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Examining pre-service teachers' subject matter knowledge of school mathematics concepts

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Abstract

This article explores the nature of the subject matter knowledge that pre-service mathematics teachers' possess by analysing its components, namely, common content knowledge and specialised content knowledge of the high school mathematics curriculum. The data was generated from 59 final year pre-service mathematics teachers' written responses, which tested three sets of competencies: solving problems on Functions and Inequalities; analysing and interpreting learners' errors; and their expertise in allocating marks to questions. Analysis of the written responses identified five categories of response patterns and thereafter individual interviews were conducted with four participants to discuss he responses. The results revealed that while the participants were competent solvers of school mathematics problems, they were unable to analyse and interpret learners' errors for diagnostic purposes. This suggests that teacher preparation should develop pre-service mathematics teachers' specialised content knowledge. The findings also confirmed the value of learner error analysis as an alternative strategy to cultivate the specialised content knowledge of future mathematics teachers.

Introduction

This article explores pre-service mathematics teachers' (PMTs) subject matter knowledge (SMK) of school mathematics. The research was motivated by South African learners' poor performance in mathematics as evident in the Annual National Assessment findings and the Grade 12 moderators' report (Department of Basic Education, 2014). The report suggests the need for a nuanced examination of the enablers or inhibitors of teaching and learning of mathematics, especially at secondary school level. We use the example of a section on Functions and Inequalities comprising of two research questions to answer the questions: (i) What is the nature of final year PMTs' SMK of school mathematics? (ii) What is the relationship (if any) between PMTs' common content knowledge (CCK) and specialised content knowledge (SCK)? Ball, Thames and Phelps' (2008) dissection of SMK into two parts, CCK and SCK, framed the data analysis. The importance of identifying PMTs' SMK of mathematics is outlined hereunder, followed by an exposition of the theoretical underpinnings by tracing its links to the Shulman (1986) conception and the Ball *et al*. (2008) refinement thereof.

Mathematical knowledge for teaching

Mathematical knowledge for teaching (MKfT) is crucial for successful teaching and learning outcomes and learner attainment (see e.g. Ball *et al.*, 2008; McAuliffe, 2013; Bansilal, Brijlall & Mkhwanazi, 2014; Pournara, Hodgen, Adler & Pillay; 2015, Livy, Vale & Herbert, 2016; Aksu & Umit, 2016; Pournara, 2016). However, there is a paucity of in-depth research on what this knowledge for teaching entails, and how it is acquired during the learning to teach phase, especially in South Africa. Driven by an interest in school learner performance (e.g. Mji & Makgato, 2006; Bansilal *et al.*, 2014; Pournara *et al.*, 2015), researchers have explored the MKfT of in-service teachers with scant attention to future mathematics teachers' MKfT. Furthermore, pre-service teacher education research has focused on the quality of curriculum design of teacher training programmes (see e.g. McAuliffe, 2013; Sapire, Shalem, Wilson-Thompson & Paulsen, 2016), with limited attention to the nature of the requirements for future teachers' MKfT *per se*.

Like those in the international arena (see e.g. Lucus, 2006; Ball *et al.*, 2008; Aksu & Umit, 2016), South African studies on MKfT (see e.g. Sorto & Sapire, 2011; Bansilal *et al.*, 2014; Pournara *et al.*, 2015) have tended to focus on the MKfT of in-service mathematics teachers. In contrast, McAuliffe's (2013) study explored South African primary school PMTs' knowledge of the subject matter of early algebra. The findings revealed that SMK for teaching early algebra was not fully established as there were shortcomings in their ability to respond to three significant aspects, namely, describing the procedures used by learners; interpreting learner productions; and analysing learner errors to improve teaching. Similarly, Pournara's (2016) study revealed that engaging with peers' mathematical contributions deepened pre-service content knowledge. Beyond the borders of South Africa, Lucus (2006) investigated the SMK of Composite Functions of both PMTs and in-service Canadian teachers. Both groups showed poor conceptual knowledge of Composite Functions. While they could perform the relevant procedures, they were unable to conceptualise the rules meaningfully to teach the concepts. While the aforementioned studies analysed some aspects of SMK among PMTs, they neglected the negotiation of the teaching and learning space that leads to teachers' attainment of MKfT.

MKfT refers specifically to the "mathematical knowledge that teachers need to carry out their work as teachers of mathematics" (Ball *et al.*, 2008, p.4). However, current debates on what MKfT entails tend to under-represent the scope originally proposed by Ball *et al.* (2008) who suggested that mathematical knowledge in teacher education courses should be more expansive than simply the development of the disciplinary content knowledge of future mathematics teachers. A more careful reading of the theoretical model they proposed demonstrates that MKfT embodies both SMK and PCK (pedagogical content knowledge). Beginning with Shulman's (1986) idea that good teaching is characterised by SMK and PCK, Ball *et al.* (2008) made explicit the components of SMK and PCK as they apply to future teachers' MKfT. Ball *et al.*'s (2008) model disaggregated SMK into two types (common and specialised) and PCK into three types (content and students, content and teaching, and curriculum) as represented in Fig 1.

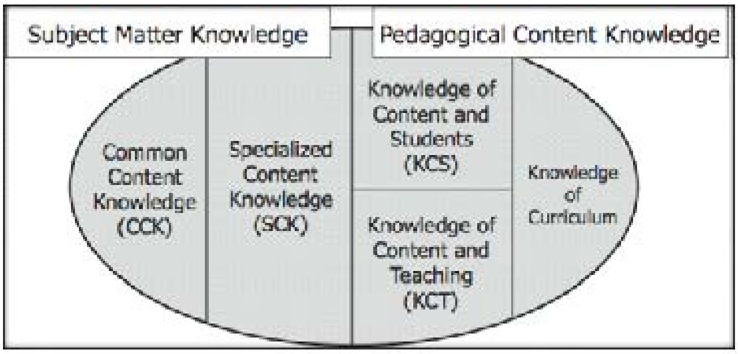

Figure 1: Model of mathematical knowledge for teaching (Source, Ball *et al*., 2008, p.5)

Beyond Shulman's (1986) definition of teacher knowledge, Ball *et al*. (2008) made a clear distinction between each strand of knowledge and the strands of knowledge PMTs should possess at the exit level to ensure their successful deployment as mathematics teachers at schools. In this instance, we delimited the focus to the SMK components (i.e. CCK and SCK). The expanded version represented in Fig. 1 serves to contextualise the relationship between the Shulman and the Ball *et al.* categories of description.

The dual purposes of teaching mathematics and, more specifically, clarifying the mathematical knowledge needed by pre-service teachers underpins the refined model. Ball et al (2008) argued that at the exit level, PMTs should have acquired sufficient mastery of SMK, "SMK is the mathematical knowledge that teachers need to carry out their work as teachers of mathematics, i.e. knowing how knowledge is generated and structured in the discipline" (2008, p.4). In many parts of the world, however, the teaching of SMK is dominated by a focus on CCK (knowing about mathematics) while SCK (mathematics teaching or representing mathematical ideas) has been marginalised (see e.g. Bansilal *et al*., 2014; Hine, 2015; Ndlovu & Brijlall, 2015). However, both are crucial, as knowing and doing mathematics are integrated and inter-related phenomena (McAuliffe, 2013).

Other than disciplinary content knowledge, SMK entails additional strategic abilities to analyse learners' errors and to identify remedial teaching and pedagogical actions (diagnostic and interventionist approaches) to activate mathematics learning. The latter, contextualised within classroom-based interactive knowledge, is termed SCK because it pays unambiguous attention to teachers' negotiation of learner responses to their pedagogy. Therefore, the purpose of this article is to explore final year mathematics PMTs' competencies and deficiencies, not just in terms of their CCK, but also of their SCK, i.e. a combination of their knowing, doing and diagnoses of leaners' learning engagements and errors during in-classroom settings. The assumption that underpins the inquiry is that the SMK (CCK and SCK) of the teacher education curriculum influences the advancement of MKfT. We suggest that future mathematics teachers can be inspired to acquire specialised knowledge to operate as master crafts-persons who can teach and work with learners in school settings through the use of error analysis pedagogy.

Methodology

Context

At the institution where the data for this study was produced, the unit, Mathematics for Educators Method 3, was offered to pre-service Further Education and Training mathematics teachers who were completing a Bachelor of Education (BEd.) degree. The eight-credit points unit was offered for ten weeks in the second semester with one double lecture period of 90 minutes per week. In this module, PMTs engaged with theories of learning and assessment with three foci: trigonometry, functions and calculus graphs. Other than theories of learning and assessment, students were exposed to approaches to teach the content of mathematics.[1]

At the study site, four of the twenty-nine weeks of each academic year are reserved for the in-school teaching practicum component. The teaching practicum begins in the second year, which means that for the four-year degree, a total of twelve weeks are spent in actual teaching settings. Despite the participants' short period of classroom teaching experience, and our concern about whether they had had sufficient preparation to be employed as mathematics teachers, there was consensus that deep analysis was necessary.

Participants

A cohort of 59 PMTs in their final year participated in the study. The group had already completed and passed all the required content and methods modules offered from year one to the first semester of year four. Unstructured interviews were conducted with four of the 59 participants that were purposively selected based on their written responses.

Methods of data generation

Data to gauge pre-service teachers' knowledge of mathematical concepts was obtained from the participants' responses to a competency test, which was followed by four unstructured interviews. The competency test, comprising items from past Grade 12 papers and the 2014 Department of Education diagnostic report of learners' errors on Graphs and the interpretation of a Hyperbola was conducted at the end of semester two of their final year. Two questions were set, each comprising of five sub-questions. Six sub-questions assessed PMTs' SMK of Functions while four sub-questions focused on their CCK (procedural as well as conceptual understanding) and their ability to analyse learners' misconceptions (see Table 1).

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[1] We acknowledge that the generic orientation to teaching and learning, curriculum design and classroom management is part of the pre-service programme, and that it could activate some conceptions of knowledge for teaching. However, the generic input is most often presented as separate from the specifics of the discipline of mathematics, and, therefore, is not the focus of this article.

While Ball *et al.* (2008) categorise knowledge of teaching strategies under knowledge of content and teaching, in this article, knowledge of teaching strategies is associated with SCK because the ability to analyse and interpret learners' errors automatically required the pre-service teachers to consider remedial teaching strategies. The literature on teacher knowledge does not clearly specify the aspects they (PMTs) should possess to design marking guides and analyse errors of assessed tasks. However, Ball *et al.*'s (2008) definition of SCK includes teachers' "need to unpack elements of mathematics to make its features apparent to students and need expertise with certain mathematical practises" (p.10). Adler (2010) also argued that among other aspects of mathematics for teaching, task design and attention to mathematical content, object and processes are also important. The expertise envisaged for SMK goes beyond mere ability to solve the problem; it also requires the ability to explain the skills and knowledge required to solve a mathematical problem. Therefore, a question (Q1.4) was included to explore PMTs' expertise to identify the skills and knowledge assessed in the question related to Functions and Inverse Functions as well as their expertise in allocating marks to questions.

Question two (see Table 1) explored PMTs' SMK of Inequalities. The sub-questions adhered to the same pattern as question one except for designing a marking guide as in question Q1.4. Question 2.1 was taken from the 2015 June examination paper. For the purpose of this study, three additional sub-questions (Q2.1.1– Q2.1.3) were added because CCK is not just about carrying out procedures; it also includes the conceptualisation of such procedures as knowledge of Functions and Inequalities is imperative for teaching and learning in secondary schools. Following Adler's (2017) call, revisiting key concepts like Functions and Inequalities (secondary school mathematics) is important for prospective teachers (p.3).

Table 1: Examples of questions on CCK and SCK

Assessing CCK	Assessing SCK
2.1 *Solve for x*: $5^x\ (x + 8) < 0$	**2.2** Here are the responses of two learners who answered this question. Explain each learner's misconception and suggest at least one way in which you will address the learners' misconception in your teaching of this concept.

Question 2.1 aimed to assess PMTs' competence to solve the problem learners were asked to solve and question 2.2 aimed to assess their competence to analyse and interpret learners' errors to the same questions.

Finally, the data from the interviews was used to further explore PMTs' SMK and allowed them to clarify their written responses. This was a useful exercise because in many instances, PMTs do not thoughtfully engage with what they write (Ndlovu & Brijlall, 2015). The reason for prompting PMTs along these lines of analyses was two-fold: firstly, a focus on key concepts and skills and, secondly, awareness of their own ability to solve and assess learners' solutions, both of which are fundamental to improving knowledge of subject matter (Chinnappan & White, 2015). To ensure trustworthiness of the data, four PMTs' actual written responses are presented in the analysis section. However, some of the responses had to be written over to make them clearer as the scanned copies of the originals were hazy. The 'R' in the transcripts refers to the researcher who interviewed the participants.

Ethical considerations

Permission to conduct the study was obtained from the registrar of the institution and from pre-service teachers. While the competency test was part of continuous assessment for the module, PMTs had to consent to have their scripts analysed and to participate in the interview as well as to have the interview recorded. The purposes of the study were explained at the beginning of the semester and PMTs were aware that participation was voluntary. To protect the participants' identities, all names used in this article are pseudonyms. Permission to use learners' scripts was sought from a Grade 12 Mathematics teacher. For purposes of confidentiality, the name of the school and those of learners were removed from the test papers.

Analysis of data

The data from the competency test was analysed inductively through the identification of patterns in the participants' written responses. Five response categories emerged from the data:

(i) *Complete response*, which indicated mathematically correct and accurate responses on Functions and Inequalities concepts;
(ii) *Partial response*, which referred to computational errors which led to an incorrect answer but evident understanding of the concept;
(iii) *Satisfactory*, which indicated incomplete reasoning, that is, only a partial explanation was offered;
(iv) *Incorrect response*; inability to explain the reason for an incorrect response;
(v) *No response*, which indicated that no attempt was made to answer the question.

The analysis of learners' errors for CCK and SCK was informed by Olivier's (1989) framework for differentiating between the types of mistakes learners make. He distinguished between slips, errors and misconceptions thus:

> Slips are wrong answers due to *processing*; they are not systematic, but are sporadically carelessly made by both experts and novices; they are easily detected and are spontaneously corrected. Errors are wrong answers due to *planning*; they are systematic in that they are applied regularly in the same circumstances. Errors are the symptoms of the underlying conceptual structures that are the cause of

errors. It is these underlying beliefs and principles in the cognitive structure that are the cause of systematic conceptual errors that are referred to as misconceptions.

For the competency test, the same response categories were used for the sub-questions, which assessed CCK and SCK.

(i) *Complete response* meant that PMTs displayed competence in both CCK and SCK;
(ii) *Partial response* indicated that the attempted responses consisted of slips;
(iii) *Satisfactory response* indicated misconception about CCK or SCK;
(iv) *Incorrect response* indicated that either CCK or SCK is lacking;
(iv) *No response* related to not being able to identify the extent to which either CCK or SCK had developed unless a PMT was asked to explain the reason behind not attempting the questions. Evidence from previous studies (Ndlovu & Brijlall, 2015; 2016) revealed that reasons other than not knowing the answer preclude students from attempting a question, e.g. time factors.

As noted previously, four participants were interviewed, one per category. No-one was interviewed for the category, no response, as this category did not allow for probing. A summary of the respondents' written responses appears in Table 1.

Table 2: Pre-service teachers' written responses to questions testing CCK and SCK of Functions and Inequalities

	Responses to CCK questions (see Fig. 2)						**Responses to SCK questions (see Fig.3)**				
Categories of responses	Q.1.1.1	Q.1.1.2	Q.1.1.3	Q.1.1.4	Q.2.1.1	Q.2.1.2	Q.1.2	Q.1.3	Q.1.4	Q.2.1.3	Q.2.2
Complete responses	42	42	30	16	42	41	8	7	19	23	12
Partial response	15	6	17	5	3	4	14	14	8	18	14
Satisfactory response	1	0	0	5	3	1	30	33	25	1	28
Incorrect response	1	11	11	22	6	7	2	1	4	10	2
No responses	0	0	1	11	5	6	5	4	3	7	3
Total	59	59	59	59	59	59	59	59	59	59	59

Responses to common content knowledge questions

In this section, we present the questions that appeared in the competency test on CCK (Fig. 2) and analyse PMTs' responses to the test items. Three errors were identified, and are discussed in the following order: squaring slips, misconceptions about the axis of symmetry, and inability to integrate concepts and to recognise interrelationships between concepts. Table 1 indicates that more respondents answered three questions (Q1.1.1; Q2.1.1; Q2.1.2) correctly than all the other questions. Correct responses were probably due to PMTs being able to draw on CCK because these questions were at levels one and two of Bloom's taxonomy and merely required recall of familiar processes. However, several responses to Q1.1.3 were characterised by misconceptions (incorrect beliefs) of CCK. The solution for Q1.1.3 required interpretation and transformation, which are integrated processes and may also explain why students encountered difficulties. In the examples below, we present the analyses of three responses that exemplified inattention to important details and misconceptions.

1.1 Given: $h(x) = 4^x$ and $f(x) = 2(x-1)^2 - 8$.

1.1.1 Sketch the graphs of h and f on the diagram sheet provided. Indicate ALL intercepts with the axes and any turning points.

1.1.2 Without any further calculations, sketch the graph of $y = \log_4 x$ on the same system of axes.

1.1.3 The graph of f is shifted 2 units to the LEFT. Write down the equation of the new graph.

2.1 The following question appeared in the grade 12 June paper 2015

solve for x: $5^x(x+8) < 0$

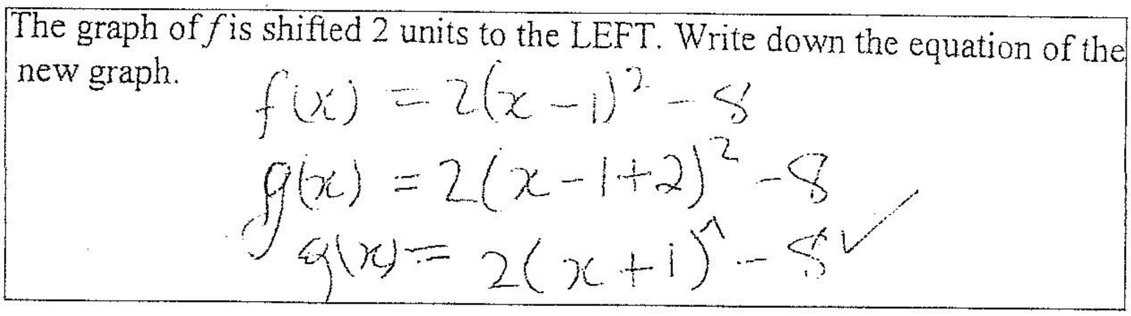
2.1.1 Write down three different ways the problem could be solved

2.1.2 Solve the problem using one of the methods and verify your answer using one of the two other methods.

Figure 2: Questions on common content knowledge in the competency test

Table 1 indicates that more respondents answered three questions (Q1.1.1; Q2.1.1; Q2.1.2) correctly than all the other questions. Correct responses were probably due to PMTs being able to draw on CCK because these questions were at levels one and two of Bloom's taxonomy and merely required recall of familiar processes. However, several responses to Q1.1.3 were characterised by misconceptions (incorrect beliefs) of CCK. The solution for Q1.1.3 required interpretation and transformation, which are integrated processes and may also explain why students encountered difficulties. In the examples below, we present the analyses of three responses that exemplified inattention to important details and misconceptions.

Squaring slips


The graph of f is shifted 2 units to the LEFT. Write down the equation of the new graph.

$f(x) = 2(x-1)^2 - 8$

$g(x) = 2(x-1+2)^2 - 8$

$g(x) = 2(x+1) - 8$

Figure 3: Thabo's response

Thabo's response (Fig. 3) was classified as partially correct because he made a slip, which was indicative of carelessness or inattention to detail rather than a misconception. The error was considered to be a slip because all the necessary procedures were carried out accurately except that the square in $a\ (x + p)^2$ in the final answer was left out, leaving the answer a $(x + p) + q$. Five PMTs whose responses were also assessed as partially correct executed similar mistakes. Others did not write the value of 'a' in the final answer. The interview response below confirmed that Thabo knew how to solve the problem:

Thabo: The equation was about horizontal shift, which means every point of f(x) will shift 2 units to the left. So the axis of symmetry is x=1 for f(x), so shifting 2 units will be at x=-1. When writing my final answer I forget to indicate the square.

Misconceptions about the axis of symmetry

Shirley's response was classified as incorrect as it exposed her misconception about the axis of symmetry. Her interpretation of the value of p in the expression $f\,(x) = a\ (x \pm p)^2 + q$ was that $(x \pm p)$ is the axis of symmetry instead of $x = \pm\, p$,indicating a fundamental misconception about axis of symmetry. Hence, she generated the new equation thus: $f\,(\text{x}) = 2(x – 3)^2 – 8$. While she understood the meaning of a horizontal shift, the incorrect conception of an axis of symmetry led to the incorrect answer. She could not make the connections between the sketched graph in Q1.1.1 and the new equation she generated in Q1.1.3. Shirley’s response revealed a fundamental knowledge gap that some PMTs might have:

> *Shirley: Shifting means moving the graph to another point. If they say 2 units to the left it means it will shift on the x-axis to the left. So in this graph, the axis of symmetry is (-1;0) and (3;0). Is it? I am not sure, no the turning point (-1;8) so the axis of symmetry should be -1. Therefore, in this case, shifting the axis of symmetry 2 units to the left will give the new axis of symmetry . . . Um, I think I am confused now but wherever the axis of symmetry is, it will shift to left. If I could remember the formula of calculating the axis of symmetry, I would get the correct answer. Here I made a mistake.*
>
> R: Where did you actually make a mistake?
>
> *Shirley: My answer is marked wrong. I know 2 and 8 should remain the same because the shift is on the x-axis so it will not affect the value of ‘a’ and that of ‘q’. It means I made a mistake when I am shifting ‘p’. Can I do the sum again? Perhaps I will see where my mistake is.*
>
> R: As a teacher, how would you help learners not to make the same mistake?
>
> *Shirley: For learners, it’s easy. They are given the formula sheet with formulas (sic), we were not given the sheets.*

While Shirley’s response seemed to indicate that PMTs’ CCK lacked insight, it should be noted that her misconception was an uncommon one. In general, while the written responses revealed that PMTs’ procedural knowledge of Functions and Inequalities was sufficient as they could correctly answer the same problems that learners had to solve, attention to Shirley’s response is important as it reveals that the mathematics pre-service curriculum did not address the specific CCK gap of a PMT. On an optimistic note, the ability to solve problems meant that some PMTs were able to identify incorrect responses. While this is insufficient for teaching, it is the basic understanding needed to teach mathematics at school level because teachers should be able to solve problems and to spot incorrect responses with ease.

Inability to integrate concepts and recognise inter-relationships between concepts

1.1.4 Show, algebraically, that $h(x + \frac{1}{2}) = 2hx$

2.1.3 How can you tell that your answer makes sense in the context of this problem?

Figure 4: Question 1.1.4

Eleven respondents did not attempt Q1.1.4 on Functions (see Fig. 4) and of those who answered, only a few provided the correct responses. These were the same individuals who provided incorrect responses to Q1.1.3. A similar trend was observed for the answers to Q2.1.3 on Inequalities, although seven respondents did not attempt to answer the question. The reasons might be that these questions required integration of different concepts and explanations for the answers given.

Based on the examples of a few individuals, the lack of CCK with respect to interpretation and transformation of Functions and Inequalities, offers insight into the importance of CCK for future mathematics teachers. Mastery of concepts is critical for successful teaching. This means that some PMTs are capable of carrying out procedures but have difficulties in explaining the processes and interrelationships between concepts. The 11 PMTs who provided incorrect responses to Q1.1.3 also provided incorrect responses to Q1.1.4, confirming our hunch that knowledge of procedures does not always translate into being able to conceptualise the concepts. To clarify, we offer the example of Shirley’s response (see Fig. 5):

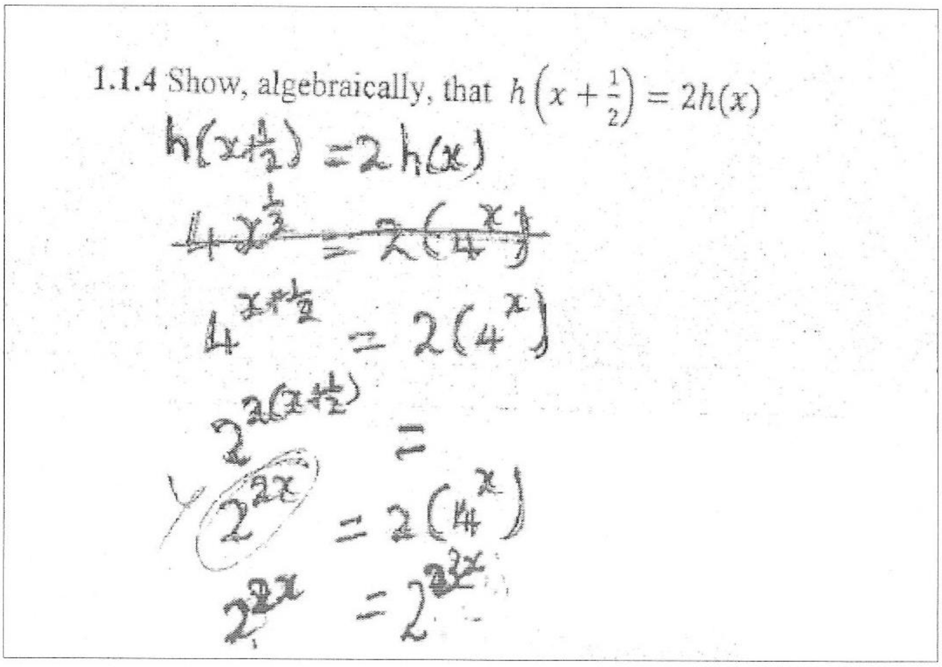


Figure 5: Shirley's response

Q1.1.4 required knowledge of exponential laws and transformation of graphs. From the written response it is evident that Shirley was unable to integrate the two concepts (knowledge of the rules of exponents and the transformation of graphs). She was able to relate the two expressions to an exponential graph of h (x) = 4^x, but lack of knowledge of the laws of exponents hindered her ability to simplify the two expressions. As a result, she equated 2(4^x) with 2^{4x}, revealing misconceptions about bases and exponents. During the interview, Shirley confirmed her inability to recognise interrelationships between the concepts.

R: What is your interpretation of question 1.1.4?
Shirley: I am supposed to prove that $h\left(x+\frac{1}{2}\right) = 2h(x)$?
R: How can you prove this?
Shirley: I know $h(x) = 4^x$ *now I must shift it horizontal, but shifting the exponential graph algebraically is not easy. I can sketch the shifted graph because I can substitute values of* x *then get the coordinates to plot.*
R: What prerequisite knowledge do you need in order to solve 1.1.4?
Shirley: Laws of exponents but I am not sure which one to apply in this questions (sic) because it's not straight forward. This question is quite confusing. If it was simplified I could have answered it.
R: In what way could it have been simplified?
Shirley: Not sure but make it simply rather ask to show by graph sketching.

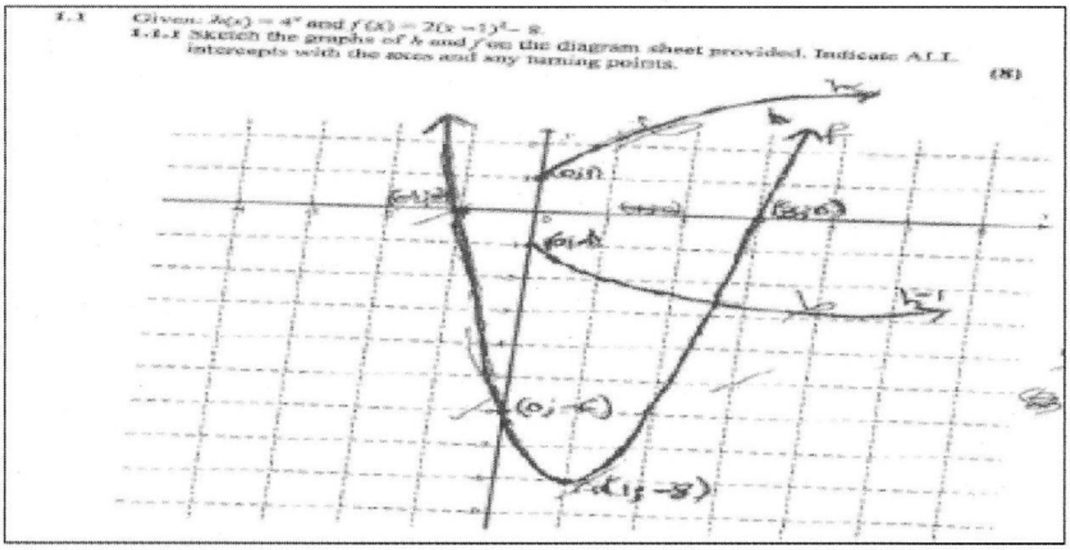
We infer from Shirley's response that the prerequisites and essentials regarding integrated concepts need to be factored into pre-service teaching programmes. Shirley is emblematic of PMTs who do not grasp that inability to recognise interrelationships between concepts would weaken their ability to teach mathematics and could result in setting easier problems for students to solve. It might also be one of the reasons that misconceptions are acquired in school; teaching of higher order thinking is avoided due to the misguided notion that mathematics must be simplified or dumbed down (instead of diagnostic intervention) for learners.

The responses to the questions on CCK are insightful, as they show the important role played by mastery of the fundamental knowledge of mathematics for pre-service teachers. Furthermore, CCK can be regarded as an essential building block and gateway to the acquisition of SCK. In the next section the vital connection between CCK and SCK becomes more apparent.

Specialised content knowledge questions and participants' responses

In this section, we present the PMTs' responses to the questions on SCK (Fig. 6), which were analysed for correctness and inadequacies. Five kinds of inadequacies were identified, four in accordance with the analytical framework and the fifth based on a task requiring assessment design. The inadequacies were: identification of minor errors, not misconceptions; inability to identify misconceptions; inability to suggest remedial teaching strategies; limitations regarding suggestions for remedial teaching strategies, and insufficient knowledge of designing assessment tasks.

1.2 Here are the responses of learners who answered **1.1.1to 1.1.4.** Explain each learner's misconception and suggest at least one way in which you will address the learners' misconception in your teaching of this concept.



Learner response to Q 1.1.3

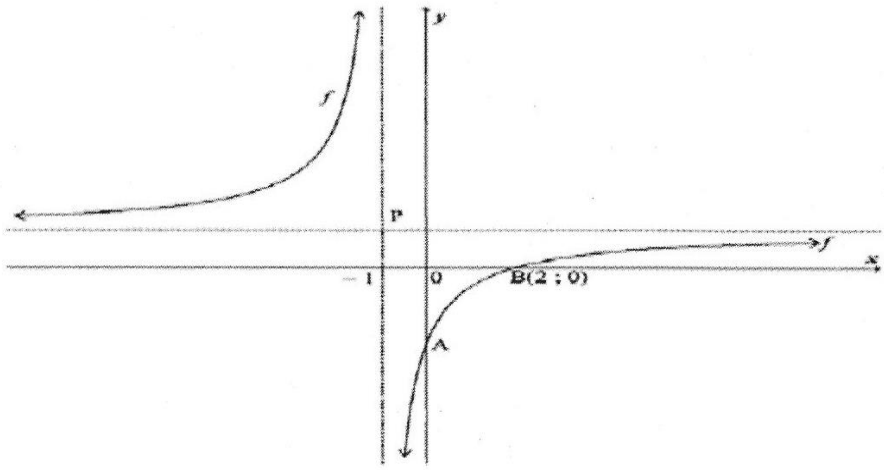
$$y = 2x^2 - 4x - 6 - 2.$$

Learner response to Q1.1.4

1.1.4 Show, algebraically, that $h\left(x + \frac{1}{2}\right) = 2h(x)$

$$h\left(x + \tfrac{1}{2}\right)$$
$$4^{\left(x + \frac{1}{2}\right)} = 2\left(4^x\right)$$
$$4^x \cdot 4^{\frac{1}{2}} = 8^x$$
$$4^x \cdot 2 = 8^x$$
$$8^x = 8^x$$
$$h\left(x + \tfrac{1}{2}\right) = 2h(x)$$

A sketch of the hyperbola $f(x) = \frac{x-d}{x-p}$, where d and p are constants, is given below. The dotted lines are the asymptotes. The asymptotes intersect at P and B(2 ; 0) is a point on f.



6.1.1	Determine the values of d and p.	(2)
6.1.2	Show that the equation of f can be written as $y = \frac{-3}{x+1} + 1$.	(2)
6.1.3	Write down the coordinates of P.	(2)
6.1.4	Write down the coordinates of the image of $B(2 ; 0)$ if B is reflected about the axis of symmetry $y = x + 2$.	(2)

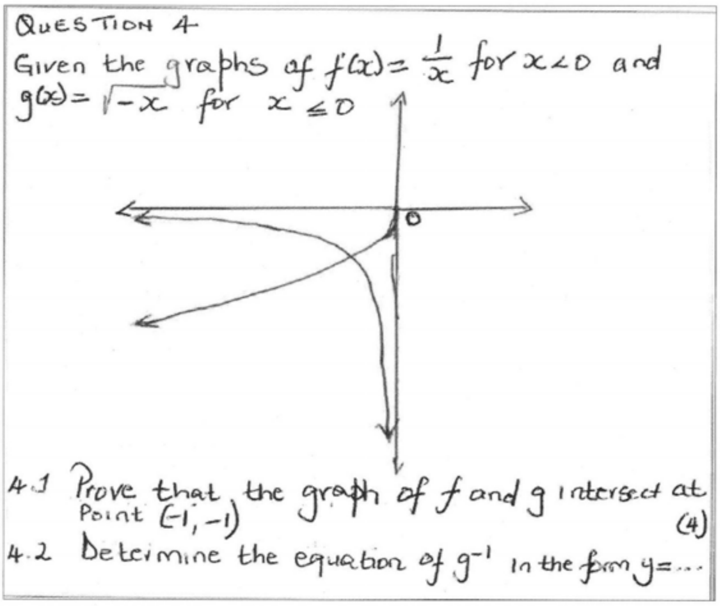
6.2 The exponential function, $g(x) = p.2^x + q$ has a horizontal asymptote at $y = 1$ and passes through $(0 ; -2)$. Determine the values of p and q. (3)
[11]

Below is the diagnostic report from the internal moderators. Read it carefully and answer the question that follows

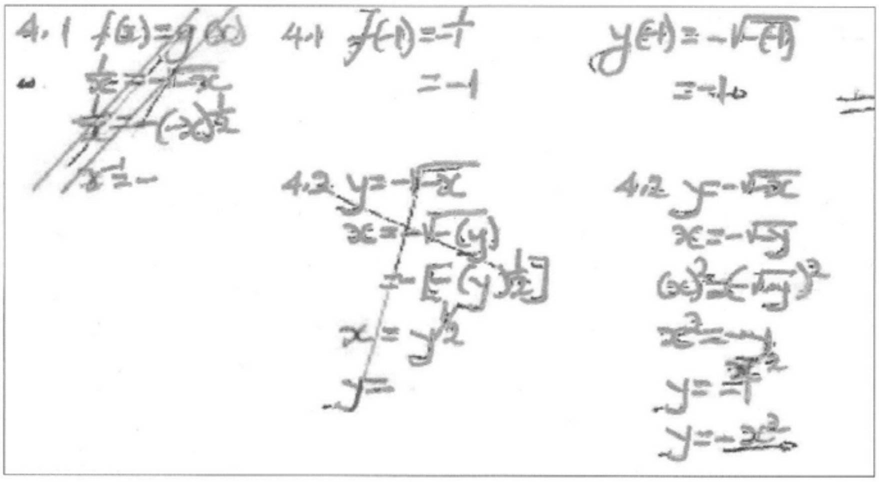
- **In question 6.1.1, learners could generally determine the value of p, but were less successful with determining the value of d.**
- **Learners did not understand the format of the function and did not recognise that this was a hyperbola. This was an unfamiliar question about a hyperbola with two unknown values and it has never been tested before. It is also problematic that learners use one point to prove the equation of the sketched graph whereas the equation has two unknown values.**
- **Learners do not understand that the point (2 ; 0) can lie on many different graphs and not merely this one particular graph. For that reason, the use of the one point to prove that the equation of the graph is as it was given, was awarded no marks.**
- **Learners coped fairly well with Question 6.2 and lost marks only for incorrect substitution or calculations**

1.3.1 Make three suggestions/ recommendations that teachers should take into cognisance when teaching this concept.

1.4 Below is a question taken from the same paper and two learners' responses are provided. Mark each response, using ticks to show if a mark was allocated and indicate how many marks each learner will get out of a score of 7. Then explain why you gave them the mark you did.



Two learners responses



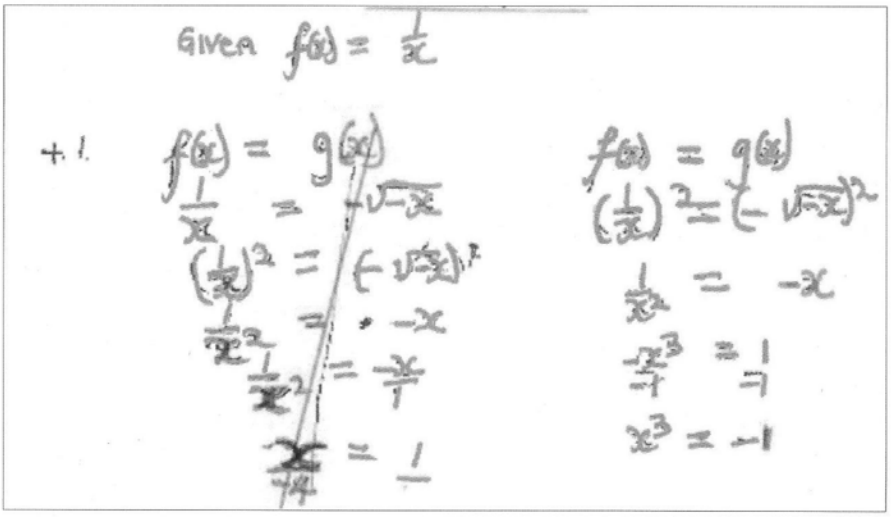


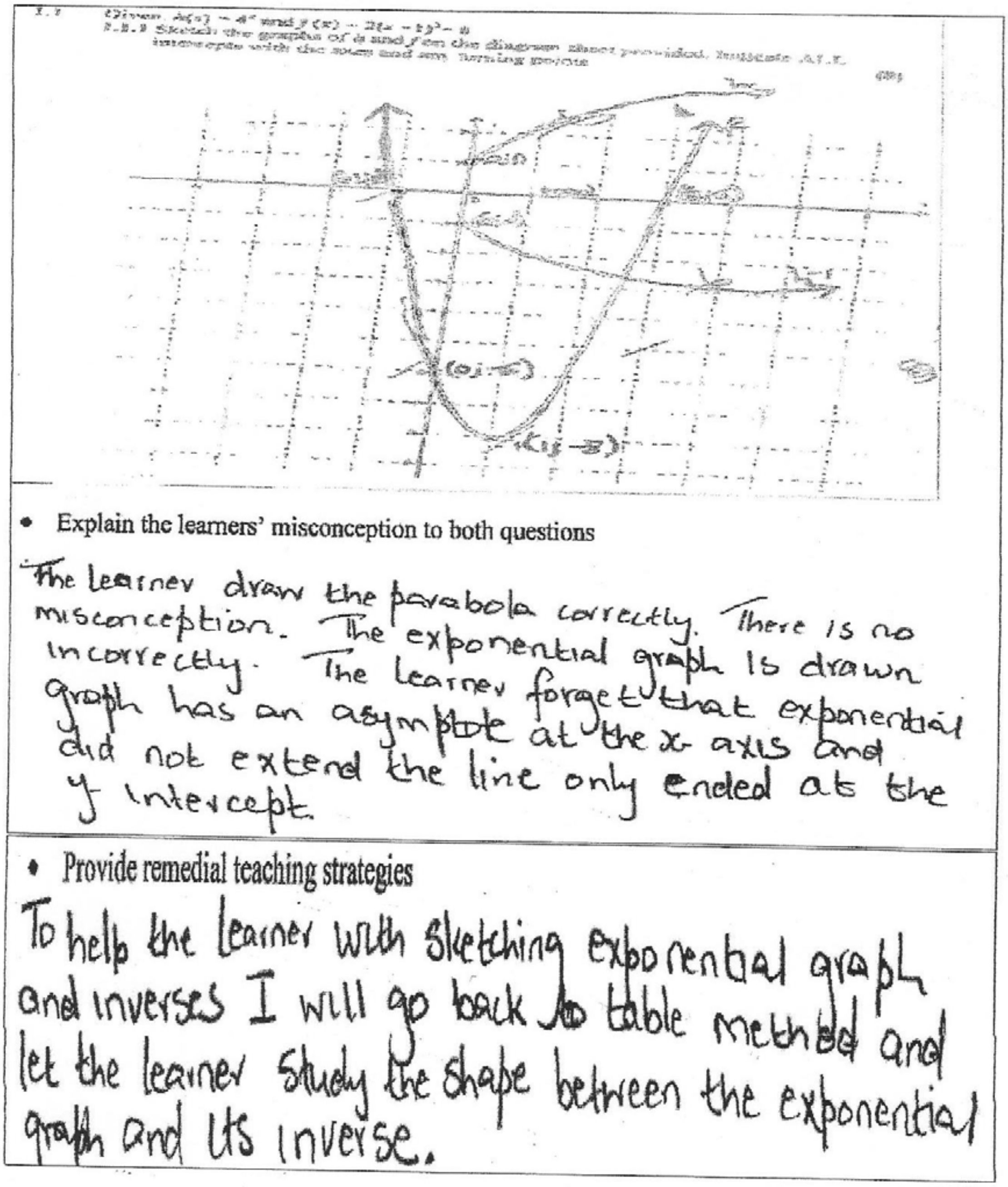
Figure 6: Questions on specialised content knowledge

Ability to identify minor errors, not misconceptions

Q1.2 to Q1.4 and Q2.2 explored PMTs' mastery of SCK. In Q1.2 and Q1.3 (see Table 2), only 15 PMTs provided complete responses to the questions that assessed SCK. Furthermore, the participants' ability to analyse and interpret learner errors was most successful in the identification of slips which are minor mistakes that can be easily detected and corrected by learners themselves without the help of teachers (Siyepu, 2013). In contrast, SCK required the PMTs to go beyond the identification of slips to the identification of concepts to prevent learner confusion. They needed to unpack learners' conceptions and concept images and to suggest appropriate remediation if necessary. From the aforementioned analysis, it is evident that aspects of knowledge seemed to be lacking among the PMTs as they could not identify learner errors, echoing the trend in Sapire *et al*.'s (2016) study.

Inability to identify misconceptions

Below are Shirley's responses to Q1.2, her analysis of learners' solutions to Q1.1.1 and Q1.1.2 (Fig. 7) and Andrew's response to Q1.1.3 (Fig. 8).



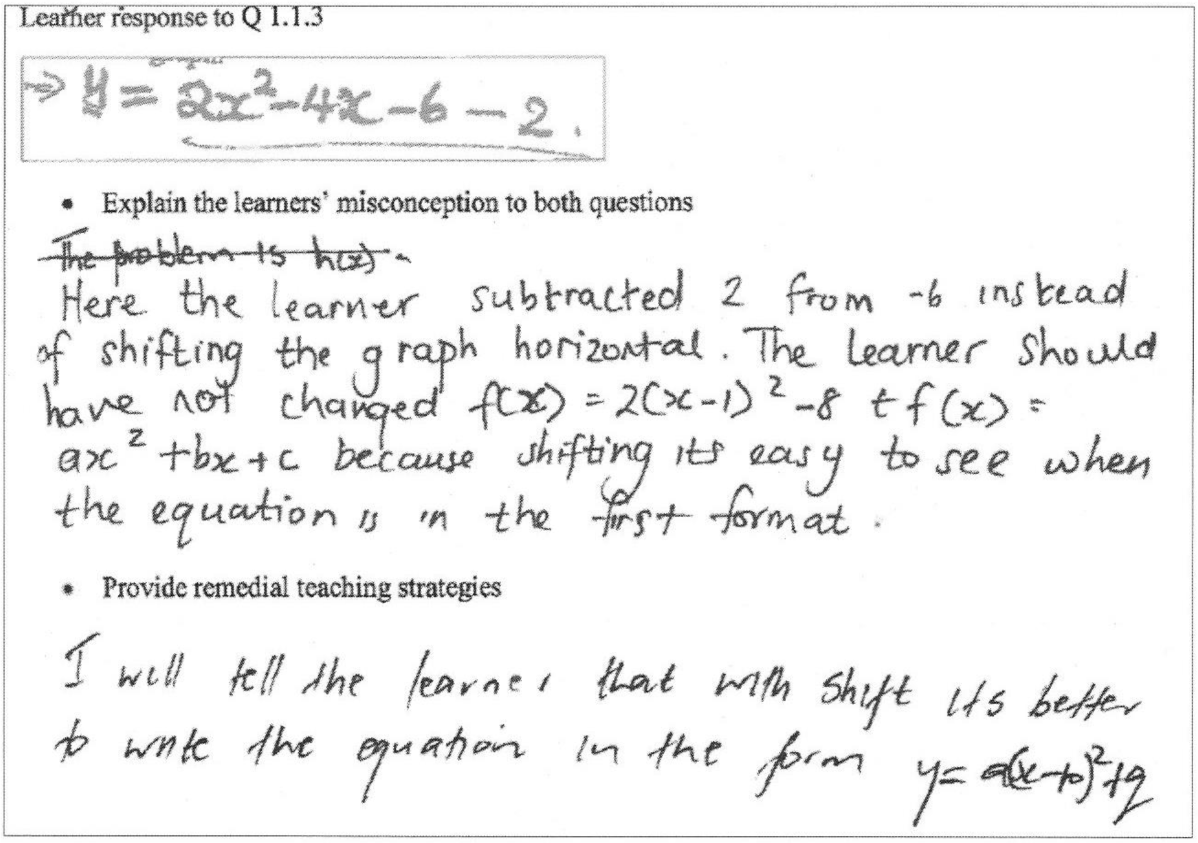
- Explain the learners' misconception to both questions

The learner draw the parabola correctly. There is no misconception. The exponential graph is drawn incorrectly. The learner forget that exponential graph has an asymptote at the x axis and did not extend the line only ended at the y intercept.

- Provide remedial teaching strategies

To help the learner with sketching exponential graph and inverses I will go back to table method and let the learner study the shape between the exponential graph and its inverse.

Figure 7: Shirley's written response


Learner response to Q 1.1.3

→ $y = 2x^2 - 4x - 6 - 2$.

- Explain the learners' misconception to both questions

~~The problem is h(x).~~
Here the learner subtracted 2 from -6 instead of shifting the graph horizontal. The learner Should have not changed $f(x) = 2(x-1)^2 - 8$ t $f(x) = ax^2 + bx + c$ because shifting its easy to see when the equation is in the first format.

- Provide remedial teaching strategies

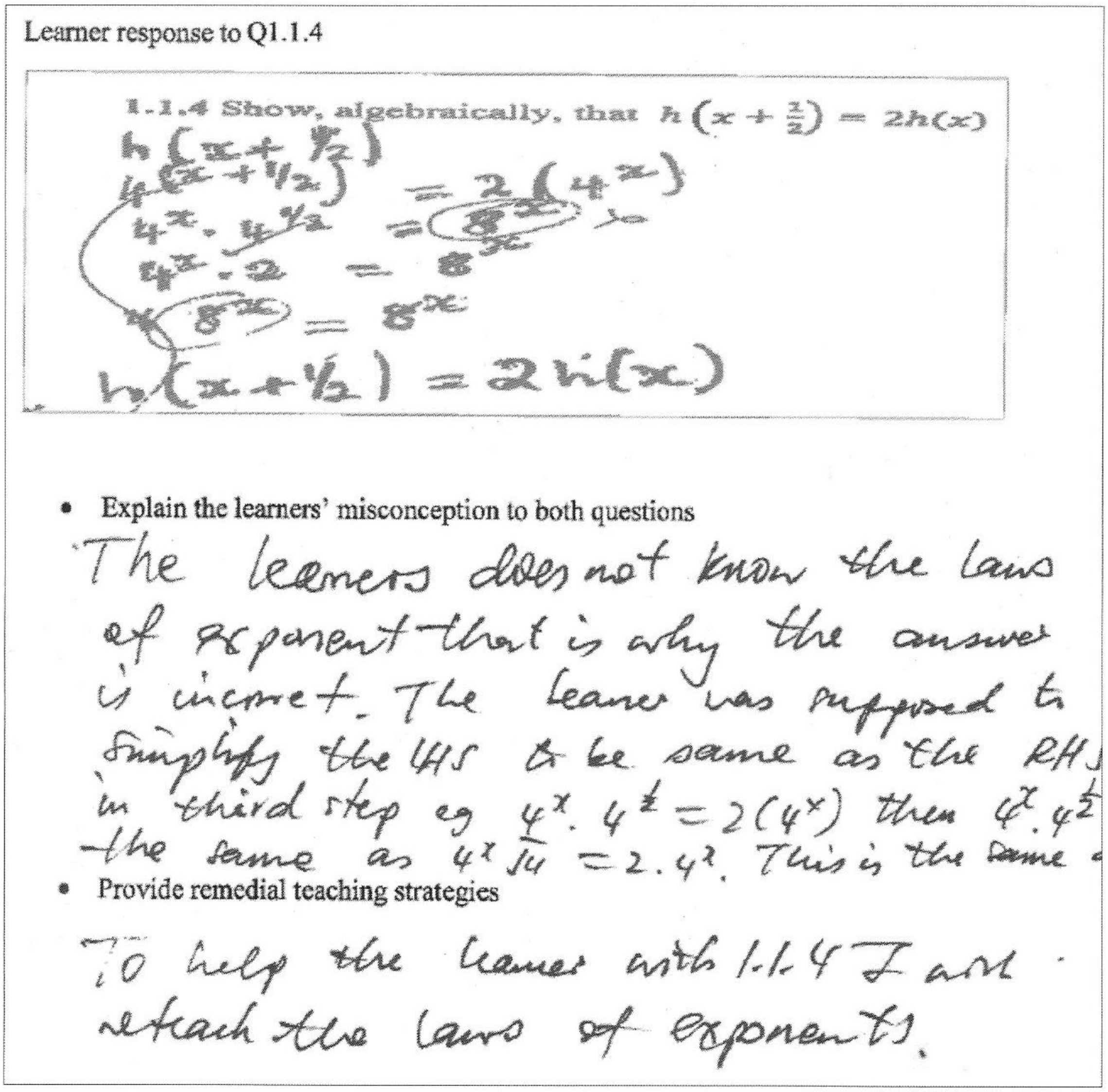
I will tell the learner that with shift its better to write the equation in the form $y = a(x-p)^2 + q$

Figure 8: Andrew's written response

Shirley's response to a graph sketched by a learner showed that she did not recognise that an inverse graph of a parabola was drawn instead of an exponential graph. She identified the misconception (a major systemic misconception leading to errors) as a slip (a minor error) as she said that the 'learner forgets'. Similarly, Andrew' response indicated that his explanation was based on the format of the equation and inappropriate application of rules instead of the learner's misunderstanding. In Shirley's case, it is interesting to note that she also did not provide a correct response to the question assessing CCK, implying that PMTs who struggle to solve problems are likely to find it difficult to recognise and interpret learners' errors. This finding suggests that CCK is important for diagnoses of learners' misconceptions, slips, and misunderstandings or non-recognition of mathematical concepts. These examples show that although the PMTs were at the exit level of the professional qualification, their capacity to analyse and interpret learners' errors and knowledge of specialised content was inadequate and was not being addressed in the BEd. curriculum.

Inability to suggest remedial teaching strategies

In different vein to Shirley and Andrew, Thabo lacked expertise to engage adequately with learners' errors but was able to identify the two obstacles (knowledge gap of the laws of exponents and misunderstanding of carrying out proofs) that prevented the learner from solving the problem in Q1.1.4.


Learner response to Q1.1.4

1.1.4 Show, algebraically, that $h\left(x + \frac{1}{2}\right) = 2h(x)$

$h(x + \frac{1}{2})$
$4^{(x + \frac{1}{2})} = 2(4^x)$
$4^x . 4^{\frac{1}{2}} = (8^x)$
$4^x . 2 = 8^x$
$8^x = 8^x$
$h(x + \frac{1}{2}) = 2h(x)$

- Explain the learners' misconception to both questions

The learners does not know the laws of exponent that is why the answer is incorrect. The learner was supposed to simplify the LHS to be same as the RHS in third step eg $4^x . 4^{\frac{1}{2}} = 2(4^x)$ then $4^x . 4^{\frac{1}{2}}$ the same as $4^x \sqrt{4} = 2.4^x$. This is the same

- Provide remedial teaching strategies

To help the learner with 1.1.4 I will reteach the laws of exponents.

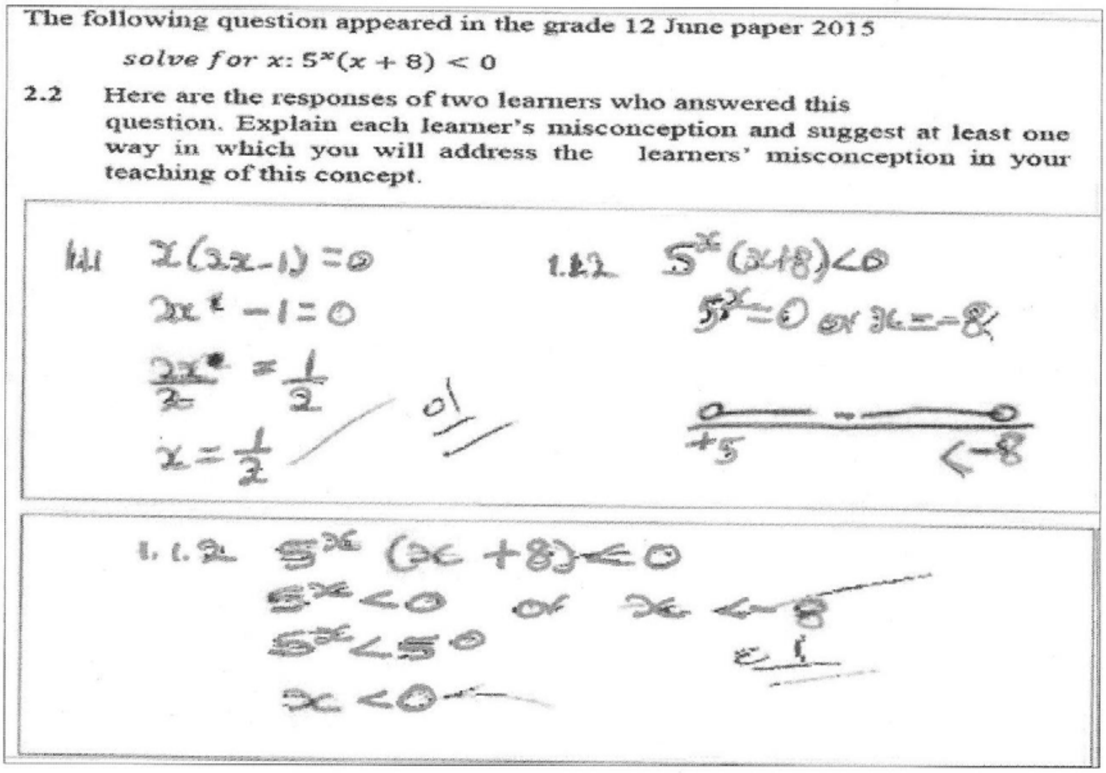
Figure 9: Thabo's response to Q1.1.4

However, identification of knowledge gaps was insufficient in the absence of providing remedial measures, indicating once again, that mastery of SCK is vital to improve learning and academic performance.

Since the majority of PMTs did not interpret learner errors of concepts in the section on Functions, it was not surprising that only seven could provide effective remedial strategies to teach Hyperbolas. Ability to interpret learner errors should logically hint at effective remedial teaching strategies since one cannot come up with a remedy to a problem without knowing its cause. Furthermore, even though the learner errors were provided in Q3.1 (see Fig. 3) the PMTs could not suggest alternative teaching strategies. Indeed, the results reveal that they lacked critical SCK. It was also apparent that the number of PMTs who provided an incorrect response or did not attempt to answer the questions on SCK was less than the total number for sub-questions on CCK. The sub-questions on SCK required them to give detailed explanations rather than merely finding a correct answer.

The responses to the sub-questions on SCK suggest that, as with the questions on CCK, there is a need to emphasise learners' conceptual understanding rather than the given answer. In addition to solving the questions, the PMTs should perhaps be asked to explain their thought processes or to use different methods to solve a question. This could force them to engage with the concept image and concept definition from a specialist's perspective. As with the questions on Functions where only 15 PMTs provided correct responses to questions requiring diagnoses of learner errors and devising remedial teaching strategies, only 12 provided the correct response to questions on Inequalities. This means that only 12 PMTs out of 59 (about 20%) could both diagnose learners' errors and suggest appropriate remedial teaching strategies. Limitations regarding suggestions for remedial teaching strategies

Question 2.2 aimed to exploring the students' ability to analyse errors to questions that deal with inequality. More than simply analysing the misconception, it explored their ability to provide remedial teaching strategies. This aspect of knowledge is critical for teachers' work in general, rather than the specific errors identified for the teaching of Functions.


The following question appeared in the grade 12 June paper 2015

solve for x: $5^x(x+8) < 0$

2.2 Here are the responses of two learners who answered this question. Explain each learner's misconception and suggest at least one way in which you will address the learners' misconception in your teaching of this concept.

Figure 10: Question 2.2 on specialised content knowledge

Learner 1. Misconceptions	Learner 2. Misconceptions
There misconception of the mathematics sign such as less than or greater than sign to same as equal when it came to solving the equation.	There is a misconception of ~~Exponents~~ laws of exponents and knowing that any num. or variable to the power of zero is not equal to zero but is equal to one
Suggestion for teaching	**Suggestion for teaching**
Teaching learners different types of sign and the difference between each of them and also teach learners that equal is not the same as (x>y) or y>x / y=x	Teach laws of exponent also by including the one where by variable has exponent of zero, also improve exponent laws to learner

Figure 11: Andiswa's written response

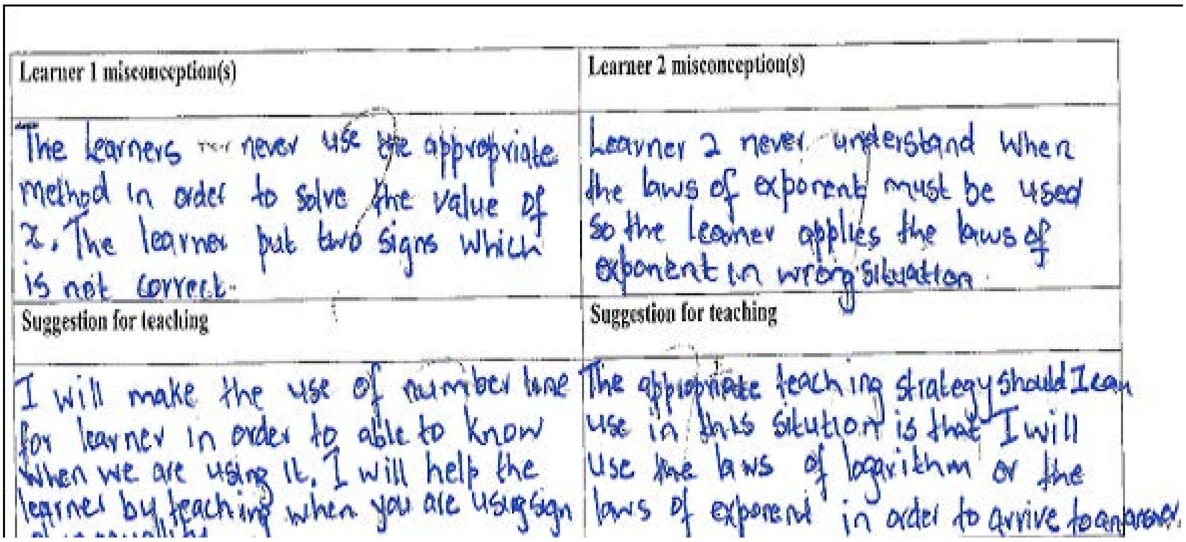
Andiswa's response had limitations but, at the same time, her explanation did not just attend to what learners did incorrectly; it also attempted to link the errors to mathematical concepts. The same limitation in Q2.2 was evident in Oliver's written response (Fig. 12).

Based on Andiswa's response on the remedial teaching strategies she would implement, we infer that some PMTs are unable to identify generic errors.

Andiswa: In the teaching of functions, especially Hyperbola all the format/representations needs to be used in class not just $y = \frac{k}{x}$. *Perhaps instead of asking learners to draw accurate graphs then ask them to sketch rough sketches when k> 0 or k<0. I think it will help learners understand. Also, the use of technology like Sketchpad will be effective.* The response targeted specific errors identified in the learners' responses and provided an alternative teaching resource like Sketchpad that could be more effective than traditional methods of teaching. Like Andiswa who could not offer ways to use Sketchpad, Thabo could not articulate remedial teaching strategies when probed further. Similar to his response to Q1.2, the response was about Functions *in showing different forms of Hyperbolas or any other Functions. This will help address the errors of shifting of graphs.*

R: How would you explain the relationship between the two forms of representation of Hyperbola$f(x) = \frac{k}{x}$ and$f(x) = \frac{x-d}{x-p}$ to learners? Also, how will you use technology to help learners understand Functions especially Hyperbola?

Thabo: Not quite sure right now how to explain it but I know I can transform $f(x) = \frac{x-d}{x-p}$ into $f(x) = \frac{k}{x}$ *by division. Using technology would be better suitable in schools with computers where you can get learners to sketch graphs in the computer and show the transformations in few seconds. Schools without technology have to come up with another solution.*



Learner 1 misconception(s)	Learner 2 misconception(s)
The learners never use the appropriate method in order to solve the value of x. The learner put two signs which is not correct.	Learner 2 never understand when the laws of exponent must be used so the leaner applies the laws of exponent in wrong situation.
Suggestion for teaching	Suggestion for teaching
I will make the use of number line for learner in order to able to know when we are using it. I will help the learner by teaching when you are using sign	The appropriate teaching strategy should I can use in this situation is that I will use the laws of logarithm or the laws of exponent in order to arrive to an answer.

Figure 12: Oliver's written responses

During the interview, Oliver offered explanations for the mistakes made by the learner with some attempt to diagnose the errors:

Oliver: Here, as I have said in my response, the misconception is about calculating the value of x. The learner need to learn the rules of exponent in order to be able to solve this one. Learner one incorrectly used algebraic symbols (= instead of inequality) and made errors in plotting the number line. The solution demonstrate logical 0=5x because it makes no logic to say zero equals something. How a Function can equals 0?

R: What do you think made learner write this solution even if there is no logic as you say?

Oliver: I think, you know when you just learn something but you did not really understand it; these things happen. Just applying the rules without attaching meaning to the rule. Also, the problem here, the learner memorised the rules and now he over used them even where it's not needed because he can recall them although he does not understand.

R: What would you suggest the teachers should do to eradicate this misconception?

Oliver: Not sure yet but I am thinking when I become a teacher I should try and integrate concepts. Perhaps when introducing a concept, use examples I used, another concept that I have already taught. Like in this case of inequalities, use the examples I used in functions and in exponents. I think so.

Although Oliver's written response is not clear, it does indicate that he could identify the misunderstood concepts. Oliver's mastery of CCK enabled him to diagnose and explain the errors in the learner's response even though his grasp of SMK is insufficient when one subjects it to Ball's interpretation of mathematical knowledge for teaching.

Insufficient knowledge of designing assessment tasks

Question 1.4 required respondents to provide a memorandum, which explored their knowledge of assessing learning. It was evident that the majority had insufficient knowledge of identifying concepts and skills. Only 19 PMTs could analyse knowledge and skills in questions related to Functions. Although it appeared that the number of correct responses to Q1.4 was greater than the number for Q1.2; Q1.3 and Q2.2, the majority still had trouble with questions that required SCK. These findings reveal incomplete learning to teach mathematics as specialists and also signal weaknesses in the BEd. mathematics curriculum. Teacher educators' failure to address PMTs' weaknesses before they are employed as teachers could result in failure to diagnose misconceptions and incapacity to think of alternative teaching strategies in actual teaching situations. The evidence is clear that despite being at the exit level, the PMTs were not adequately prepared to engage with learners' mathematical errors. Thus, their SMK of school mathematics was not at the level expected of specialist mathematics teachers. Whilst this inquiry did not interrogate the pre-service mathematics curriculum it does raise concerns about higher education institutions' responsibility to ensure that the next generation of mathematics specialists are adequately equipped with competencies to teach mathematics effectively.

From the analysis of the assessment aspects, it could be argued that most of the PMTs were unable to diagnose misconceptions and that the suggested strategies were not pedagogically informed. The results did show, however, that some demonstrated competency to teach and diagnose errors in the sections on Functions and Inequalities taught in schools.

Subject matter knowledge: the relationship between common content knowledge and specialised content knowledge

Theoretically, CCK and SCK are integral components of SMK. However, in this case, the results do not necessarily show a specific relationship between pre-service teachers' competencies of CCK and SCK. However, they do suggest that CCK is the first step in the acquisition of SCK. It is also evident that knowledge of procedures is necessary for the development of SCK. Based on the analysis of the results for both questions, it is apparent that the PMTs who provided correct responses to questions of high cognitive demand as well as being able or partially able to diagnose learners' errors were those who provided correct responses to almost all the questions that assessed procedural knowledge. Those who did not even attempt the questions or provided incorrect responses to questions that required analysis and interpretation of learners' errors provided unsatisfactory and incorrect responses to questions about the application of procedures. In some cases, it was troubling to note that final year PMTs' expertise is at an inadequate level to teach mathematics as they could solve problems correctly and analyse errors but they could not diagnose misconceptions for remediation purposes.

Mastery of SCK should enable PMTs to identify whether learners make minor errors, misunderstand concepts or are unable to link concepts for problem solving. While we concede that procedural knowledge is not sufficient to execute the work of teaching, the findings show that CCK is the basis for the development of SCK and that deepening PMTs' SCK has the potential to extend their mastery of SMK.

Drawing on the results, some PMTs were found to be competent with respect to CCK. We therefore, conclude that the challenge is not only about developing content knowledge; attention should also be directed to intensifying their abilities and making explicit the importance of combining CCK and SCK to solve and diagnose mathematics learners' misconceptions and errors for effective teaching. We argue that, this could facilitate the deepening of SMK of school mathematics of specialist mathematics teachers-to-be.

Conclusion

This article explored pre-service teachers' competency to solve problems of some of the concepts they will have to teach at school, their capacity to understand and explain learners' misconceptions of these concepts, and their ability to suggest ways to remedy the identified misconceptions. The purpose of the inquiry was to explore the level of preparedness of FET final year PMTs in terms of CCK and SCK as they are both linked to the development of SMK. The evidence revealed that the PMTs were competent in carrying out procedures to solve problems related to Functions and Inequalities. While some of the written responses showed knowledge gaps, the interviews with four participants revealed that, the PMTs could recall the rules and procedures to be followed. The main challenge lay in their ability to analyse and diagnose learners' errors. The PMTs seemed to lack the competency to correct learners' errors and misconceptions. In most cases, the diagnoses were at an operational level (explaining the rules that learners did not follow) instead of identifying misunderstood concepts. Therefore, the lack of SCK exposed SMK gaps, which means that this cohort of final year PMTs was not yet equipped in the BEd. programme to successfully execute the work of teaching Functions and Inequalities in schools.

These findings have crucial implications for the teaching and learning of mathematics. When PMTs become in-service professionals, they are likely to be grappling with the pedagogical approaches of teaching mathematics while simultaneously grappling with SMK. Such challenges could have long lasting negative effects on learner outcomes and attainment. In view of these findings, we suggest that the BEd. mathematics curriculum should factor in three foci: first, establishing a solid grounding in CCK, then, advancing SCK through an emphasis on the usefulness of diagnosing learners' errors, followed by the importance of remedial action. Indeed, the BEd. programme should prepare pre-service teachers to make connections between the concepts learnt at university and their implementation at school as mathematics teachers.

Providing PMTs with opportunities to develop the acquisition of SMK depends on the pedagogy and course design at pre-service level. Therefore, if the aim is to ensure that PMTs have adequate knowledge of the subject matter, more time should be allocated to learner error analysis approaches. Continually exposing PMTs to activities where they engage with learners' errors would probably improve their knowledge of school mathematics and their SMK. The use of a learner error analysis approach is also useful for those who teach future teachers as it reveals specific misconceptions, slips and inabilities that should be corrected before final year students are sent out as specialist mathematics teachers.

While limited conclusions can be drawn from a case of one institution, this study highlighted both the final year PMTs' lack of readiness and gaps in the higher education institution's preparation of school mathematics specialists. It further highlighted the need to restructure the curriculum in order to produce better-prepared mathematics teachers. As McAuliffe (2013) pointed out, there is a crucial need to rethink mathematics education curricula.

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