 1} = \frac{1}{2}\left[c_1\zeta + \left(c_2 - \frac{c_1^2}{2}\right)\zeta^2 + \ldots\right] \quad (7)$$

From Equation (5) and (7), we obtain

$$\varphi(w(\zeta)) = 1 + \frac{1}{2}B_1c_1\zeta + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right]\zeta^2 + \ldots \quad (8)$$

Now, by utilising Equation (1) and (2), we obtain

$$\begin{aligned} &1 + \frac{1}{b}\left[\frac{\zeta D_q(f(\zeta))}{f(\zeta)} + \frac{\epsilon q \zeta^2 D_q(D_q(f(\zeta)))}{f(\zeta)} - 1\right] \\ &= 1 + \frac{q(\varepsilon[2]_q + 1)a_2\zeta}{b} \\ &+ \frac{q\left([2]_q(\varepsilon[3]_q + 1)a_3 - (\varepsilon[2]_q + 1)a_2^2\right)\zeta^2}{b} + \ldots \end{aligned} \quad (9)$$

By utilising Equation (8) and (9), we compare the coefficients of ζ and ζ^2,

$$\frac{q(\varepsilon[2]_q + 1)a_2}{b} = \frac{1}{2}B_1c_1$$

$$\frac{q([2]_q(\varepsilon[3]_q + 1)a_3 - (\varepsilon[2]_q + 1)a_2^2)}{b} = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2$$

or equivalently,

$$a_2 = \frac{B_1c_1b}{2q(\epsilon[2]_q + 1)} \quad (10)$$

$$\begin{aligned} a_3 = &\frac{B_1c_2b}{2q[2]_q(\varepsilon[3]_q + 1)} - \frac{B_1c_1^2b}{4q[2]_q(\varepsilon[3]_q + 1)} \\ &+ \frac{B_2c_1^2b}{4q[2]_q(\epsilon[3]_q + 1)} + \frac{B_1^2c_1^2b^2}{4q^2[2]_q(\epsilon[2]_q + 1)(\epsilon[3]_q + 1)} \end{aligned} \quad (11)$$

Next, by utilising Equation (10) and (11), we obtain

$$a_3 - \mu a_2^2 = \frac{B_1 b}{2q[2]_q(\varepsilon[3]_q + 1)} \left[c_2 - \frac{1}{2}\left[1 - \frac{B_2}{B_1} - \frac{B_1 b}{q(\epsilon[2]_q + 1)}\left(1 - \frac{[2]_q(\epsilon[3]_q + 1)}{\epsilon[2]_q + 1}\mu\right)\right]c_1^2\right] \quad (12)$$

If we take

$$\upsilon = \frac{1}{2}\left[1 - \frac{B_2}{B_1} - \frac{B_1 b}{q(\epsilon[2]_q + 1)}\left(1 - \frac{[2]_q(\epsilon[3]_q + 1)}{\epsilon[2]_q + 1}\mu\right)\right]$$

then, from Equation (12),

$$|a_3 - \mu a_2^2| = \frac{|B_1 b|}{2q[2]_q(\epsilon[3]_q + 1)}\left|c_2 - \upsilon c_1^2\right| \tag{13}$$

As we demonstrate, the Fekete-Szegö inequality is derived for the class $H_{q,b}(\varphi)$ by applying inequality (4) of Lemma 1 in Equation (13). Theorem 1 has been successfully proved.

By setting $b = 1$, Theorem 1 has a corollary as below.

Corollary 1.
Let $\varphi(\zeta) = 1 + B_1\zeta + B_2\zeta^2 + \cdots$ with $B_1 \neq 0$. If f provided by Equation (1) is in class $H_q(\varphi)$ with $\varepsilon \geq 0$, then

$$|a_3 - \mu a_2^2| \leqslant \frac{|B_1|}{q[2]_q(\varepsilon[3]_q + 1)} \max\left\{1, \left|\frac{B_2}{B_1} + \frac{B_1}{q(\varepsilon[2]_q + 1)}\left(1 - \frac{[2]q(\varepsilon[3]_q + 1)}{\varepsilon[2]_q + 1}\mu\right)\right|\right\}$$

Theorem 2.
Let $\varphi(\zeta) = 1 + B_1\zeta + B_2\zeta^2 + \cdots$ with $B_1 > 0$ and $B_2 \geq 0$. Let

$$\sigma_1 = \frac{(\varepsilon[2]_q + 1)B_1^2 b + q(\varepsilon[2]_q + 1)^2(B_2 - B_1)}{[2]q(\varepsilon[3]_q + 1)B_1^2 b}$$

$$\sigma_2 = \frac{(\epsilon[2]_q + 1)B_1^2 b + q(\epsilon[2]_q + 1)^2(B_2 + B_1)}{[2]_q(\epsilon[3]_q + 1)B_1^2 b}$$

and

$$\sigma_3 = \frac{(\epsilon[2]_q + 1)B_1^2 b + q(\epsilon[2]_q + 1)^2 B_2}{[2]q(\epsilon[3]_q + 1)B_1^2 b}$$

If f is provided by Equation (1) in class $H_{q,b}(\varphi)$ with $\epsilon \geq 0$ and $b > 0$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_2 b}{q[2]_q(\epsilon[3]_q+1)} + \frac{B_1^2 b^2}{q^2[2]_q(\epsilon[2]_q+1)(\epsilon[3]_q+1)} \\ \left(1 - \frac{[2]q(\epsilon[3]_q+1)}{\epsilon[2]_q+1}\mu\right), \quad \mu \leq \sigma_1 \\ \frac{B_1 b}{q[2]_q(\epsilon[3]_q+1)}, \quad \sigma_1 \leq \mu \leq \sigma_2 \\ -\frac{B_2 b}{q[2]_q(\epsilon[3]_q+1)} - \frac{B_1^2 b^2}{q^2[2]_q(\epsilon[2]_q+1)(\epsilon[3]_q+1)} \\ \left(1 - \frac{[2]q(\epsilon[3]_q+1)}{\epsilon[2]_q+1}\mu\right), \quad \mu \geq \sigma_2 \end{cases}$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{q(\epsilon[2]_q + 1)^2}{[2]_q(\epsilon[3]_q + 1)B_1^2 b} \left[B_1 - B_2 - \frac{B_1^2 b}{q(\epsilon[2]_q + 1)}\left(1 - \frac{\mu[2]_q(\epsilon[3]_q + 1)}{\epsilon[2]_q + 1}\right)\right]|a_2|^2 \leq \frac{B_1 b}{q[2]_q(\epsilon[3]_q + 1)}$$

and if $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{q(\epsilon[2]_q + 1)^2}{[2]_q(\epsilon[3]_q + 1)B_1^2 b} \left[B_1 + B_2 + \frac{B_1^2 b}{q(\epsilon[2]_q + 1)}\left(1 - \frac{\mu[2]_q(\epsilon[3]_q + 1)}{\epsilon[2]_q + 1}\right)\right]|a_2|^2 \leq \frac{B_1 b}{q[2]_q(\epsilon[3]_q + 1)}$$

Proof. By utilising Lemma 2 in Equation (12), we can obtain the results in Theorem 2.

By setting $b = 1$, Theorem 2 has a corollary as below.

Corollary 2.
Let $\varphi(\zeta) = 1 + B_1\zeta + B_2\zeta^2 + \cdots$ with $B_1 > 0$ and $B_2 \geq 0$. Let

$$\sigma_1 = \frac{(\varepsilon[2]_q + 1)B_1^2 + q(\varepsilon[2]_q + 1)^2(B_2 - B_1)}{[2]q(\varepsilon[3]_q + 1)B_1^2}$$

$$\sigma_2 = \frac{(\epsilon[2]_q + 1)B_1^2 + q(\epsilon[2]_q + 1)^2(B_2 + B_1)}{[2]_q(\epsilon[3]_q + 1)B_1^2}$$

and

$$\sigma_3 = \frac{(\epsilon[2]_q + 1)B_1^2 + q(\epsilon[2]_q + 1)^2 B_2}{[2]q(\epsilon[3]_q + 1)B_1^2}$$

If f provided by Equation (1) is in class $H_q(\varphi)$ with $\epsilon \geq 0$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_2}{q[2]_q(\epsilon[3]_q+1)} + \frac{B_1^2}{q^2[2]_q(\epsilon[2]_q+1)(\epsilon[3]_q+1)} \\ \left(1 - \frac{[2]q(\epsilon[3]_q+1)}{\epsilon[2]_q+1}\mu\right), \quad \mu \leq \sigma_1 \\ \frac{B_1}{q[2]_q(\epsilon[3]_q+1)}, \quad \sigma_1 \leq \mu \leq \sigma_2 \\ -\frac{B_2}{q[2]_q(\epsilon[3]_q+1)} - \frac{B_1^2}{q^2[2]_q(\epsilon[2]_q+1)(\epsilon[3]_q+1)} \\ \left(1 - \frac{[2]q(\epsilon[3]_q+1)}{\epsilon[2]_q+1}\mu\right), \quad \mu \geq \sigma_2 \end{cases}$$

Further, if $\sigma_1 \le \mu \le \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{q(\epsilon[2]_q + 1)^2}{[2]_q(\epsilon[3]_q + 1)B_1^2}\left[B_1 - B_2 - \frac{B_1^2}{q(\epsilon[2]_q + 1)}\left(1 - \frac{\mu[2]_q(\epsilon[3]_q + 1)}{\epsilon[2]_q + 1}\right)\right]|a_2|^2 \le \frac{B_1}{q[2]_q(\epsilon[3]_q + 1)}$$

and if $\sigma_3 \le \mu \le \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{q(\epsilon[2]_q + 1)^2}{[2]_q(\epsilon[3]_q + 1)B_1^2}\left[B_1 + B_2 + \frac{B_1^2}{q(\epsilon[2]_q + 1)}\left(1 - \frac{\mu[2]_q(\epsilon[3]_q + 1)}{\epsilon[2]_q + 1}\right)\right]|a_2|^2 \le \frac{B_1}{q[2]_q(\epsilon[3]_q + 1)}$$

In addition, as shown above, the Fekete-Szegö inequality is discovered for the class $I_{q,b}(\varphi)$. We get the required outcome by carrying out the processes as in Theorem 1.

Theorem 3.
Let $\varphi(\zeta) = 1 + B_1\zeta + B_2\zeta^2 + \cdots$ with $B_1 \neq 0$. If f is provided by Equation (1) in class $\mathcal{J}_{q,b}(\varphi)$ with $\epsilon \ge 0$, then

$$|a_3 - \mu a_2^2| \le \frac{|B_1 b|}{q\delta_3[2]_q}\max\left\{1, \left|\frac{B_2}{B_1} + \frac{B_1 b}{q\delta_3^2}(\rho + \mu\delta_3[2]_q)\right|\right\}$$

where

$$\rho = \epsilon - \frac{\epsilon(\epsilon - 1)}{2}q([2]_q^2 + 1) + (1 - \epsilon)[2]_q(\epsilon q - [2]_q)$$

and

$$\delta_\tau = \epsilon + (1 - \epsilon)[\tau]_q.$$

By setting $b = 1$, Theorem 3 has a corollary as below.

Corollary 3.
Let $\varphi(\zeta) = 1 + B_1\zeta + B_2\zeta^2 + \cdots$ with $B_1 \neq 0$. If f is provided by Equation (1) in class $\mathcal{J}_q(\varphi)$ with $\epsilon \ge 0$, then

$$|a_3 - \mu a_2^2| \le \frac{|B_1|}{q\delta_3[2]_q}\max\left\{1, \left|\frac{B_2}{B_1} + \frac{B_1}{q\delta_3^2}(\rho + \mu\delta_3[2]_q)\right|\right\}$$

where

$$\rho = \epsilon - \frac{\epsilon(\epsilon - 1)}{2}q([2]_q^2 + 1) + (1 - \epsilon)[2]_q(\epsilon q - [2]_q)$$

and

$$\delta_\tau = \epsilon + (1 - \epsilon)[\tau]_q.$$

Theorem 4.
Let $\varphi(\zeta) = 1 + B_1\zeta + B_2\zeta^2 + \cdots$ with $B_1 > 0$ and $B_2 \ge 0$. Let

$$\sigma_4 = \frac{\rho B_1^2 b + q\delta_2^2(B_2 - B_1)}{\delta_3[2]_q B_1^2 b},$$

$$\sigma_5 = \frac{\rho B_1^2 b + q\delta_2^2(B_2 + B_1)}{\delta_3[2]_q B_1^2 b},$$

and

$$\sigma_6 = \frac{\rho B_1^2 b + q\delta_2^2 B_2}{\delta_3[2]_q B_1^2 b}.$$

If f is provided by Equation (1) in class $I_{q,b}(\varphi)$ with $\epsilon \ge 0$ and $b > 0$, then

$$|a_3 - \mu a_2^2| \begin{cases} \frac{B_2 b}{\delta_3 q[2]_q} + \frac{B_1^2 b}{q^2\delta_2^2\delta_3[2]_q}(\rho - \mu\delta_3[2]_q), & \mu \le \sigma_4 \\ \frac{B_1 b}{\delta_3 q[2]_q}, & \sigma_4 \le \mu \le \sigma_5 \\ -\frac{B_2 b}{\delta_3 q[2]_q} - \frac{B_1^2 b}{q^2\delta_2^2\delta_3[2]_q}(\rho - \mu\delta_3[2]_q), & \mu \ge \sigma_5 \end{cases}$$

Further, if $\sigma_4 \le \mu \le \sigma_6$, then

$$|a_3 - \mu a_2^2| + \frac{q\delta_2^2}{\delta_3[2]_q B_1^2 b}\left[B_1 - B_2 - \frac{B_1^2 b}{q\delta_2^2}(\rho - \mu\delta_3[2]_q)\right]$$
$$|a_2|^2 \le \frac{B_1 b}{\delta_3 q[2]_q}$$

and if $\sigma_6 \le \mu \le \sigma_5$, then

$$|a_3 - \mu a_2^2| + \frac{q\delta_2^2}{\delta_3[2]_q B_1^2 b}\left[B_1 + B_2 + \frac{B_1^2 b}{q\delta_2^2}(\rho - \mu\delta_3[2]_q)\right]$$
$$|a_2|^2 \le \frac{B_1 b}{\delta_3 q[2]_q}$$

where

$$\rho = \epsilon - \frac{\epsilon(\epsilon - 1)}{2}q([2]_q^2 + 1) + (1 - \epsilon)[2]_q(\epsilon q - [2]_q)$$

and

$$\delta_\tau = \epsilon + (1-\epsilon)[\tau]_q.$$

By setting $b = 1$, Theorem 4 has a corollary as below.

Corollary 4.
Let $\varphi(\zeta) = 1 + B_1\zeta + B_2\zeta^2 + \cdots$ with $B_1 > 0$ and $B_2 \geq 0$. Let

$$\sigma_4 = \frac{\rho B_1^2 + q\delta_2^2(B_2 - B_1)}{\delta_3[2]_q B_1^2},$$

$$\sigma_5 = \frac{\rho B_1^2 + q\delta_2^2(B_2 + B_1)}{\delta_3[2]_q B_1^2}$$

and

$$\sigma_6 = \frac{\rho B_1^2 + q\delta_2^2 B_2}{\delta_3[2]_q B_1^2}$$

If f is provided by Equation (1) in class $I_q(\varphi)$ with $\epsilon \geq 0$ then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_2}{\delta_3 q[2]_q} + \frac{B_1^2}{q^2\delta_2^2\delta_3[2]_q}(\rho - \mu\delta_3[2]_q), & \mu \leq \sigma_4 \\ \frac{B_1}{\delta_3 q[2]_q}, & \sigma_4 \leq \mu \leq \sigma_5 \\ -\frac{B_2}{\delta_3 q[2]_q} - \frac{B_1^2}{q^2\delta_2^2\delta_3[2]_q}(\rho - \mu\delta_3[2]_q), & \mu \geq \sigma_5 \end{cases}$$

Further, if $\sigma_4 \leq \mu \leq \sigma_6$, then

$$|a_3 - \mu a_2^2| + \frac{q\delta_2^2}{\delta_3[2]_q B_1^2}\left[B_1 - B_2 - \frac{B_1^2}{q\delta_2^2}(\rho - \mu\delta_3[2]_q)\right]$$
$$|a_2|^2 \leq \frac{B_1}{\delta_3 q[2]_q}$$

and if $\sigma_6 \leq \mu \leq \sigma_5$, then

$$|a_3 - \mu a_2^2| + \frac{q\delta_2^2}{\delta_3[2]_q B_1^2}\left[B_1 + B_2 + \frac{B_1^2}{q\delta_2^2}(\rho - \mu\delta_3[2]_q)\right]$$
$$|a_2|^2 \leq \frac{B_1}{\delta_3 q[2]_q}$$

where

$$\rho = \epsilon - \frac{\epsilon(\epsilon-1)}{2}q([2]_q^2 + 1) + (1-\epsilon)[2]_q(\epsilon q - [2]_q)$$

and

$$\delta_\tau = \epsilon + (1-\epsilon)[\tau]_q.$$

3. CONCLUSION

This study demonstrates two discoveries of subclasses belonging to the analytic functions of complex order by applying the q-derivative operator, $H_{q,b}(\varphi)$ and $I_{q,b}(\varphi)$, for finding results for the Fekete-Szegö inequalities.

4. ACKNOWLEDGMENT

We thank the research funding (SBK0485-2021) provided by Universiti Malaysia Sabah, the reviewers' time with constructive feedback, and all the references.

REFERENCES

Aldweby and Darus (2016). Coefficient Estimates of Classes of Q-starlike and Q-convex Functions. *Advanced Studies in Contemporary Mathematics*, **26**(1); 21–26

Aldweby, H. and M. Darus (2017). On Fekete-Szegö Problems for Certain Subclasses Defined by-Derivative. *Journal of Function Spaces*, **2017**; 1–5

Ali, E. and R. El Ashwah (2021). Convolution Properties of Bounded Analytic Function Classes with Higher and Complex Order Defined by q-derivatives Operator. *Journal of Physics: Conference Series*, **1943**; 012115

Alsoboh, A. and M. Darus (2019). On Fekete-Szegö Problem Associated with q-derivative Operator. *Journal of Physics: Conference Series*, **1212**; 012003

Altinkaya, Ş. and S. Yalçin (2017). On The Fekete-Szegö Problem For Analytic Functions Defined by Using Symmetric q-derivative Operator. *Konuralp Journal of Mathematics*, **5**(1); 176–186

Aral, A., V. Gupta, and R. P. Agarwal (2013). *Applications of q-calculus in Operator Theory*. Springer

Bulut, S. (2017). Certain Subclasses of Analytic and Bi-univalent Functions Involving the q-derivative Operator. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, **66**(1); 108–114

Hern, A. L. P., A. Janteng, and R. Omar (2020). Hankel Determinant $H_2(3)$ for Certain Subclasses of Univalent Functions. *Mathematics and Statistics*, **8**(5); 566–569

Hern, A. L. P., A. Janteng, and R. Omar (2022). Subclasses of Analytic Functions with Negative Coefficients Involving q-Derivative Operator. *Sciences and Technology Indonesia*, **7**(3); 327–332

Ibrahim, M., A. Senguttuvan, D. Mohankumar, and R. G. Raman (2020). On Classes of Janowski Functions of Complex Order Involving a q-Derivative Operator. *International Journal of Mathematics and Computer Science*, **15**(4); 1161–1172

Jackson, F. H. (1908). On q-Functions and a Certain Difference Operator. *Transactions of the Royal Society of Edinburgh*, **46**; 253–281

Jackson, F. H. (1910). On q-definite Integrals. *Quarterly Journal of Pure and Applied Mathematics*, **41**; 193–203

Janteng and Halim (2020). Fekete-Szegö Functional of Classes of Analytic Functions Involving the q-Derivative Operator. *Applied Mathematical Sciences*, **14**(10); 481–488

Janteng, A. and S. A. Halim (2009a). Coefficient Estimate for a Subclass of Close-to-convex Functions with Respect to Symmetric Points. *International Journal of Mathematical Analysis*, **3**(7); 309–313

Janteng, A. and S. A. Halim (2009b). A Subclass Quasi-convex Functions with Respect to Symmetric Points. *Applied Mathematical Sciences*, **3**(12); 551–556

Jeyaraman, M. and T. Suresh (2014). Subordination Properties of Multivalent Functions Defined by Generalized Multiplier Transformation. *Journal of Complex Analysis*, **2014**; 1–12

Karahuseyin, A. S. Y. S., Z. (2017). On $H_2(1)$ Hankel Determinant for Univalent Functions Defined by Using q-DerivOn $H_2(1)$ Hankel Determinant for Univalent Functions Defined by Using q-Deriv. *Turkish Journal of Mathematics and Mathematics Education*, **9**(1); 25–33

Lashin, A. M. Y., B. M. Algethami, and A. O. Badghaish (2021). Convolution Properties of Certain Classes of Analytic Functions Defined by Jackson q-Derivative. *Mathematics*, **10**(1); 105

Ma, W. and M. Minda (1992). A Unified Treatment of Some Special Classes of Univalent Functions. *International Conference on Complex Analysis at the Nankai Institute of Mathematics*; 157–169

Olatunji, S. and H. Dutta (2018). Fekete-Szegö Problem for Certain Analytic Functions Defined By q-Derivative Operator with Respect to Symmetric and Conjugate Points. *Proyecciones (Antofagasta)*, **37**(4); 627–635

Piejko, K. and J. Sokół (2020). On Convolution and q-calculus. *Boletín de la Sociedad Matemática Mexicana*, **26**(2); 349–359

Purohit, S. and R. Raina (2014). Some Classes of Analytic and Multivalent Functions Associated with q-derivative Operators. *Acta Universitatis Sapientiae, Mathematica*, **6**(1); 5–23

Ramachandran, C., T. Soupramanien, and B. Frasin (2017). New Subclasses of Analytic Function Associated with Q-difference Operator. *European Journal of Pure and Applied Mathematics*, **10**(2); 348–362

Selvaraj, C., K. Karthikeyan, and S. Lakshmi (2017). Coefficient Estimates of New Classes of q- Starlike and q- Convex Functions of Complex Order with Respect to (j, k)- Symmetric Points. *Omanian Journal of Mathematics and Computer Science*, **7**; 29–40

Seoudy, T. and M. Aouf (2016). Coefficient Estimates of New Classes of q-starlike and q-convex Functions of Complex Order. *Journal of Mathematical Inequalities*, **10**(1); 135–145

Shamsan, H., R. S. Qahtan, and S. Latha (2021). Coefficient Estimates for Bi-univalent Ma-minda Type Functions Associated With q-derivative. *TWMS J. App. and Eng. Math*, **11**(4); 1100–1109

Shilpa, N. (2022). Fekete-Szegö Inequalities for Certain Analytic FunctionsAssociated with q Derivative Operator. *Advances and Applications in Mathematical Sciences*, **21**(4); 2125–2135

Srivastava, H. M. and S. M. El Deeb (2020). A Certain Class of Analytic Functions of Complex Order Connected with a q-analogue of Integral Operators. *Miskolc Mathematical Notes*, **21**(1); 417–433

Srivastava, R. and H. M. Zayed (2019). Subclasses of Analytic Functions of Complex Order Defined By q-derivative Operator. *Stud. University Babes-Bolyai Math*, **64**(1); 71–80

