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Research Paper

Best Proximity Point Results in Fuzzy Normed Spaces

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Abstract
Fixed point (briefly FP) theory is a potent tool for resolving several actual problems since many problems may be simplified to the FP problem. The idea of Banach contraction mapping is a foundational theorem in FP theory. This idea has wide applications in several fields; hence, it has been developed in numerous ways. Nevertheless, all of these results are reliant on the existence and uniqueness of a FP on some suitable space. Because the FP problem could not have a solution in the case of nonself-mappings, the idea of the best proximity point (briefly Bpp) is offered to approach the best solution. This paper investigates the existence and uniqueness of the Bpp of nonself-mappings in fuzzy normed space(briefly F_N space) to arrive at the best solution. Following the introduction of the definition of the Bpp, the existence, and uniqueness of the Bpp are shown in a F_N space for diverse fuzzy proximal contractions such as $\mathfrak{B}\tilde{\psi}$ - fuzzy proximal contractive mapping and $\mathfrak{B}\hbar$ $\mathfrak{J}\hbar$ - fuzzy proximal contractive mapping.

Keywords
Fuzzy Normed Space, Fuzzy Proximal Contractive, Best Proximity Point, Fuzzy Banach Space, Cauchy Sequence

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1. INTRODUCTION

One of the most important branches of modern mathematics is functional analysis. It is crucial in the theory of differential equations, particularly partial differential equations, representation theory, and probability, as well as in the study of numerous properties of different spaces such as normed space, Hilbert space, Banach space, and others (Sabri and Ahmed, 2023; Nemah, 2017; Eiman and A.Mustafa, 2016; Zeana and K.Assma, 2016; Dakheel and Ahmed, 2021).

Numerous applications in mathematics and allied fields, such as inverse problems, are made possible by the well-known Banach FP theorem (Banach, 1922), which concerns the presence and uniqueness of FP of self-mappings defined on a complete metric space (see (Lin et al., 2018; Zhang and Hofmann, 2020)). The Banach FP theorem has drawn many academics to expand the reach of metric FP theory because of its wide variety of applications (see (Abbas et al., 2012; Latif et al., 2015; Mustafa et al., 2014)).

On the other hand, if $\mathscr{U}$ and $\mathscr{V}$ are both nonempty subsets of ($\mathscr{L}$; d) (where ($\mathscr{L}$; d) is a metric space), consequently in the situation of a mapping $\mathbf{T}$ $\mathscr{U}\rightarrow$ $\mathscr{V}$, it is possible that there is not a point u in $\mathscr{U}$ such that u=Tu: where d is the distance between $\mathscr{U}$ and $\mathscr{V}$. In these kinds of instances, it is preferable to locate an element u in $\mathscr{U}$ such that the distance between u and $\mathbf{T}$ u is as low as possible. If there is such an element u in $\mathscr{U}$, consequently it represents the Bpp of $\mathbf{T}$.

Zadeh developed and examined the concept of a fuzzy set in his groundbreaking research in Zadeh (1978). The exploration of fuzzy sets has resulted in the fuzzification of several distinct mathematical ideas. It has potential applications in many different domains. Kramosil and Michálek (1975) initially presented the idea of fuzzy metric spaces. Then the concept of fuzzy metric spaces was modified by George and Veeramani (1994). Numerous articles have been published on fuzzy metric spaces (Hussain et al., 2020; Paknazar, 2018; Gregori et al., 2020; Zainab and Kider, 2021 Sabri, 2021; Li and Zhang, 2023). F_N space was subsequently introduced in different methods by a large number of other mathematicians. F_N space has been the topic of a considerable number of publications; for instance, see (Miheţ and Zaharia, 2014; Sabri and Ahmed, 2022; Kider and Kadhum, 2019; Sharma and Hazarika, 2020; Sabre, 2012; Konwar and Debnath, 2023; Sabri and Ahmed, 2023; Raghad, 2021).

Examining the presence and uniqueness of the Bpp in a F_N space is the goal of this work, offering an approach to expand and fuzzify results in normed spaces. To that end, three theorems are presented that demonstrate the presence and uniqueness of the Bpp under various circumstances. In addition, an example is provided to demonstrate the use of the main theorem.

2. EXPERIMENTAL SECTION

2.1 Preliminaries

This section defines the terminology and outcomes which is going to be utilized throughout the paper. Materials :

2.1.1 Definition 3.1 Nadaban and Dzitac (2014)

Let **D** represent the vector space over the field R. A triplet (**D**, F_N, ⊗) is termed a F_N space where ⊗ is a t-norm and F_N represent a fuzzy set on **D**×R satisfies the requirements below for every p,q∈**D**:

$$(F_{N1}) F_N(p, 0) = 0,$$
$$(F_{N2}) F_N(p, \tau) = 1\forall > 0 \text{ if and only p = 0}$$
$$(F_{N3}) F_N(rp, \tau) = F_N\left(p, \frac{\tau}{|r|}\right), \forall r \in R, \text{ where r } \neq 0 \text{ and } \tau \geq 0$$
$$(F_{N4}) F_N(p, \tau) \otimes F_N(q, s) \leq F_N(p+q, \tau+s)\forall\tau \geq 0$$
$$(F_{N5}) F_N(p, .) \text{is the continuous for each p} \in \mathbf{D} \text{ and } \lim_{\tau\to\infty} F_N(p, \tau) = 1$$

2.1.2 Definition 3.2 Bag and Samanta (2003)

Consider (**D**,F_N,⊗) be a F_N space. then

1. A sequence p_n is called a convergent if $\lim_{n\to\infty} F_N$ (p_n- p, τ) = 1;∀ > 0 and p ∈**D**.
2. A sequence p_n is called Cauchy if $\lim_{n\to\infty} F_N$ (p_{n+1}- p_n, τ) = 1;∀ > 0 and j = 1,2,...

2.1.3 Definition 3.3 Bag and Samanta (2003)

A F_N space (**D**, F_N ,⊗) is called complete if every Cauchy sequence in **D** is convergent in **D**. In a F_N space (**D**,F_N,⊗), Sabri and Ahmed (2022) presented the notion of fuzzy distance. Consider $\tilde{U}$ and $\tilde{V}$ be subsets of (**D**,F_M,⊗) which are nonempty and $\tilde{U}\circ(\tau), \tilde{V}\circ(\tau)$ denoted by the following sets :

$$\tilde{U}\circ(\tau) = \{P \in U : F_N(p-q, \tau) = N_a(\tilde{U}, \tilde{V}, \tau) \text{ for some } q \in \tilde{v}\}$$
$$\tilde{V}\circ(\tau) = \{q \in \tilde{V} : F_N(p-q, \tau) = N_a(\tilde{U}, \tilde{V}, \tau) \text{ for some } p \in \tilde{U}\}$$
$$\text{Where} N_a(\tilde{U}, \tilde{V}, \tau) = sup\left\{F_N(p-q, \tau) : p \in \tilde{U}, q \in \tilde{V}\right\}$$

3. RESULTS AND DISCUSSION

In this section, the definition $\mathfrak{B}\tilde{\psi}$- fuzzy proximal contractive mapping and $\mathfrak{B}\hbar$ $\mathfrak{I}\hbar$- fuzzy proximal contractive mapping is presented, then our main results are proved. In a fuzzy metric space, Guria et al. (2019) proposed the notion of *B*pp. In the following, the notion of the *B*pp in the framework of F_N space is introduced.

3.1 Definition 4.1

Let (**D**, F_N, ⊗) be a F_N space and $\tilde{U}, \tilde{V}$ are nonempty subsets of **D** . An element P* ∈ $\tilde{V}$ is called the *B*pp of a mapping **T**:$\tilde{U}\to\tilde{V}$ if F_N (P*-Tp*, τ)= $N_a(\tilde{U}, \tilde{V}, \tau)$ for all $\tau > 0$

Next, the definition of $\mathfrak{B}\tilde{\psi}$- fuzzy proximal contractive mapping is presented. Consider ψ represents the collection of all functions $\tilde{\psi}$:[0,1] → [0,1], having the properties below:

1. $\tilde{\psi}$ is decreasing
2. $\tilde{\psi}$ is continuous
3. $\tilde{\psi}$ (μ)= 0 if and only if μ = 1

3.2 Definition 4.2

Assume that (**D**, F_N, ⊗) is a F_N space and let $\tilde{U}, \tilde{V}$ subsets of **D** which are nonempty.Let **T** $\tilde{U}\to\tilde{V}$ be a mapping. Then **T** $\mathfrak{B}\tilde{\psi}$ - fuzzy proximal contractive mapping where $\tilde{\psi} \in \psi$ if for all p,q,u,v ∈ $\tilde{U}$ we have,

$$\left.\begin{array}{l} F_N(u-\mathbf{T}p, \tau) = N_a(\tilde{U}, \tilde{V}, \tau) \\ F_N(u-\mathbf{T}q, \tau) = N_a(\tilde{U}, \tilde{V}, \tau) \end{array}\right\} \Rightarrow \tilde{\psi}(F_N(u-v, \tau)) \leq \vartheta(\tau)\mathfrak{B}\tilde{\psi}(p, q, \tau) \quad (1)$$

where ϑ:(0,∞)→(0,1) is a function and $\mathfrak{B}\tilde{\psi}$ (p,q,τ) = max $\tilde{\psi}(F_N$(p-q,τ))$\tilde{\psi}(F_N$(p-u,τ)) ⊗ $\tilde{\psi}(F_N$(p-u,τ)), $\tilde{\psi}(F_N$(q-u,τ))

3.3 Theorem 4.3

Assume that (**D**, F_N, ⊗) be a fuzzy Banach space (brefily FB space) where ⊗ is min t-norm and let $\tilde{U}, \tilde{V}$ subsets of **D** (where $\tilde{U}$ and $\tilde{V}$ closed). suppose $\tilde{U}\circ(\tau)$ is non-empty and **T** : $\tilde{U}\to\tilde{V}$ is nonself mapping fulfilling the following requirements:

1. **T** ($\tilde{U}\circ(\tau)$) ⊆ $\tilde{v}\circ(\tau)$, ∀ $\tau > 0$
2. **T** is $\mathfrak{B}\tilde{\psi}$ - fuzzy proximal contractive mapping
3. if a sequence qn is in $\tilde{V}\circ(\tau)$ and p ∈ $\tilde{U}$ such that F_N (p-qn,τ)= Nd ($\tilde{U}$,$\tilde{V}$,τ) as n → ∞ p ∈ $\tilde{U}\circ(\tau)$, ∀ $\tau > 0$

Then T possesses a unique *B*pp.

proof: Consider p∘ in $\tilde{U}\circ(\tau)$ since **T** ($\tilde{U}\circ(\tau)$) ⊆ $\tilde{V}$ (τ), there exists p1 ∈ $\tilde{U}\circ(\tau)$ such that F_N (p1-T∘,τ)=N_a ($\tilde{U}$,$\tilde{V}$,τ)∀ > 0

The process is repeated, and we get a sequence p_n in $\tilde{U}\circ(\tau)$ fulfilling

$$\begin{array}{l} F_N(Pn-\mathbf{T}p_{-1}, \tau) = N_a(\tilde{U}, \tilde{V}, \tau) \\ F_N(P_{n+1}-\mathbf{T}p_n, \tau) = N_a(\tilde{U}, \tilde{V}, \tau) \end{array} \quad (2)$$

If for any n∘ ∈ N, P_{n+1}=P_n, then according ton(2), is a *B*pp of **T**. Consequently, suppose $P_{n+1} \neq P_n$ for each n ∈ N. Now for each τ>0 and n ∈ N ∪ 0 define L_n (τ) = F_N (P_n-P_{n+1}, τ). From (1) we get

$$\tilde{\psi}(L_n(\tau)) = \tilde{\psi}(F_N(P_n{-}P_{n+1}, \tau)) \leq \vartheta(\tau)\mathfrak{B}\tilde{\psi}(P_{n-1}{-}P_n, \tau) \quad (3)$$

Where
$\mathfrak{B}\tilde{\psi}$ (P_{n-1}-$P_{n'}$, τ) = max $\tilde{\psi}$ (F_N (P_{n-1}-$P_{n'}$, τ)), $\tilde{\psi}$ (F_N (P_{n-1}-$P_{n'}$,

$\tau) \otimes \tilde{\psi}$ F_N (P_n-$P_{n'}$, τ)), $\tilde{\psi}$ (F_N (P_n-P_{n+1}, τ)
= max {$\tilde{\psi}$ (F_N (P_{n-1}-$P_{n'}$, τ)),$\tilde{\psi}$ (F_N (P_{n-1}-$P_{n'}$, τ) $\tilde{\psi}$ (F_N (P_n-P_{n+1}, τ) }
= max {$\tilde{\psi}$ (F_N (P_{n-1}-$P_{n'}$, τ)), $\tilde{\psi}$ (F_N (P_n-P_{n+1}, τ)) }

If max {$\tilde{\psi}$ (F_N (P_{n-1}-$P_{n'}$, τ)), $\tilde{\psi}$ (F_N (P_n-P_{n+1}, τ))}= $\tilde{\psi}$ (F_N (P_n-P_{n+1}, τ)) then $\tilde{\psi}$ (L_n)(τ)) $\leq \vartheta$ (τ) $\tilde{\psi}$(L_n)(τ) < $\tilde{\psi}$(L_n)(τ))
but this a contradiction since 0<$\vartheta(\tau)$<1. Hence

$$\tilde{\psi}(L_n)(\tau)) \leq \vartheta(\tau)\tilde{\psi}(L_{n-1})(\tau) < \tilde{\psi}(L_{n-1})(\tau))$$

Therefore L_n (τ) is increasing, so there is L(τ)∈ (0,1] with $\lim_{n-\infty} L_n(\tau)$ = L(τ) ∀τ> 0. Now, it will be established that L(τ)=1; ∀τ> 0 Suppose there is $\tau\circ$ > 0 with 0 <L($\tau\circ$) < 1. Then

$$\begin{aligned}\tilde{\psi}(L)(\tau\circ)) &= \lim_{n-\infty}\tilde{\psi}(L)(\tau\circ)) \\ &\leq \vartheta(\tau\circ)\lim_{n-\infty}\tilde{\psi}(L_{n-1})(\tau\circ)) \\ &\leq \vartheta(\tau\circ)\lim_{n-\infty}\tilde{\psi}(L)(\tau\circ)) \\ &\leq \tilde{\psi}(L)(\tau\circ))\end{aligned}$$

a contradiction. Thus, L(τ)=1 and conclude that

$$\lim_{n-\infty} L_n = 1 \tag{4}$$

After that, in order to demonstrate that p_n is a Cauchy sequence. Consider p_n is not Cauchy. Consequently there is ∍ ∈(0,1) such that ∀k∈N, there are m(k),n(k)∈N with $m_k > n_k \geq$ k and

$$F_N(P_{mk} - P_{nk}, \tau\circ) \leq 1- \ni \text{ where} \tau\circ > 0 \tag{5}$$

Suppose m(k) is the smallest number that is larger than n(k), and meets the Equation (5)

$$F_N(P_{mk-1} - P_{nk}, \tau\circ) > 1- \ni$$

which indicates

$$\begin{aligned}1- \ni &\geq F_N(P_{mk} - P_{nk}, \tau\circ) \\ &\geq F_N(P_{mk} - P_{mk-1}, \tau\circ) \otimes F_N(P_{mk-1} - P_{nk}, \tau\circ) \\ &> F_N(P_{mk} - P_{mk-1}, \tau\circ) \otimes 1- \ni\end{aligned}$$

As a result, we find

$$\lim_{k-\infty} F_N(P_{mk} - P_{nk}, \tau\circ) = 1- \ni \tag{6}$$

Now from

$$F_N(P_{mk-1} - P_{nk}, \tau\circ) \geq F_N(P_{mk+1} - P_{mk}, \tau\circ) \otimes F_N (P_{mk} - P_{nk}, \tau\circ) \otimes F_N(P_{nk} - P_{nk+1}, \tau\circ)$$

Taking limit as k → ∞, we arrive

$$\lim_{k-\infty} F_N(P_{m(k)+1} - P_{n(k)+1}, \tau\circ) \geq 1- \ni \tag{7}$$

Now, combining the results of (4) and (6), we yield

$$F_N(P_{mk} - P_{nk}, \tau\circ) \geq F_N(P_{mk} - P_{mk+1}, \tau\circ) \otimes F_N (P_{mk+1} - P_{nk+1}, \tau\circ) \otimes F_N(P_{nk+1} - P_{nk}, \tau\circ)$$

Thus, it follows

$$\lim_{k-\infty} F_N(P_{mk+1} - P_{nk+1}, \tau\circ) = 1- \ni$$

Additionally

$$F_N(P_{mk} - P_{nk}, \tau\circ) \geq F_N(P_{mk} - P_{mk+1}, \tau\circ) \otimes F_N (P_{mk+1} - P_{nk+1}, \tau\circ) \otimes F_N(P_{nk+1} - P_{nk}, \tau\circ)$$

Indicates

$$\lim_{k-\infty} F_N(P_{mk} - P_{nk+1}, \tau\circ) \geq 1- \ni$$

Likewise,

$$\lim_{k-\infty} F_N(P_{nk} - P_{mk+1}, \tau\circ) \geq 1- \ni$$

Now

$$F_N(P_{mk+1} - \mathbf{T}p_{mk}, \tau) = N_a(\tilde{U}, \tilde{V}, \tau)$$
$$F_N(P_{nk+1} - \mathbf{T}p_{nk}, \tau) = N_a(\tilde{U}, \tilde{V}, \tau) \tag{8}$$

Indicates

$$\begin{aligned}&\tilde{\psi}(F_N(P_{mk+1} - P_{nk+1}, \tau) \leq \vartheta(\tau\circ)\mathfrak{B}\tilde{\psi}(P_{mk}, P_{nk}, \tau\circ) \\ &\leq \vartheta(\tau\circ)max\left\{\tilde{\psi}(F_N(P_{mk} - P_{nk}, \tau\circ)\tilde{\psi}(F_N(P_{mk}, P_{mk+1}, \tau\circ) \otimes (F_N(P_{nk} - P_{mk+1}, \tau)\tilde{\psi}(F_N(P_{nk}, P_{nk+1}, \tau\circ))\right\}\end{aligned}$$

As k goes to ∞ in above, we obtain

$$\tilde{\psi}(1- \ni) \leq \vartheta(\tau\circ)max\left\{\tilde{\psi}(1- \ni), \tilde{\psi}(1 \otimes (1- \ni)), \tilde{\psi}(1)\right\} = \vartheta(\tau\circ)\tilde{\psi}(1- \ni)$$

if $\tilde{\psi}$ (1-∍) = 0 then ∍=0 but this contradiction.

If$\psi(1- \ni)\tilde{}$ > 0then$\tilde{\psi}(1- \ni) \leq \vartheta(\tau\circ)\tilde{\psi}(1- \ni) < \tilde{\psi}(1- \ni)$

A contradiction since 0<$\vartheta(\tau\circ)$<1 therefore p_n is a Cauchy. Because (D,F_N),⊗) is complete then p_n converges to p* ∈ **D**,

$$\lim_{n-\infty} F_N(P_n - P*, \tau) = 1 \tag{9}$$

Furthermore,

$N_a(\tilde{U}, \tilde{V}, \tau) = (F_N(P_{n+1} - \mathbf{T}P_n, \tau)$(by Equation 3)
$\geq F_N(P_{n+1} - P_*, \tau) \otimes F_N(P^* - \mathbf{T}P_n, \tau)$
(applying condition(F_{N4}))
$\geq F_N(P_{n+1} - P^*, \tau) \otimes F_N(P^* - P_{n+1}, \tau) \otimes \mathbf{T}P_n, \tau)$
(applying(F_{N4}))
$= F_N(P_{n+1} - P^*, \tau) \otimes F_N(P^* - P_{n+1}, \tau) \otimes N_a(\tilde{U}, \tilde{V}, \tau)$

which indicates
$N_a(\tilde{U}, \tilde{V}, \tau) \geq F_N(P_{n+1} - P_*, \tau) \otimes F_N(P^* - \mathbf{T}P_n, \tau)$
$\geq F_N(P_{n+1} - P_*, \tau) \otimes F_N(P^* - P_{n+1}, \tau) \otimes N_a(\tilde{U}, \tilde{V}, \tau)$

Using limit as n→∞ in the preceding inequality, the following result is obtained:
$N_a(\tilde{U}, \tilde{V}, \tau) \geq 1 \otimes F_N(P^* - \mathbf{T}P_n, \tau)$
$\geq 1 \otimes 1 \otimes N_a(\tilde{U}, \tilde{V}, \tau)$

that is,

$$\lim_{n-\infty} F_N(P^* - \mathbf{T}P_n, \tau) = N_a(\tilde{U}, \tilde{V}, \tau) \quad (10)$$

Now, to demonstrate that **T** has a Bpp. Note that (c) and (9) imply p*∈ $\tilde{U}\circ(\tau)$ and hence **T**P*($\tilde{U}\circ(\tau)$). **T**($\tilde{U}\circ(\tau)$) ⊆ $\tilde{V}\circ(\tau)$ assures the existence of $\mu \in \tilde{U}\circ(\tau)$ for which

$$F_N(\mu - \mathbf{T}P^*, \tau) = N_a(\tilde{U}, \tilde{V}, \tau) \quad (11)$$

We claim that μ=P*. Contrary to this, suppose that μ≠P* By (1), (2), and (11), obtain $\tilde{\psi}$ (F_N (μ-P_{n+1}, τ)) ≤ ϑ (τ) $\mathfrak{B}\tilde{\psi}$ (P^n-P^*, τ)

$$\leq \vartheta(\tau)\max\left\{\tilde{\psi}F_N(P_n - P^*, \tau)\tilde{\psi}(F_N(P_n - \mu, \tau)\otimes F_N(P^* - \mu, \tau))\tilde{\psi}(F_N(P^* - P_{n+1}, \tau))\right\}$$

Employing limit as n approaches to ∞ in the above equation, one obtain

$$\tilde{\psi}\ F_N(\mu - P^*, \tau) \leq \vartheta(\tau)\max\left\{\tilde{\psi}(F_N(P^* - P^*, \tau)\ \tilde{\psi}F_N(P^* - \mu, \tau)) \otimes (F_N(P^* - \mu, \tau)), \tilde{\psi}(F_N(P^* - P^*, \tau)\right\}$$
$$\leq \vartheta(\tau)\tilde{\psi}(F_N(P^* - \mu, \tau))$$
$$< \tilde{\psi}(F_N(P^* - \mu, \tau))$$

a contradiction 0<$\vartheta(\tau)$<1. Therefore, μ=P*and as a result F_N (p^*-μ,τ)=N_a ($\tilde{U}$, $\tilde{V}$, τ) thus p* is the Bpp of T. If α is another Bpp of T with α≠p*, then 0< (F_N(P*-α, τ))<1 and F_N(P*TP*, τ) = $N_a(\tilde{U}, \tilde{V}, \tau)$ and $F_N(\alpha \mathbf{T}\alpha, \tau) = N_a(\tilde{U}, \tilde{V}, \tau)$

$$\tilde{\psi}\ F_N(\alpha - P^*, \tau) \leq \vartheta(\tau)\mathfrak{B}\tilde{\psi}(\alpha - P^*, \tau)$$
$$\leq \vartheta(\tau)\max\left\{\tilde{\psi}(F_N(\alpha - P^*, \tau)\tilde{\psi}(F_N(\alpha - \alpha, \tau)\otimes F_N(P^* - \alpha, \tau))\tilde{\psi}(F_N(P^* - P^*, \tau))\right\}$$
$$= \vartheta(\tau)\tilde{\psi}(F_N(\alpha - P^*, \tau)$$
$$< \tilde{\psi}(F_N(\alpha - P^*, \tau)$$

a contradiction. Thus the Bpp is unique.

3.4 Example 4.4:

Let **D** = **R**. Suppose F_N:**D** ×**R**→[0,1] is a fuzzy norm, defined by: F_N (p,τ)= $\frac{\tau}{\tau+p}$; ∀p ∈ **D** τ>0, where ‖p‖: R→[0,∞) with ‖p‖=|p|. Let $\tilde{U}$= {1,2,3,4,5} and $\tilde{V}$ = {6,7,8,9,10} So that N_a($\tilde{U}$, $\tilde{V}$, τ) = supF_N (p-q,τ): p ∈$\tilde{U}$, q ∈$\tilde{U}$ = $\frac{\tau}{\tau+1}$ Define **D** : $\tilde{U}$→ $\tilde{V}$ by

$$\mathbf{T}(P) = \begin{cases} 6 & if P = 5 \\ P + 5 & otherwise \end{cases}$$

We have $\tilde{U}\circ\tau$=5 and $\tilde{V}\circ\tau$=6, T ⊆ $\tilde{V}\circ\tau$ Since F_N (μ-Tp,τ)= N_a($\tilde{U}$, $\tilde{V}$, τ) =$\frac{\tau}{\tau+1}$ implies (u,p)=(5,5) or (u,p)=(5,1) Now, let $\tilde{\psi}$ defined by $\tilde{\psi}$ (μ)=1-μ for each μ∈[0,1]. Then from (1), we have $\tilde{\psi}$(F_N (u-ϑ,τ)≥ ϑ (τ) $\mathfrak{B}\tilde{\psi}$ (p,q,τ)

$$(F_N(u - \vartheta, \tau) = \frac{\tau}{\tau + ||u - v||} = \frac{\tau}{\tau + |5 + 5|} = 1$$

which implies $\tilde{\psi}$(F_N (u-ϑ,τ)=1- (F_N (u-ϑ,τ)=1-1=0 and this shows that $\tilde{\psi}$(F_N (u-ϑ,τ)≤$\vartheta(\tau)$ $\mathfrak{B}\tilde{\psi}$(p,q,τ)

hold for each p,q, u,v ∈ $\tilde{U}$ and for all τ >0 and $\vartheta(\tau)$ ∈(0,1) .Therefore, each of the requirements of Theorem 4.3 are met, and there exists a uniquep*∈ $\tilde{U}$ such that F_N (p*-Tp*,τ)= N_a($\tilde{U}$, $\tilde{V}$, τ) for all τ >0. In this example p*=5 is a unique Bpp. Now if we assume that $\tilde{U}\circ(\tau)$ is a nonempty closed set, then we may reduce some requirements in Theorem 4.3 as shown below.

3.5 Theorem 4.5:

Assume that (**D**, F_N, ⊗) is a FB space where ⊗ is min t-norm. Suppose that $\tilde{U}\circ(\tau)$ is a closed subset of **D** and **T** $\tilde{U}$ →$\tilde{V}$ is a mapping meeting the following conditions:

1. **T** ($\tilde{U}\circ(\tau)$) ⊆ $\tilde{v}\circ(\tau)$ for each $\tau > 0$
2. there is a function $\tilde{\psi} \in \psi$ for which
3.

$$\left.\begin{array}{l} F_N(u - \mathbf{T}p, \tau) = N_a(\tilde{U}, \tilde{V}, \tau) \\ F_N(u - \mathbf{T}q, \tau) = N_a(\tilde{U}, \tilde{V}, \tau) \end{array}\right\} \quad (12)$$

$$\Rightarrow \tilde{\psi}\ (F_N(u-v,\tau)) \leq \vartheta\ (\tau)\ \mathfrak{B}\tilde{\psi}\ (p,q,\ \tau)$$

holds for each p,q,u,v $\in \tilde{U}$, and τ>0, where ϑ:(0,∞)→(0,1) is a function and $\mathfrak{B}\tilde{\psi}$(p,q,τ)=max $\tilde{\psi}$(F_N(p,q,τ), $\tilde{\psi}$(F_N(p,q,τ) ⊗ (F_N(p,q,τ), $\tilde{\psi}$(F_N(p,q,τ)))

Then **T** possesses a unique Bpp

Proof: Similar to the proof of Theorem 4.3, construct a Cauchy sequence p_n in $\tilde{U}\circ(\tau)$. The sequence p_n is convergent to some p^* in $\tilde{U}\circ(\tau)$ because $\tilde{U}\circ(\tau)$ is a closed set, and the completeness of (D, F_N, ⊗) guarantees this. The remainder of the proof is identical to the proof given for Theorem 4.3. Now the notion of $\mathfrak{B}\hbar$ $\mathfrak{J}\hbar$- fuzzy proximal contractive mapping is introduced as follows:

3.6 Definition 4.6:

Let (**D**, F_N, ⊗) be a F_N space and let $\tilde{U}$,$\tilde{V}$ nonempty subsets of **D**. Assume that **T**:$\tilde{U}\to\tilde{V}$ is a given mapping. Then T is termed as $\mathfrak{B}\hbar$ $\mathfrak{J}\hbar$- fuzzy proximal contractive mapping if for each p,q,u,v $\in\tilde{U}$ and τ>0 we have,

$$\left.\begin{array}{l} F_N(u-\mathbf{T}p,\tau)=N_a(\tilde{U},\tilde{V},\tau) \\ F_N(u-\mathbf{T}q,\tau)=N_a(\tilde{U},\tilde{V},\tau) \end{array}\right\}$$

$$\Rightarrow (F_N(u-v,\tau)) \geq \mathfrak{J}\hbar(p,q,\tau)+\mathfrak{B}\hbar(p,q,\tau) \tag{13}$$

where $\hbar$:I→I with $\hbar(\mu)$>0 for each $\mu\in(0,1]$ and $\mathfrak{B}\hbar$ (p,q,τ)= min { $\hbar$(p-q,τ), (F_N(p-u,τ)⊗ F_N(p-u,τ), $\hbar F_N$(p-ϑ,τ))} $\mathfrak{J}\hbar$ (p-q,τ) + {F_N(p-q,τ),F_N(p-u,τ)}

The next theorem employs a different contraction condition than the previous results.

3.7 Theorem 4.7:

Assume that (**D**, F_N, ⊗) is a FB space where ⊗ is min t-norm and let $\tilde{U}$,$\tilde{V}$ closed and nonempty subsets of **D** where $\tilde{U}\circ\tau$ is non-empty. Let **T** $\tilde{U}\to\tilde{V}$ be nonself mapping meeting the following conditions:

1. **T** ($\tilde{U}\circ(\tau)$) ⊆ $\tilde{v}\circ(\tau)$
2. **T** $\mathfrak{B}\hbar$ $\mathfrak{J}\hbar$ fuzzy proximal contractive mapping.
3. If sequence {q_n} is in $\tilde{V}\circ(\tau)$ and p $\in\tilde{U}$ with F_N (p-$q^n\tau$)= $N_a(\tilde{U},\tilde{V},\tau)$ as n → ∞ then p $\in\tilde{U}\circ(\tau)$

Then**T** possesses a unique Bpp.

Proof: Consider p∘ in $\tilde{U}\circ(\tau$. since **T** ($\tilde{U}\circ(\tau)$) ⊆ $\tilde{V}\circ(\tau$, there exists p1 $\in\tilde{U}\circ(\tau)$ such that F_N(p1-TP∘τ) = $N_a(\tilde{U},\tilde{V},\tau)$: for each τ > 0

The process is repeated, and we get a sequence p_n in $\tilde{U}\circ(\tau)$ fulfilling

$$\begin{aligned} &F_N(P_{n1}-\mathbf{T}p_{n-1},\tau)=N_a(\tilde{U},\tilde{V},\tau) \\ &F_N(P_{nk+1}-\mathbf{T}p_n,\tau)=N_a(\tilde{U},\tilde{V},\tau) \end{aligned} \tag{14}$$

From (13) and (14) obtain, F_N (P_n-P_{n+1}, τ) ≥ $\mathfrak{J}\hbar$ (P_{n-1}, p_n,τ) $\mathfrak{B}\hbar$ (P_{n-1}, p_n,τ)

$$\begin{aligned} &\geq min\{F_N(P_{n-1}-P_n,\tau)F_N(P_n-P_n,\tau)+min\hbar \\ &\{F_N(P_{n-1}-P_n,\tau)\hbar F_N(P_{n-1}-P_n,\tau) \\ &\otimes F_N(P_n-P_n,\tau),F_N(P_n-P_{n-1},\tau))))\} \end{aligned} \tag{15}$$

≥ F_N (P_{n-1}-P_n, τ)+min { $\hbar$ (F_N (P_{n-1}-P_n,τ)), $\hbar$ (F_N (P_n-P_{n+1},τ))}

which implies F_N (P_n-P_{n+1},τ) ≥ F_N (P_{n-1}-P_n,τ)

That is {F_N (P_n-P_{n+1},τ)} is is increasing, so there is $\mathfrak{j}(\tau)$ ∈(0,1] with $\lim_{n-\infty}$ F_N (P_{n+1}-P_n,τ) = $\mathfrak{j}(\tau)$∀ τ > 0

Now, it will be established that $\mathfrak{j}(\tau)$=1 for each τ > 0. Suppose there is $\tau\circ$> 0 such that 0 <$\mathfrak{j}$ ($\tau\circ$) < 1. Considering limit as n goes to ∞ in (15), then

$\mathfrak{j}$ (τ ∘)≥j(τ∘)+min$\hbar$($\mathfrak{j}$(τ∘)),$\hbar$(1)}

If min $\hbar$($\mathfrak{j}$(τ∘)),$\hbar$(1)=$\hbar$($\mathfrak{j}$(τ∘)) then obtain $\mathfrak{j}$(τ∘)≤($\mathfrak{j}$(τ∘) + $\hbar$($\mathfrak{j}$(τ∘)) implies thath ($\mathfrak{j}$(τ∘)=0 and this is a contradiction. If min$\hbar\mathfrak{j}$(τ∘)),$\hbar$(1)=$\hbar$(1) then obtain $\mathfrak{j}$(τ∘) ≥ $\mathfrak{j}$(τ∘)+$\hbar$(1) implies that $\hbar$(1)=0 and this is a contradiction. This demonstrates that for all τ> 0, $\mathfrak{j}(\tau)$ = 1.

After that, to demonstrate p_n is a Cauchy sequence. Consider p_n is not Cauchy. Then there is ∍ ∈ (0,1) such that ∀k∈N, there are m(k),n(k)∈N with m_k > n_k ≤k and

$$F_N(P_{mk}-P_{nk},\tau) \leq 1-\ni \tag{16}$$

Suppose m(k) is the smallest number that is larger than n(k), that meeting (16),

$$F_N(P_{mk}-P_{nk},\tau) > 1-\ni$$

In a manner analogous to the proof of Theorem 4.3, get

$$\lim_{n-\infty} F_N(P_{mk}-P_{nk},\tau)=1-\ni,\ \lim_{n-\infty} F_N(P_{mk+1}-P_{nk+1},\tau) = 1-\ni$$

$$\lim_{n-\infty} F_N(P_{mk}-P_{nk+1},\tau)\geq 1-\ni,\ \lim_{n-\infty} F_N(P_{nk}-P_{mk+1},\tau) \geq 1-\ni$$

From (14) we get

$$\begin{aligned} &F_N(P_{mk+1}-TP_{mk},\tau\circ)=N_a(\tilde{U},\tilde{V},\tau\circ)\text{and} \\ &F_N(P_{nk+}-TP_{nk},\tau\circ)=N_a(\tilde{U},\tilde{V},\tau\circ) \end{aligned}$$

Hence, (13) implies

$$\begin{aligned} &F_N(P_{mk+1}-P_{nk+1},\tau\circ)\geq\mathfrak{J}\hbar(P_{mk},P_{nk},\tau\circ)+\mathfrak{B}\hbar(P_{mk}, P_{nk},\tau\circ) \\ &\geq min\{F_N(P_{mk}-TP_{nk},\tau\circ),F_N(P_{nk}-TP_{mk+1},\tau\circ)\} \\ &+min\{\hbar(F_N(P_{mk}-P_{nk},\tau\circ)) \end{aligned}$$

If k tends to ∞ in above inequality, one obtains

1-϶ ≥(1-϶)+min {$\hbar$(1-϶)$\hbar$(1)}

Thus

0≥ min {$\hbar$(1-϶)$\hbar$(1)}

then, either $\hbar$(1-϶)=0 or $\hbar$(1)=0 but this is in both cases a contradiction, therefore p_n is a Cauchy. Because (D,F_N ,⊗) is complete then p_n converges to p* ∈ D,

$$\lim_{n-\infty} F_N(P_n - P*, \tau) = 1 foreach \tau > 0 \tag{17}$$

Now to demonstrate that T has Bpp. Like Theorem 4.3, obtain P* ∈ $\tilde{U}\circ(\tau)$. As ($\tilde{U}\circ(\tau)$) ⊆ $\tilde{V}\circ(\tau)$ assures the existence og μ ∈$\tilde{U}\circ(\tau)$

$$F_N(\mu - TP^*, \tau) = N_a(\tilde{U}, \tilde{V}, \tau) \tag{18}$$

We claim that μ=p*. Contrary to this, suppose that μ≠p* By (14) and (18), obtain

$$F_N(P_{nk+1} - \mu, \tau) \geq \mathfrak{J}\hbar(P_n, P^*, \tau) + \mathfrak{B}\hbar(P_n, P^*, \tau$$
$$\geq min\{F_N(P_n - P^*, \tau), F_N(P^* - P_{n+1}, \tau)\}$$
$$+ min\{\hbar(F_N(P_n - P^*, \tau)), \hbar(F_N(P_n - P_{n+1}, \tau)\otimes$$
$$(F_N(P^* - P_{n+1}, \tau), \hbar(F_N(P^* - \mu, \tau)$$

Using limit as n approaches to ∞ in above inequality, one gets

$$F_N(P^* - \mu, \tau) \geq 1 + min\{\hbar(1)F_N(P^*, \mu, \tau))\}$$
$$So 1 \geq F_N(P^* - \mu, \tau) \geq 1, \text{which implies} F_N(P^* - \mu, \tau)$$
$$= 1, \text{that is} \mu = P^* and F_N(P^* - TP^*, \tau) = N_a(\tilde{U}, \tilde{V}, \tau)$$

To demonstrate the uniqueness, assume α is another Bpp of T such that α≠p*, that is 0<

$$F_N(P^* - \alpha, \tau) < 1 \text{for} \tau > 0 \text{as} F_N(P^* - TP^*, \tau) = N_a$$
$$(\tilde{U}, \tilde{V}, \tau) \text{and} F_N(\alpha - T\alpha, \tau) = N_a(\tilde{U}, \tilde{V}, \tau)$$

hence, from (13) we obtain

$$F_N(P^* - \alpha, \tau) \geq \mathfrak{J}\hbar(P^*, \alpha, \tau) + \mathfrak{B}\hbar(P^*, \alpha, \tau)$$
$$\geq min\{F_N(P^*, \alpha, \tau), F_N(\alpha, P^* \tau)\}$$
$$+ min\{\hbar(F_N(P^*, \alpha, \tau)), \hbar(F_N(P^*, P^*, \tau)\otimes$$
$$(F_N(\alpha, P^*, \tau), \hbar(F_N(\alpha, \alpha, \tau)\}$$
$$(F_N(P^*, \alpha, \tau) + min\{\hbar(F_N(P^*, \alpha, \tau))\hbar(1)\}$$

Therefore

$$(F_N(P^*, \alpha, \tau) \geq (F_N(P^*, \alpha, \tau) + min\{\hbar(F_N(P^*, \alpha, \tau))\hbar(1)\} \tag{19}$$

Which implies $\hbar$ F_N (P*,α,τ)=0 or $\hbar$ which is in both cases a contradiction as $\hbar(\mu)$>0 for each μ ∈ (0,1]. Thus F_N (P*,α,τ)=1 and so p*=α.

4. CONCLUSION

The Bpp theorem for various non-self fuzzy contractive mapping types such as $\mathfrak{B}\hbar$- fuzzy proximal contractive mapping and $\mathfrak{B}\hbar$ $\mathfrak{J}\hbar$- fuzzy proximal contractive mapping in a F_N space is established in this study. An example is presented to show the significance of the results that were obtained. The generalizations of these kinds of contraction mappings and studies of their applications in the F_N space needs more investigation in upcoming work.

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