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Research Paper

Bi-Univalent Function Classes Defined by Using an Einstein Function and a New Generalised Operator

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Abstract
Let A be the class of all analytic and univalent functions $f(z) = z+\Sigma_{k=2}^{\infty} a_k z^k$ in the open unit disc $\mathbb{D} = \{z:|z|<1\}$. S then represents the classes of every function in A that is univalent in $\mathbb{D}$. For every $f \in S$, there is an inverse f^{-1}. A function $f \in A$ in $\mathbb{D}$ is categorised as bi-univalent if f and its inverse $g = f^{-1}$ are both univalent. Motivated by the generalised operator, subordination principle, and the first Einstein function, we present a new family of bi-univalent analytic functions on the open unit disc of the complex plane. The functions contained in the subclasses are used to account for the initial coefficient estimate of $|a_2|$. In this study, we derive the results for the covering theorem, distortion theorem, rotation theorem, growth theorem, and the convexity radius for functions of the class $N_{\lambda,\alpha}^{s,m,k}(\Sigma, E)$ of bi-univalent functions related to an Einstein function and a generalised differential operator $D_{\lambda,\alpha}^{s,m,k} f(z)$. We use the elementary transformations that preserve the class $N_{\lambda,\alpha}^{s,m,k}(\Sigma, E)$ in order to attain the intended results. The required properties are then obtained.

Keywords
Einstein Function, Subordination, Distortion Theorem, Covering Theorem, Radius of Convexity, Bi-Univalent Functions

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1. INTRODUCTION

For the open unit disc $\mathbb{D} = \{z \in \mathbb{C}:|z|<1\}$, we set A as denoting the class of

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \text{ for } z \in \mathbb{D} \tag{1}$$

analytic functions. S represents the class of all functions in A that are univalent in $\mathbb{D}$ (for further detail on univalent functions, see Duren (2001)) and satisfy the standard normalisation condition $f(0) = f'(0) - 1 = 0$. The Koebe one-quarter theorem (Duren, 2001) demonstrates that each univalent function $f \in S$ having a disc with a radius of $\frac{1}{4}$ possesses an inverse function f^{-1} that can be defined by

$$f^{-1}(f(z)) = z, \text{ for } z \in \mathbb{D},$$

and

$$f(f^{-1}(w)) = w, \, |w| < r_0(f), \, r_0(f) \leq \frac{1}{4}$$

The function $f \in S$ is deemed as bi-univalent if f and f^{-1} are both univalent in $\mathbb{D}$. Let the class of bi-univalent functions in $\mathbb{D}$ of the form (1) be denoted by $\sum$. Moreover, it is easily demonstrated that the series expansion of the inverse function can be written as follows:

$$\begin{aligned} g(w) &= f^{-1}(w) \\ &= w - a_2 w^2 + (2a_2^2 - a_3)w^3 \\ &\quad - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots, \text{ for } w \in \mathbb{D}. \end{aligned} \tag{2}$$

Class $\sum$ includes functions such as $\frac{z}{1-z'} - \log(1-z)$, and $\frac{1}{2}\log(\frac{1+z}{1-z})$. Nevertheless, the well-known Koebe function does not belong to $\sum$. Other typical instances of functions in S such as $z - \frac{z^2}{2}$, and $\frac{z}{1-z^2}$, do not belong to $\sum$. For references to related works on bi-univalent functions, see the revival paper by Srivastava et al. (2010), as well as several other studies (Ali et al., 2012; Oros and Cotîrlă, 2022; Srivastava et al., 2018).

Normalised analytic function operators are commonly utilised in the field of Geometric Function Theory (GFT), particularly differential and integral operators. A wide range of authors have written numerous articles on a variety of topics, including operators and novel generalisation. The differential operator, which was first introduced in 1975 by Ruscheweyh (1975), was a particularly major breakthrough. Differential and integral operators were then presented in a different version by Salagean (1983). From there on, many academics have developed new operators and used them in numerous research topics involving GFT. They include Rossdy et al. (2022), Wanas (2019), Yunus et al. (2017), Elhaddad and Darus (2021), and Frasin (2020).

In this paper, we provide some information regarding the differential operator that is applied to examine our new subclasses. According to Rossdy et al. (2022), the differential operator is defined by:

Definition 1.1 For $f \in A$, $0 < \lambda < 1$, $0 < \alpha < 1$, $m \in \mathbb{N} = \{1, 2, \cdots\}$, $b \in \mathbb{C}\backslash Z_0^-$, $s \in \mathbb{C}$, $k \in \mathbb{N}_0$,

$$D^{s,m,k}_{\lambda,\alpha} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^s [1 + \lambda(n-1)(1-\lambda)^m]^k a_n z^n \tag{3}$$

We can see that when two functions of the class $\sum$ are linked in a convex combination, it need not be bi-univalent. Even though the two functions of $f_1(z) = \frac{z}{1-z}$ and $f_2(z) = \frac{z}{1+iz}$ are examples of bi-univalent functions, their sum, $f_1 + f_2$, is not univalent because its derivative no longer exists at $\frac{1}{2}(1+i)$. Nevertheless, several elementary transformations preserve the class $\sum$, as seen below (Wei, 2017; Sivasubramanian et al., 2014):

i. Rotation: If $f \in \sum$, $0 \in \mathbb{R}$, and $g(z) = e^{-i\theta} f(e^{i\theta} z)$, then $g \in \sum$;

ii. Dilation: If $f \in \sum$, $0 < r < 1$, and $g(z) = \frac{1}{r} f(rz)$, then $g \in \sum$;

iii. Conjugation: If $f \in \sum$ and $g(z) = \overline{f(\overline{z})}$, then $g \in \sum$;

iv. Disk automorphism: If $f \in \sum$, $\zeta \in \mathbb{D}$, and $g(z) = \frac{f\left(\frac{z+\zeta}{1+\overline{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2) f'(\zeta)}$, then $g \in \sum$.

v. Omitted value transformation: If $f \in \sum$ with $f(z) \pm w$ for all $z \in \mathbb{D}$, and $g(z) = \frac{wf(z)}{w - f(z)}$, then $g \in \sum$.

In GFT, determining coefficient estimates $|a_n| (n \in \mathbb{N})$ is essential because this allows details of these functions' geometric properties to be obtained. The evaluation of analytic function coefficients determines the structural characteristics and particulars of GFT. For example, in the univalent function set, the second coefficient $|a_2|$ implies the covering theorems, growth and distortion bounds. The renowned Bieberbach Conjecture, as proven by Louis de Branges (De Branges, 1985), posits that the coefficient inequality as written below is true for each $f \in S$ provided by the Taylor-Maclaurin series expansion (1):

$$|a_n| \leq n \quad (n \in \mathbb{N}\backslash\{1\}), \tag{4}$$

where $\mathbb{N}$ represents the set of all positive integers. Lewin in his research Lewin (1967) on bi-univalent functions of the class $\sum$, discovered the bound $|a_2| < 1.51$. Brannan and Clunie (1980) in their subsequent work proposed that $|a_2| \leq \sqrt{2}$. Additionally, Netanyahu (1969) demonstrated that $max_{f \in \sum} |a_2| = \frac{4}{3}$. In addition to estimating the coefficients for $|a_2|$ and $|a_3|$, Brannan and Taha (1988) proposed the concepts of strongly bi-starlike functions of the order α and strongly bi-convex functions of the order α. Following the lead of Brannan and Taha (1988), other researchers (Rossdy et al., 2021; Soni et al., 2018; Xu et al., 2012) have studied numerous subclasses of $\sum$ and determined the coefficient bounds for $|a_2|$ and $|a_3|$.

The geometric theory of bi-univalent functions shows more applications of Lewin's inequality (Lewin, 1967). An important implication is the distortion theorem. This theorem gives non-sharp upper and lower bounds for $|f'(z)|$ as f ranges over the class $\sum$.

The idea of subordination is then employed as defined below:

Definition 1.2 (Miller and Mocanu, 2000) Given that $f(z) \prec g(z)$, with f being a subordinate to g, and both functions taken to be analytic. This indicates that $f(z) = g(w(z))$, where w is taken as analytic in $\mathbb{D}$, which corresponds to $|w(z)| < 1$ and $w(0) = 0$.

Ma (1992) presented the subset of functions $S^*(\phi) = \{f \in A : \frac{zf'(z)}{f(z)} \prec \phi(z), \phi \in P, z \in \mathbb{D}\}$ in 1994, in which the symbol "$\prec$" corresponds to the subordination stated in Definition 1.2 above. Ma (1992) studied several relevant topics, such as covering, growth, and distortion theorems. Then, by inserting certain functions for ϕ in $S^*(\phi)$, we obtain various subclasses of A with distinct geometric analyses, such as those from the work by Janowski (1970), Mendiratta et al. (2015), and Cho et al. (2019). Many characteristics of the analytic univalent functions are connected with differential and integral operators, such as coefficient bound, covering theorems, distortion theorems, growth theorems, inclusion properties, and radius of convexity; all of these have been investigated (Omar and Abdul Halim, 2012; Zhang et al., 2021; Saheb and Al-Khafaji, 2021; Kumar and Sahoo, 2021; Awasthi, 2017). Yet, there has been relatively little research and discovery on the relevant features involving bi-univalent function subclasses. However, much attention has been focused on bi-univalent functions' initial coefficients (Al-Ameedee et al., 2020; Rossdy et al., 2021; Soni et al., 2018).

Gradshteyn and Ryzhik (2014) published a formulation for the Bernoulli polynomials in 1980, which has substantial uses in number theory and classical analysis. The Bernoulli polynomials are featured in differentiable periodic functions in the integral form of the functions because they are used for polynomial approximation of these functions. The polynomials are used to represent the remainder term of the Euler-Maclaurin quadrature rule in its composite form as well. The Bernoulli polynomials $B_n(x)$ are commonly defined (Natalini and Bernardini, 2003) using the generating function:

$$G(x,t) := \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n, \ |t| < 2\pi,$$

where for each nonnegative integer n, $B_n(x)$ are polynomials in x.

Since

$$\sum_{j=0}^{n-1} \begin{pmatrix} n \\ j \end{pmatrix} B_j(x) = nx^{n-1}, \ \ n = 2, 3, \cdots ,$$

the Bernoulli polynomials can be calculated readily via recursion.

The initial Bernoulli polynomials are

$$B_0(x) = 1,$$

$$B_1(x) = x - \tfrac{1}{2},$$

$$B_2(x) = x^2 - x + \tfrac{1}{6},$$

$$B_3(x) + x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x, \cdots$$

Moreover, Bernoulli numbers $B_n := B_n(0)$ can be directly generated by setting $x = 0$ in the Bernoulli polynomials. The initial Bernoulli numbers are

$$B_0(x) = 1,$$

$$B_1(x) = -\tfrac{1}{2},$$

$$B_2(x) = \tfrac{1}{6},$$

$$B_4(x) = -\tfrac{1}{30}$$

$$B_{2n+1} = 0, \ \forall_n = 1, 2, \cdots$$

Furthermore, Bernoulli numbers B_n can be produced using the Einstein function $E(z)$:

$$E(z) := \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

The name of Einstein function is sometimes applied in mathematics for one of the functions (see (Abramowitz and Stegun, 1972; Weisstein, 2022)):

$$E_1(z) := \tfrac{z}{e^z - 1},$$

$$E_2(z) := \tfrac{z^2 e^z}{(e^z-1)^2},$$

$$E_3(x) := \log(1 - e^{-z}),$$

$$E_4(x) := \tfrac{z}{e^z-1} - \log(1 - e^{-z}).$$

Both E_1 and E_2 exhibit these desirable properties. E_1 and E_2 (convex functions) have a symmetric range along the real axis and starlike range about $E_1(0) = E_2(0) = 1$ and $\mathbb{R}(E_1(z)) > 0, \mathbb{R}(E_2(z)) > 0, \forall_z \in \mathbb{D}$. The series representation is given by

$$E_1(z) = 1 + \sum_{n=1}^{\infty} \frac{B_n}{n!} z^n$$

$$E_2(z) = 1 + \sum_{n=1}^{\infty} \frac{(1-n)B_n}{n!} z^n,$$

where B_n denotes the n^{th} Bernoulli number. However, $E_1'(0)$ and $E_2'(0) \ngtr 0$, therefore we must establish new functions $E(z) := E_1(z) + z$ and $\mathbb{E}(z) := E_2(z) + \frac{1}{2}z$. A significant function class will be known as P and P defines the function family ϕ that is restricted by the image domain of ϕ (ϕ is a convex function with Re $(\phi) > 0$ in $\mathbb{D}$) being symmetric along the real axis and starlike about $\phi(0) = 1$ with $\phi'(0) > 0$. We can now say that $E, \mathbb{E} \in P$. The following are the series representations:

$$E(z) = 1 + z + \sum_{n=1}^{\infty} \frac{B_n}{n!} z^n,$$

and

$$\mathbb{E}(z) = 1 + \frac{1}{2}z + \sum_{n=1}^{\infty} \frac{(1-n)B_n}{n!} z^n,$$

The contour integral (see (Arfken and Weber, 1999)) can be used to define the n^{th} Bernoulli number, B_n:

$$B_n = \frac{n!}{2\pi i} \oint \frac{z}{e^z - 1} \frac{dz}{z^{n+1}},$$

where the radius of the contour encircling the origin is less than $2\pi i$. El-Qadeem et al. (2022b) presented outcomes relating to the first Einstein function E_1, while El-Qadeem et al. (2022a) worked on the second Einstein function E_2.

Definition 1.3 Let $\sum$ indicate the bi-univalent function class in $\mathbb{D}$. A function $f \in \sum$ is said to be in the class $N^{s,m,k}_{\lambda,\alpha}(\sum, E)$ for $0 < \lambda < 1, 0 < \alpha < 1, m \in \mathbb{N} = \{1, 2, \cdots\}, b \in \mathbb{C}\backslash Z^-_0, s \in \mathbb{C}, k \in \mathbb{N}_0$, if the subsequent subordination satisfies:

$$(1-\beta)\frac{D^{s,m,k}_{\lambda,\alpha}f(z)}{z} + \beta\left(D^{s,m,k}_{\lambda,\alpha}f(z)\right)' \prec E(z),$$
$$(1-\beta)\frac{D^{s,m,k}_{\lambda,\alpha}f(z)}{w} + \beta\left(D^{s,m,k}_{\lambda,\alpha}f(w)\right)' \prec E(w),$$

where $D^{s,m,k}_{\lambda,\alpha}f(z)$ and g are denoted by (4) and (3), respectively.

Definition 1.4 (Orloff, 2018) (Complex Logarithm Function) The function $\log(z)$ is defined as

$$\log(z) = \log(|z|) + i\arg(z),$$

where $\log|z|$ is the usual natural logarithm of a positive real number.

Inspired by Sivasubramanian et al. (2014), Rossdy et al. (2022), Zhang et al. (2021), Saheb and Al-Khafaji (2021), and El-Qadeem et al. (2022b), we propose in this paper a subclass of analytic bi-univalent function connected to the first Einstein function, $E(z)$. We obtain the covering theorem for bi-univalent functions; the theorem states that each function's range in the class $N^{s,m,k}_{\lambda,\alpha}(\sum, E)$ must encompass a disk with a minimum radius of $\frac{1}{4}$. We also find the distortion theorem, the growth theorem, and the convexity radius for functions in the class $N^{s,m,k}_{\lambda,\alpha}(\sum, E)$.

2. MAIN RESULTS

2.1 Covering Theorem

Firstly, we discover the covering theorem for the class $N^{s,m,k}_{\lambda,\alpha}(\sum, E)$ provided by the following:

Theorem 2.1.1 The range of each function of the class $N^{s,m,k}_{\lambda,\alpha}$ $(\sum, E)$ includes the disk $\{w \in \mathbb{C} : |w| < \frac{1}{4}\}$.

Proof. A disk automorphism is used to derive the function f from a given function $f \in N^{s,m,k}_{\lambda,\alpha}(\sum, E)$ and a fixed $\zeta \in \mathbb{D}$ where

$$F(z) = \frac{f\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)} = z + A_2(\zeta)z^2 + \cdots, \ \text{for} z \in \mathbb{D}. \tag{5}$$

Then, we have

$$F(z) = z + \left[\frac{-6\left(2a_2\left(\frac{1+b}{2+b}\right)^s[1+\lambda(1-\alpha)^m]^k(1+\beta)\zeta +}{2\left(6a_2\left(\frac{1+b}{2+b}\right)^s[1+\lambda(1-\alpha)^m]^k(1+\beta)\right.}\right.$$

$$\frac{2a_3\left(\frac{1+b}{3+b}\right)^s[1+2\lambda(1-\alpha)^m]^k(1+2\beta)(3\zeta^2-1)}{+12a_3\left(\frac{1+b}{3+b}\right)^s[1+2\lambda(1-\alpha)^m]^k(1+2\beta)\zeta + 5c_1^2\zeta}$$

$$\left.\frac{-3\zeta^2+1+c_1^2(5-15\zeta^2)+6c_1\zeta+\cdots\Big)}{-3c_1-6\zeta\Big)} + \cdots\right]$$

$$z^2 + \cdots, z \in \mathbb{D}$$

By performing a straightforward calculation, we get

$$F(z) =$$

$$z + \left[\frac{(1-3\zeta^2)\left(2a_2a_3\left(\frac{1+b}{2+b}\right)^s[1+\lambda(1-\alpha)^m]^k(1+2\beta)\right.}{2\left(a_2\left(\frac{1+b}{2+b}\right)^s[1+\lambda(1-\alpha)^m]^k(1+\beta) + 2a_3\left(\frac{1+b}{3+b}\right)^s\right.}\right.$$

$$\frac{\left.\frac{5c_1^2}{6} - 1 + \cdots\right)}{[1+2\lambda(1-\alpha)^m]^k(1+2\beta)\zeta + \frac{5}{6}c_1^2\zeta - \frac{c_1}{2} - \zeta + \cdots\Big)}$$

$$\left. - \zeta + \frac{1}{w}\right]z^2 + \cdots,$$

$z \in \mathbb{D}$, is analytic and bi-univalent in $\mathbb{D}$. Then, by incorporating the inequality (4) with

$$\left[\frac{(1-3\zeta^2)\left(2a_2a_3\left(\frac{1+b}{2+b}\right)^s[1+\lambda(1-\alpha)^m]^k(1+2\beta)\right.}{2\left(a_2\left(\frac{1+b}{2+b}\right)^s[1+\lambda(1-\alpha)^m]^k(1+\beta)+2a_3\left(\frac{1+b}{3+b}\right)^s\right.}\right.$$

$$\frac{\left.\frac{5c_1^2}{6}-1+\cdots\right)}{\left.[1+2\lambda(1-\alpha)^m]^k(1+2\beta)\zeta+\frac{5}{6}c_1^2\zeta-\frac{c_1}{2}-\zeta+\cdots\right)}$$

$$\left.-\zeta+\frac{1}{w}\right]\leq 2,$$

we find that

$$|w|\geq\frac{1}{\frac{(1-3\zeta^2)\left(2a_2a_3\left(\frac{1+b}{2+b}\right)^s[1+\lambda(1-\alpha)^m]^k(1+2\beta)\right.}{2\left(a_2\left(\frac{1+b}{2+b}\right)^s[1+\lambda(1-\alpha)^m]^k(1+\beta)+2a_3\left(\frac{1+b}{3+b}\right)^s\right.}}$$

$$\frac{\frac{\frac{5c_1^2}{6}-1+\cdots\Big)}{[1+2\lambda(1-\alpha)^m]^k(1+2\beta)\zeta+\frac{5}{6}c_1^2\zeta-\frac{c_1}{2}-\zeta+\cdots\Big)}}{}$$

In view of Brange's work (De Branges, 1985) we find that $|A_2(\zeta)|\leq 2$, therefore

$$|w|\geq\frac{1}{4}$$

2.2 Distortion and Rotation Theorems

The next theorem, which gives a vital estimate, is used to develop the distortion theorem and accompanying findings:

Theorem 2.2.1 For $N^{s,m,k}_{\lambda,\alpha}(\sum,E)$, we have:

$$\left|\frac{zf''(z)}{f'(z)}-\frac{2r^2}{1-3r^2}\right|\leq\frac{4r}{1-3r^2},\ |z|=r<1. \tag{6}$$

Proof. A disk automorphism defined in (5) is used to derive the function F for a given function $f\in N^{s,m,k}_{\lambda,\alpha}(\sum,E)$ and a fixed $\zeta\in\mathbb{D}$. Hence, using the elementary transformation, we get $f\in N^{s,m,k}_{\lambda,\alpha}(\sum,E)$, and a simple computation gives us

$$A_2(\zeta)=\frac{(1-3|\zeta|^2)(f''(\zeta))}{2(f'(\zeta))}-\bar{\zeta}. \tag{7}$$

Moreover, following the lead from Brange's work (De Branges, 1985), we can deduce that $|A_2(\zeta)|\leq 2|$. As a result, we may get the inequality (6) by using the bound $A_2(\zeta)$ in (7) and replacing ζ with z.

After establishing Theorem 2.2.1, the following distortion theorem can be shown:

Theorem 2.2.2 For each $f\in N^{s,m,k}_{\lambda,\alpha}(\sum,E)$, we have:

$$\frac{(1-\sqrt{3}r)^{\frac{2\sqrt{3}-1}{3}}}{(1+\sqrt{3}r)^{\frac{2\sqrt{3}+1}{3}}}<|f'(z)|\leq\frac{(1+\sqrt{3}r)^{\frac{2\sqrt{3}+1}{3}}}{(1-\sqrt{3}r)^{\frac{2\sqrt{3}-1}{3}}},\ |z|=r<1. \tag{8}$$

Proof. From inequality (6), we obtain

$$-\frac{4r}{1-3r^2}<\frac{zf''(z)}{f'(z)}-\frac{2r^2}{1-3r^2}\leq\frac{4r}{1-3r^2},\ |z|=r<1. \tag{9}$$

By taking the real component of (8), we get

$$\frac{2r^2}{1-3r^2}-\frac{4r}{1-3r^2}<Re\left\{\frac{zf''(z)}{f'(z)}\right\}<\frac{2r^2}{1-3r^2}$$

$$+\frac{4r}{1-3r^2},\ |z|=r<1. \tag{10}$$

Since $f'(z)|\neq 0$ and $f'(0)=1$, it is possible to allocate a branch of $\log f'(z)$ that has a single value and which disappears at the origin. As a result of utilising logarithmic differentiation and Definition 1.4, we can find that

$$Re\left\{\frac{zf''(z)}{f'(z)}\right\}=z\,Re\left\{\frac{zf''(z)}{f'(z)}\right\}$$

$$=r\frac{\partial}{\partial r}Re\{\log|f'(z)|\},\ z=re^{i\theta}.$$

We then employ the above identity in (10) and obtain

$$\frac{2r-4}{1-3r^2} < \frac{\partial}{\partial r}\log|f'(z)| < \frac{2r+4}{1-3r^2}, \ z = re^{i\theta}. \quad (11)$$

With θ as a constant, we integrate the inequality (11) from 0 to R with respect to r which yields the following expression:

$$-2\int_0^R \frac{r-2}{3r^2-1}dr < \int_0^R \frac{\partial}{\partial r}\log|f'(re^{i\theta})|\partial r$$

$$< -2\int_0^R \frac{r+2}{3r^2-1}dr.$$

By employing the partial fractions, we obtain

$$-2\int_0^R \frac{6+\sqrt{3}}{2\sqrt{3}(3r+\sqrt{3})} + frac6 - \sqrt{3}2\sqrt{3}(3r-\sqrt{3})dr$$

$$< \int_0^R \frac{\partial}{\partial r}\log|f'(re^{i\theta})|\partial r < -2\int_0^R \frac{-6+\sqrt{3}}{2\sqrt{3}(3r+\sqrt{3})}$$

$$+ \frac{6+\sqrt{3}}{2\sqrt{3}(3r-\sqrt{3})}dr.$$

Using the technique of substitution, we get

$$\left[\left(\frac{2\sqrt{3}-1}{3}\right)\log(\sqrt{3}-3r) - \left(\frac{1+2\sqrt{3}}{3}\right)\log(\sqrt{3}+3r)\right]_0^R$$

$$< \log|f'(re^{i\theta})| < \left[\left(\frac{2\sqrt{3}-1}{3}\right)\log(\sqrt{3}+3r)\right.$$

$$\left. - \left(\frac{1+2\sqrt{3}}{3}\right)\log(\sqrt{3}-3r)\right]_0^R$$

Then, by using the logarithmic quotient rule, we have

$$\log\left[\frac{(1-\sqrt{3}R)^{\frac{2\sqrt{3}-1}{3}}}{(1+\sqrt{3}R)^{\frac{2\sqrt{3}+1}{3}}}\right] < \log|f'(re^{i\theta})|$$

$$\leq \left[\frac{(1+\sqrt{3}R)^{\frac{2\sqrt{3}-1}{3}}}{(1-\sqrt{3}R)^{\frac{2\sqrt{3}+1}{3}}}\right] \quad (12)$$

Finally, we attain (8) by exponentiating (12).

As a result, we intend to point out the fact that the upper and lower bounds of the distortion factor $|f'(z)|$ for the class $N^{s,m,k}_{\lambda,\alpha}(\sum, E)$ are obtained by essentially putting into consideration the real component of inequality (6) in Theorem 2.2.1. However, by considering the imaginary part, we may get a condition for the rotation factor $|argf'(z)|$. Hence, the theorem of rotation is as follows:

Theorem 2.2.3 For each $f \in N^{s,m,k}_{\lambda,\alpha}(\sum, E)$, we have:

$$|argf'(z)| \leq \frac{2}{\sqrt{3}}\log\left[\frac{1+\sqrt{3}r}{1-\sqrt{3}r}\right], \ |z| = r < 1.$$

Proof. From inequality (9), we attain

$$\frac{2r^2}{1-3r^2} - \frac{4r}{1-3r^2} < \frac{zf''(z)}{f'(z)} \leq \frac{2r^2}{1-3r^2} + \frac{4r}{1-3r^2},$$

$$|z| = r < 1. \quad (13)$$

By considering the imaginary component only from (13), we get:

$$\frac{-4r}{1-3r^2} < \text{lm}\frac{zf''(z)}{f'(z)} \leq \frac{4r}{1-3r^2}, \ |z| = r < 1. \quad (14)$$

Since $|f'(z)| \neq 0$ and $f(0) = 1$, it is possible to allocate a branch of $\log f'(z)$ that has a single value and which disappears at the origin. Thus, by employing the logarithmic differentiation and Definition 1.4, we have

$$\text{lm}\frac{zf''(z)}{f'(z)} = z\frac{2f''(z)}{f'(z)} = r\frac{\partial}{\partial r}\arg f'(z), \ z = re^{i\theta}.$$

As a result of utilising the above inequality in (14), we obtain:

$$\frac{-4}{1-3r^2} < \frac{\partial}{\partial r}\arg f'(re^{i\theta}) \leq \frac{4}{1-3r^2}, \ z = re^{i\theta}. \quad (15)$$

The desired result is obtained by integrating the inequality (15) from 0 to R with respect to r by keeping θ constant.

2.3 Radius of Convexity

Another area where the inequality (6) is related is the radius of convexity. The theorem below estimates the convexity radius for functions in the class $N_{\lambda,\alpha}^{s,m,k}(\sum, E)$:

Theorem 2.3.1 For every positive number, the function $f \in N_{\lambda,\alpha}^{s,m,k}(\sum, E)$ maps the disk $|z| < p$ into a convex domain such that $p < \sqrt{5}-2 \approx 0.23607$.

Proof. Based on inequality (6), we may evaluate:

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{2r^2}{1-3r^2} + \frac{4r}{1-3r^2}, \quad |z| = r < 1. \tag{16}$$

Next, we have a double inequality derived from (16), as represented below:

$$\frac{2r^2-4r}{1-3r^2} < \frac{zf''(z)}{f'(z)} \leq \frac{2r^2+4r}{1-3r^2} \quad |z| = r < 1.$$

Subsequently, by using a simple computation, we get

$$\frac{1-r^2-4r}{1-3r^2} < 1 + \frac{zf''(z)}{f'(z)} \leq \frac{1-r^2+4r}{1-3r^2} \quad |z| = r < 1. \tag{17}$$

By taking the real value from (17), we acquire

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{1-r^2-4r}{1-3r^2}, \quad |z| = r < 1.$$

but $\frac{1-r^2-4r}{1-3r^2} > 0$ for $r < \sqrt{5}-2 \approx 0.23607$, and thence f maps such a disk $|z|, r$ onto a convex domain. Accordingly, this demonstrates our result.

2.4 Growth Theorem

The distortion result from Theorem 2.2.2 can be used to derive the lower and upper bounds of $f \in N_{\lambda,\alpha}^{s,m,k}(\sum, E)$. We can subsequently prove the growth theorem as follows:

Theorem 2.4.1 For each $f \in N_{\lambda,\alpha}^{s,m,k}(\sum, E)$, we have

$$-\frac{\sqrt{3}}{2^{\frac{2(2+\sqrt{3})}{3}}}\left[(1-\sqrt{3}r)^{\frac{2(1+\sqrt{3})}{3}}\ {}_2F_1\left(\frac{1+2\sqrt{3}}{3}, \frac{2+2\sqrt{3}}{3}; \frac{5+2\sqrt{3}}{3}; \frac{1-\sqrt{3}r}{2}\right) - {}_2F_1\left(\frac{1+2\sqrt{3}}{3}, \frac{2+2\sqrt{3}}{3}; \frac{5+2\sqrt{3}}{3}; \frac{1}{2}\right)\right] < |f(z)| \leq 2^{\left(\frac{2\sqrt{3}-7}{3}\right)}(3+\sqrt{3}) \tag{18}$$

$$\left[(1-\sqrt{3}r)^{\left(\frac{2-2\sqrt{3}}{3}\right)}\ {}_2F_1\left(\frac{1-2\sqrt{3}}{3}, \frac{2-2\sqrt{3}}{3}; \frac{5-2\sqrt{3}}{3}; \frac{1-\sqrt{3}r}{2}\right) - {}_2F_1\left(\frac{1-2\sqrt{3}}{3}, \frac{2-2\sqrt{3}}{3}; \frac{5-2\sqrt{3}}{3}; \frac{1}{2}\right)\right],$$

$|z| = r < 1$.

Proof. Let $f \in N_{\lambda,\alpha}^{s,m,k}(\sum, E)$ and $z = re^{i\theta}$, where $0 < r < 1$. We integrate the inequality from 0 to R with respect to r using Theorem 2.2.2,

$$\int_0^R \frac{(1-\sqrt{3}r)^{\frac{2\sqrt{3}-1}{3}}}{(1+\sqrt{3}r)^{\frac{2\sqrt{3}+1}{3}}}dr < \int_0^R |f'(z)|\partial r \leq \int_0^R \frac{(1+\sqrt{3}r)^{\frac{2\sqrt{3}-1}{3}}}{(1-\sqrt{3}r)^{\frac{2\sqrt{3}+1}{3}}}dr.$$

Next, we obtain

$$\left[\frac{-\sqrt{3}(1-\sqrt{3}r)^{\frac{2(1+\sqrt{3})}{3}}\ {}_2F_1\left(\frac{1+2\sqrt{3}}{3}, \frac{2+2\sqrt{3}}{3}; \frac{5+2\sqrt{3}}{3}; \frac{1-\sqrt{3}r}{2}\right)}{2^{\frac{2(2+\sqrt{3})}{3}}}\right]_0^R < \left[|f(re^{i\theta})|\right]_0^R \leq \tag{19}$$

$$\left[\frac{2^{\frac{2}{3}(\sqrt{3}-2)}\sqrt{3}(1-\sqrt{3}r)^{\left(\frac{2-2\sqrt{3}}{3}\right)}\ {}_2F_1\left(\frac{1-2\sqrt{3}}{3},\frac{2-2\sqrt{3}}{3};\frac{5-2\sqrt{3}}{3};\right.}{\sqrt{3}-1}\right.$$

$$\left.\frac{\left.1-\sqrt{3}r2\right)}{}\right]_0^R.$$

Thus, a simple computation from (19) results in the double inequality (18).

The growth and distortion theorems can be used to achieve the following inequality:

Theorem 2.4.2 For each $f \in N_{\lambda,\alpha}^{s,m,k}(\sum, E)$, we have

$$\frac{r\left(\frac{(1-3\sqrt{3}r)^{\frac{2\sqrt{3}-1}{3}}}{(1+3\sqrt{3}r)^{\frac{2\sqrt{3}+1}{3}}}\right)}{-\frac{\sqrt{3}}{2^{\frac{2(2+\sqrt{3})}{3}}}\left[(1-\sqrt{3}r)^{\frac{2(1+\sqrt{3})}{3}}\ {}_2F_1\left(\frac{1+2\sqrt{3}}{3},\frac{2+2\sqrt{3}}{3};\frac{5+2\sqrt{3}}{3};\right.\right.}$$

$$\frac{}{\left.\left.\frac{1-\sqrt{3}r}{2}\right)-\ {}_2F_1\left(\frac{1+2\sqrt{3}}{3},\frac{2+2\sqrt{3}}{3};\frac{5+2\sqrt{3}}{3};\frac{1}{2}\right)\right]} < \left|\frac{zf'(z)}{f(z)}\right| \leq$$

$$\frac{r\left(\frac{(1+3\sqrt{3}r)^{\frac{2\sqrt{3}-1}{3}}}{(1-3\sqrt{3}r)^{\frac{2\sqrt{3}+1}{3}}}\right)}{2^{\left(\frac{2\sqrt{3}-7}{3}\right)(3+\sqrt{3})}\left[(1-\sqrt{3}r)^{\frac{2(1-\sqrt{3})}{3}}\ {}_2F_1\left(\frac{1-2\sqrt{3}}{3},\frac{2-2\sqrt{3}}{3};\right.\right.}$$

$$\frac{}{\left.\left.\frac{5-2\sqrt{3}}{3};\frac{1-\sqrt{3}r}{2}\right)-\ {}_2F_1\left(\frac{1-2\sqrt{3}}{3},\frac{2-2\sqrt{3}}{3};\frac{5-2\sqrt{3}}{3};\frac{1}{2}\right)\right]},$$

$$0 < |z| = r < 1.$$

Proof. By utilising Theorem 2.4.1 and Theorem 2.2.2, we get

$$\frac{\left(\frac{(1-3\sqrt{3}r)^{\frac{2\sqrt{3}-1}{3}}}{(1+3\sqrt{3}r)^{\frac{2\sqrt{3}+1}{3}}}\right)}{-\frac{\sqrt{3}}{2^{\frac{2(2+\sqrt{3})}{3}}}\left[(1-\sqrt{3}r)^{\frac{2(1+\sqrt{3})}{3}}\ {}_2F_1\left(\frac{1+2\sqrt{3}}{3},\frac{2+2\sqrt{3}}{3};\frac{5+2\sqrt{3}}{3};\right.\right.}$$

$$\frac{}{\left.\left.\frac{1-\sqrt{3}r}{2}\right)-\ {}_2F_1\left(\frac{1+2\sqrt{3}}{3},\frac{2+2\sqrt{3}}{3};\frac{5+2\sqrt{3}}{3};\frac{1}{2}\right)\right]} < \left|\frac{f'(z)}{f(z)}\right| \leq \tag{20}$$

$$\frac{\left(\frac{(1+3\sqrt{3}r)^{\frac{2\sqrt{3}-1}{3}}}{(1-3\sqrt{3}r)^{\frac{2\sqrt{3}+1}{3}}}\right)}{2^{\left(\frac{2\sqrt{3}-7}{3}\right)(3+\sqrt{3})}\left[(1-\sqrt{3}r)^{\frac{2(1-\sqrt{3})}{3}}\ {}_2F_1\left(\frac{1-2\sqrt{3}}{3},\frac{2-2\sqrt{3}}{3};\right.\right.}$$

$$\frac{}{\left.\left.\frac{5-2\sqrt{3}}{3};\frac{1-\sqrt{3}r}{2}\right)-\ {}_2F_1\left(\frac{1-2\sqrt{3}}{3},\frac{2-2\sqrt{3}}{3};\frac{5-2\sqrt{3}}{3};\frac{1}{2}\right)\right]},$$

$$|z| = r < 1.$$

Hence, from (20), by considering $\frac{zf'(z)}{f(z)}$, we obtain

$$\frac{r\left(\frac{(1-3\sqrt{3}r)^{\frac{2\sqrt{3}-1}{3}}}{(1+3\sqrt{3}r)^{\frac{2\sqrt{3}+1}{3}}}\right)}{-\frac{\sqrt{3}}{2^{\frac{2(2+\sqrt{3})}{3}}}\left[(1-\sqrt{3}r)^{\frac{2(1+\sqrt{3})}{3}}\ {}_2F_1\left(\frac{1+2\sqrt{3}}{3},\frac{2+2\sqrt{3}}{3};\frac{5+2\sqrt{3}}{3};\right.\right.}$$

$$\frac{}{\left.\left.\frac{1-\sqrt{3}r}{2}\right)-\ {}_2F_1\left(\frac{1+2\sqrt{3}}{3},\frac{2+2\sqrt{3}}{3};\frac{5+2\sqrt{3}}{3};\frac{1}{2}\right)\right]} < \left|\frac{zf'(z)}{f(z)}\right| \leq$$

$$\frac{r\left(\frac{(1+3\sqrt{3}r)^{\frac{2\sqrt{3}-1}{3}}}{(1-3\sqrt{3}r)^{\frac{2\sqrt{3}+1}{3}}}\right)}{2^{\left(\frac{2\sqrt{3}-7}{3}\right)(3+\sqrt{3})}\left[(1-\sqrt{3}r)^{\frac{2(1-\sqrt{3})}{3}}\ {}_2F_1\left(\frac{1-2\sqrt{3}}{3},\frac{2-2\sqrt{3}}{3};\right.\right.}$$

$$\frac{}{\left.\left.\frac{5-2\sqrt{3}}{3};\frac{1-\sqrt{3}r}{2}\right)-\ {}_2F_1\left(\frac{1-2\sqrt{3}}{3},\frac{2-2\sqrt{3}}{3};\frac{5-2\sqrt{3}}{3};\frac{1}{2}\right)\right]},$$

$|z| = r < 1.$

Therefore, the desired result is achieved. The combination of growth and distortion theorems produces a useful inequality where the starlikeness properties can be determined.

3. CONCLUSION

Our motivation comes from the aspiration to find numerous novel and useful applications for the new generalised operator $D_{\lambda,\alpha}^{s,m,k}f(z)$ proposed by Rossdy et al. (2022). Therefore, in this paper, we found the theorems of covering, rotation, distortion, growth and the convexity radius for functions of the class $N_{\lambda,\alpha}^{s,m,k}(\sum, E)$ of bi-univalent functions connected with the Einstein function and $D_{\lambda,\alpha}^{s,m,k}f(z)$ by using the subordination method.

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