

The Definite Positive Property of Characteristic Function from Compound Geometric Distribution as The Sum of Gamma Distribution

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ABSTRACT

In this expository article it is given characterization of compound geometric distribution as the sum of gamma distribution. The characterization of this compound distribution is obtained by using the property of characteristic function as the Fourier-Stieltjes transform. The property of definite positive of characteristic function from compound geometric distribution as the sum of gamma distribution is exposed by analytical methods as the quadratic form of characteristic function.

Keyword: *compound geometric distribution, gamma distribution, characteristic function, definite positive.*

1. INTRODUCTION

Let us define S as the sum of independent and identically random variables, that is in the form

$$S = \sum_{i=1}^N X_i \quad (1.1)$$

where the number of random variable X_i , that is N , itself a random variable. The property of this random variable has been paid great attention to many years. The most popular terminology for the distribution generated by such as a sum of random variables is called compound distribution.

Compound distributions arise from many applied probability models and from insurance risk models in particular. One of the compound distributions that attracted the researcher was the compound geometric distribution. The compound geometric distribution, as a special case of the compound negative binomial distribution plays a vital role in analysis of ruin probabilities and related problems in risk theory and insurance as in Willmot and Lin (2001). The most recently theoretical approach of compound geometric distribution has started by introducing on higher-order properties of compound geometric distributions (Willmot, 2002), this development is continued on the study of convolutions of compound geometric distributions (Psarrakos, 2009). Furthermore, it has reviewed convolution on generated random variable from exponential distribution with stabilizer constant (Devianto

et. al, 2015; Devianto, 2016) and some properties of hypoexponential distribution with stabilizer constant (Devianto et. al, 2015). While in the last development of compound geometric distribution was delivered by Koutras and Eryilmaz (2016) when they studied compound geometric distributions of order k .

In the present paper, we study the property of compound geometric distribution as the sum of gamma distribution. It will be discussed characterization of compound geometric distribution as the sum of gamma distribution in the form of definite positive property of characteristic function. In Section 2, we will explain about characterization of gamma distribution and compound geometric distribution. While the some properties of compound geometric distribution as the sum of gamma distribution will be given in Section 3.

2. THE CHARACTERIZATION OF GAMMA DISTRIBUTION AND COMPOUND GEOMETRIC DISTRIBUTION

The gamma distribution is a family of continuous probability distribution with parameter α dan β . Let X be a random variable from gamma distribution with the form

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} \exp\left[-\frac{x}{\beta}\right] \quad (2.1)$$

for $x > 0$, $\alpha > 0$, and $\beta > 0$. The expectation and variance of gamma distribution are given by

$$E(x) = \alpha\beta \quad (2.2)$$

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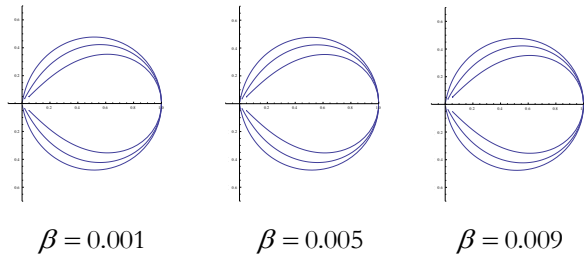


Figure 1. Parametric curves of characteristic function from gamma distribution with various parameters $\alpha = 0.5, 0.7, 0.9$

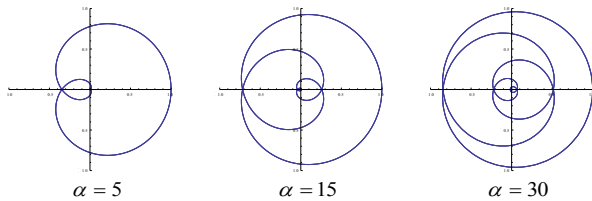


Figure 2. Parametric curves of characteristic function from gamma distribution with various parameters $\beta = 0.5, 0.7, 0.9$

$$Var(X) = \alpha\beta^2 \tag{2.3}$$

As for the moment generating function and characteristic function of the gamma distribution we can obtain by using definition of Fourier-Stieltjes transform refers to Lukacs (2009) in the following

$$M_X(t) = E[\exp(tX)] = \left(\frac{1}{1-\beta t}\right)^\alpha \tag{2.4}$$

$$\phi_X(t) = E[\exp(itX)] = \left(\frac{1}{1-\beta it}\right)^\alpha \tag{2.5}$$

The Figure 1 is the graph of parametric curves of characteristic function from gamma distribution with various parameter α . While the Figure 2 is the graph of parametric curves of characteristic function from gamma distribution with various parameter β . The graph of these parametric curves of characteristic function are described in the complex plane that shows a smooth line, this is to confirm that its characteristic function is continuous and never vanish on the complex plane.

The compound geometric distribution is the sum of independent and identically random variable, where the number of random variables has geometric distribution as follows

$$P(N = n) = p(1-p)^n \tag{2.6}$$

for $n = 0, 1, 2, \dots$ and $0 < p < 1$, p is the probability of success. The geometric distribution has expectation and variance as follows

$$E(N) = \frac{1-p}{p} \tag{2.7}$$

$$Var(N) = \frac{1-p}{p^2} \tag{2.8}$$

By using similar way with Equation (2.4) and (2.5), we have the moment generating function and characteristic function of the geometric distribution respectively as follows

$$M_N(t) = \frac{1}{1-(1-p)\exp[t]} \tag{2.9}$$

$$\phi_N(t) = \frac{p}{1-(1-p)\exp[it]} \tag{2.10}$$

The Figure 3 is the graph of parametric curves of characteristic function from geometric distribution with various parameter p . The graph of its characteristic function is described in the complex plane that shows a smooth line, this is also to confirm that this characteristic function is continuous and never vanish on the complex plane.

3. THE SOME PROPERTIES OF COMPOUND GEOMETRIC DISTRIBUTION AS THE SUM OF GAMMA DISTRIBUTION

The property of compound geometric distribution as the sum of gamma distribution will be stated on its definite positive of characteristic function. In order of the main result in this section, it is explained the property definite positive of characteristic function in Proposition 3.4, at the first we introduce the moment generating function, characteristic function and continuity property at the following propositions.

Proposition 3.1. Let the random variable $S = \sum_{i=1}^N X_i$ is defined as a compound geometric distribution as the sum of gamma distribution with parameter (p, α, β) , then moment generating function is obtained as follows

$$M_S(t) = \frac{p}{1-(1-p)(1-\beta t)^\alpha} \tag{3.1}$$

Proof. By using definition of conditional expectation and its linearity property then it is obtained the moment generating function as follows

$$M_S(t) = E\left(\exp\left(t\sum_{i=1}^N X_i\right)\right) = E\left(E\left(\exp\left(t\sum_{i=1}^N X_i\right)\middle|N\right)\right) \tag{3.2}$$

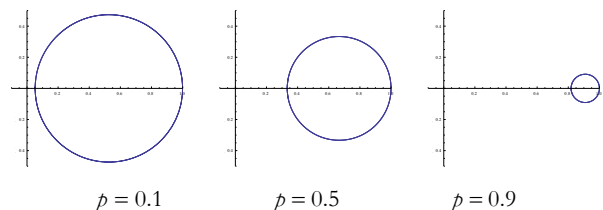


Figure 3. Parametric curves of characteristic function from geometric distribution with various parameters $p = 0.1, 0.5, 0.9$

Now, let us use the definition of moment generating function then we have

$$M_S(t) = E(M_X(t)^N) = M_N(\ln(M_X(t))) \quad (3.3)$$

Base on the Equation (2.4) and (2.9) and we substitute to Equation (3.3), then we have

$$M_S(t) = \frac{p}{1 - (1-p)(1-\beta t)^{-\alpha}}$$

Proposition 3.2. Let the random variable is defined as a compound geometric distribution as the sum of gamma distribution with parameter, then (p, α, β) characteristic function is obtained as follows

$$\phi_S(t) = \frac{p}{1 - (1-p)(1-\beta it)^{-\alpha}} \quad (3.4)$$

Proof. By using definition of conditional expectation and its linearity property then we can write the characteristic function as follows

$$\phi_S(t) = E\left(\exp\left(i t \sum_{i=1}^N X_i\right)\right) = E\left(E\left(\exp\left(i t \sum_{i=1}^N X_i\right) \middle| N\right)\right) \quad (3.5)$$

Now, let us use the definition of moment generating function and definition of characteristic function to write

$$\phi_S(t) = E(\phi_X(t)^N) = M_N(\ln(\phi_X(t))) \quad (3.6)$$

Base on the Equation (2.4) and (2.5) then we have

$$\phi_S(t) = \frac{p}{1 - (1-p)(1-\beta it)^{-\alpha}}$$

Proposition 3.3. Let the random variable $S = \sum_{i=1}^N X_i$ is defined as a compound geometric distribution as the sum of gamma distribution with parameter (p, α, β) and characteristic function

$$\phi_S(t) = \frac{p}{1 - (1-p)(1-\beta it)^{-\alpha}}$$

then $\phi_S(t)$ is continuous.

Proof. The continuity property is explained by using definition of uniform continuity, that is for every $\epsilon > 0$ there exists $\delta > 0$

such that $|\phi_S(t_1) - \phi_S(t_2)| < \epsilon$ for $|t_1 - t_2| < \delta$ where δ depends only on ϵ . Then the uniform continuity of characteristic function is obtained by the following way. First we write the following equation

$$\begin{aligned} |\phi_S(t_1) - \phi_S(t_2)| &= \left| \frac{p}{1 - (1-p)(1-\beta it_1)^{-\alpha}} - \frac{p}{1 - (1-p)(1-\beta it_2)^{-\alpha}} \right| \\ &= p \left| \frac{1}{1 - (1-p)\phi_X(t_1)} - \frac{1}{1 - (1-p)\phi_X(t_2)} \right| \end{aligned} \quad (3.7)$$

Now let us define $h = t_1 - t_2$, so that for $h \rightarrow 0$ we have the following limit

$$\lim_{h \rightarrow 0} \left| \frac{1}{1 - (1-p)\phi_X(h+t_2)} - \frac{1}{1 - (1-p)\phi_X(t_2)} \right| = 0 \quad (3.8)$$

This hold for $\delta < \epsilon$ where $|\phi_S(h+t_2) - \phi_S(t_2)| < \epsilon$ for $|t_1 - t_2| < \delta$. Then $\phi_S(t)$ is uniformly continuous.

Proposition 3.4. Let the random variable $S = \sum_{i=1}^N X_i$ is defined as a compound geometric distribution as the sum of gamma distribution with parameter (p, α, β) and characteristic function

$$\phi_S(t) = \frac{p}{1 - (1-p)(1-\beta it)^{-\alpha}}$$

then $f_S(t)$ is positively defined function with quadratic form

$$\sum_{1 \leq j \leq n} \sum_{1 \leq l \leq n} c_j \bar{c}_l \phi_X(t_j - t_l) \geq 0 \quad (3.9)$$

for any complex number c_1, c_2, \dots, c_n and real number t_1, t_2, \dots, t_n .

Proof. We will show the characteristic function $\phi_S(t)$ is positive function that is to satisfy the quadratic form

$$\sum_{1 \leq j \leq n} \sum_{1 \leq l \leq n} c_j \bar{c}_l \phi_X(t_j - t_l) \geq 0$$

for any complex numbers c_1, c_2, \dots, c_n and real numbers

t_1, t_2, \dots, t_n . It is used definition of characteristic function of compound geometric distribution as the sum of gamma distribution and a geometric series with ratio $(1-p)E(\exp(i(t_j-t_l)X))$ where $p \in (0,1)$ and $E(\exp(i(t_j-t_l)X)) \leq 1$, hence we have the following equation

$$\begin{aligned} \sum_{1 \leq j \leq n} \sum_{1 \leq l \leq n} c_j \bar{c}_l \phi_S(t_j - t_l) &= \sum_{1 \leq j \leq n} \sum_{1 \leq l \leq n} c_j \bar{c}_l \frac{p}{1 - (1-p)(1-\beta i(t_j-t_l))^{-\alpha}} \\ &= \sum_{1 \leq j \leq n} \sum_{1 \leq l \leq n} c_j \bar{c}_l \frac{p}{1 - (1-p)E(\exp(i(t_j-t_l)X))} \\ &= p \sum_{1 \leq j \leq n} \sum_{1 \leq l \leq n} c_j \bar{c}_l \left(\sum_{m=0}^{\infty} (1-p)^m E(\exp(i(t_j-t_l)X))^m \right) \end{aligned} \quad (3.10)$$

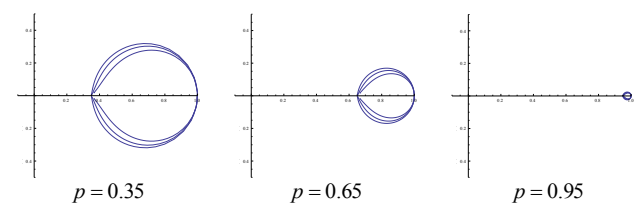


Figure 4. Parametric curves of characteristic function from compound geometric distribution as the sum of gamma distribution with various parameters $\alpha = 0.5, 0.7, 0.9$ where $\beta = 0.009$

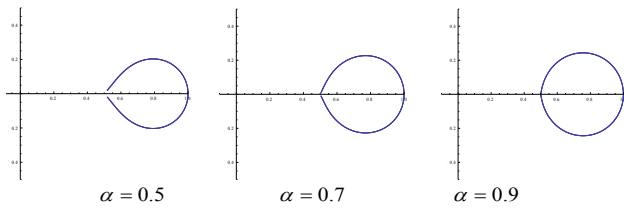


Figure 5. Parametric curves of characteristic function from compound geometric distribution as the sum of gamma distribution with various parameters $\beta = 0.05, 0.07, 0.09$ where $p = 0.5$

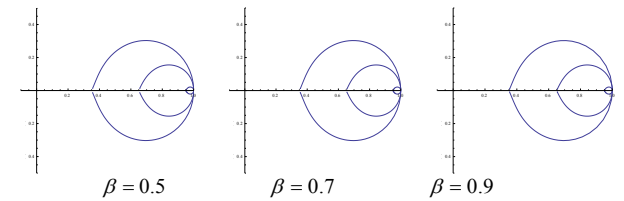


Figure 6. Parametric curves of characteristic function from compound geometric distribution as the sum of gamma distribution

We rewrite the Equation (3.10) by using its complex conjugate property and exponential function property as the follows

$$\sum_{1 \leq j, l \leq n} c_j \bar{c}_l \phi_s(t_j - t_l) = p \left(\sum_{1 \leq j \leq n} c_j \left(\sum_{m=0}^{\infty} (1-p) E(\exp(it_j X))^m \right) \right) \times \overline{\left(\sum_{1 \leq l \leq n} c_l \left(\sum_{m=0}^{\infty} (1-p) E(\exp(it_l X))^m \right) \right)} \quad (3.11)$$

Next, base on the quadratic form property of modulus complex number, then we obtain

$$\sum_{1 \leq j, l \leq n} c_j \bar{c}_l \phi_s(t_j - t_l) = p \left| \sum_{1 \leq j \leq n} c_j \left(\sum_{m=0}^{\infty} (1-p) E(\exp(it_j X))^m \right) \right|^2 = p \left| \sum_{1 \leq j \leq n} c_j \left(\sum_{m=0}^{\infty} (1-p) (1 - \beta i t_j)^{-\alpha} \right) \right|^2 \geq 0 \quad (3.12)$$

Then it is proved that $f_s(t)$ as positively defined function where the quadratic form has nonnegative values.

The graph of parametric curves of characteristic function from compound geometric distribution as the sum of gamma distribution will be presented in Figure 4, Figure 5, and Figure 6 as follows

Based on the graph presented in Figures 4, 5, and 6, it can be concluded that the characteristic function of compound geometric distribution as the sum of gamma distribution is continuous, as well as it has discussed on Proposition 3.3. In Figure 4 and 6, the shape of parametric curves form three smooth lines. While in Fig-

ure 5 the shape of parametric curves only form the same smooth line for various parameter β , and tends to be one for all various parameter α dan β when fixed on parameter p .

4. CONCLUSION

The compound geometric distribution as the sum of gamma distribution is the sum of independent and identically random variable from gamma distribution, where the number of these random variables has geometric distribution. The characterization of compound geometric distribution as the sum of gamma distribution is obtained by using the property of characteristic function, where the characteristic function is defined as the Fourier-Stieltjes transform. The property of definite positive of characteristic function from compound geometric distribution as the sum of gamma distribution is exposed by showing its quadratic form of characteristic function. The addition property of this compound geometric distribution as the sum of gamma distribution is continuity peroperty of characteristic function, where the graph of the parametric curves of characteristic function is in the smooth line and never vanish on the complex plane.

REFERENCES

Devianto, D., (2016). The uniform continuity of characteristic function from convoluted exponential distribution with stabilizer constant. AIP Conference Proceedings, American Institute of Physics.

Devianto, D., Maiyastri, Oktasari, L. & Anas, M. (2015). Convolution on generated random variable from exponential distribution with stabilizer constant. Applied Mathematical Sciences, 9(96), 4781-4789.

Devianto, D., Oktasari, L., & Maiyastri. (2015). Some properties of hypoexponential distribution with stabilizer constant. Applied Mathematical Sciences, 9(142), 7063-7070.

Koutras, M.V., & Eryilmaz, S. (2016). Compound geometric distribution of order k. Methodol. Comput. Appl. Proba, 19(2), 377-393.

Lukacs, E. (1970). Characteristic function (2nd edition). London: Griffin.

Psarrakos, G. (2009). A note on convolutions of compound geometric distributions. Statistics & Probability letters, 79(9), 1231-1237.

Willmot, G.E. (2002). On higher-order properties of compound geometric distributions. Applied Probability, 39(2), 324-340.

Willmot, G.E., & Lin, X.S. (2001). Lunberg approximations for compound distributions with insurance applications. New York, NY: Springer.