Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 5 (2023) 917

Journal of the Nigerian Society of Physical Sciences

Dirac Equation for Energy-Dependent Potential With Energy-dependent Tensor Interaction

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Abstract

The relativistic symmetries of the Dirac equation were investigated with an energy-dependent tensor potential interaction for two different energy-dependent potentials under parametric Nikiforov-Uvarov method and supersymmetric quantum mechanics and shape-invariance method. It is observed that the energy-dependent tensor interaction has stronger removal effect of the energy degeneracies in both the spin and pseudospin symmetries than the non-energy-dependent tensor interaction.

DOI:10.46481/jnsps.2022.917

Keywords: Bound state, Dirac equation, Eigen solutions, Wave equation, potential function.

Article History :
Received: 05 July 2022
Received in revised form: 31 August 2022
Accepted for publication: 01 September 2022
Published: 14 January 2023

Communicated by: Oluwatimilehin Oluwadare

1. Introduction

The relativistic spin-1/2 particles in quantum mechanics is usually described by Dirac equation. The subject has drawn attention of physicist in the theoretical domain over the years. The theoretical physicists under the Dirac equation have analysed the characteristic features of deformed nuclei, effective shell models [1-6], etc for the pseudospin symmetry (PS) for different physical potential models. The identical bands and mesons were also studied in details for the spin symmetry (SS). The theoretical reports of these studies showed that energy doublets were produced under the SS and PS for different levels. Very recently, tensor potential interaction was introduced to the Dirac equation to reduce the energy degeneracy. The reduction of the degeneracy depends on the applied tensor potential. The Coulomb tensor potential for instance, reduced some degeneracies leaving some doublets unbroken. The application of Yukawa tensor potential also breaks some degeneracy doublets and produced another form of degeneracies. Onate et al. in their recent study applied Hellmann tensor potential and they found out that the whole degeneracies were broken even when the tensor strength is taken as small as 0.2. Owing to the application of Dirac equation, different authors have studied the Dirac equation in diverse areas using different traditional techniques [7-13]. However, it is very clear that the Dirac equation under SS and PS for energy-dependent potential (EDP) has not received attention to the best of our knowledge. Hence, the call for this study. Motivated by the application of relativistic wave equations particularly the Dirac equation, this study wants

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to examine the effect of EDP potential on the eigenvalues of the SS and PS. This study will consider two sets of potentials using two different traditional methodologies. The two potential models are modified Mie-type-constant EDP and Kratzer EDP respectively in the presence of an energy-dependent tensor (EDT) interaction via parametric Nikiforov-Uvarov method and supersymmetry quantum mechanics and shape invariance method. The major aim of this study is to determine the production of energy degenerate doublets by EDP and its removal by EDT interaction. The modified Mie-type-constant EDP and the Kratzer EDP respectively are given as

$$V(r,E) = \frac{\lambda_1(1+a_1E)}{r^2} + \frac{\lambda_2(1+a_2E)}{r^{-2}} + \lambda_3(1+a_3E) \quad (1)$$

$$V(r,E) = \frac{\lambda_4(1+a_4E)}{r^2} - \frac{\lambda_5(1+a_5E)}{r}. \quad (2)$$

Here, $\lambda_i (i = 1, 2, \dots)$ are potential strengths and $a_i (i = 1, 2, \dots)$ are potential parameters. Since this paper aim at determining the production and removal of energy degenerate state in the presence and absence of energy-dependent tensor interaction, we propose a Coulomb-constant energy-dependent tensor interaction of the form

$$U(r,E) = \frac{H_1(1+b_1E)}{r} + H_2(1+b_2E)r \quad (3)$$

2. Dirac Equation (SS and PS)

The Dirac equation with spin-1/2 particles under the potentials $S(r)$ and $V(r)$ as attractive scalar potential and repulsive vector potential is of the form [14, 15, 16]

$$[C\alpha\cdot\rho + \beta\left(MC^2 + S(r) + V(r) - E\right]\psi_{n\kappa}(r) = 0, \quad (4)$$

with E and M as energy and particle mass, $\rho = -i\hbar\nabla$ defines momentum operator with α and β as 4×4 Dirac matrices, i.e.

$$\alpha = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix},\ \beta = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad (5)$$

and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (6)$$

Here, I represents the 2×2 matrix identity and, σ_i are the Pauli 3-vector spin matrices. In the nuclei spherical symmetry, the angular momentum operator J and spin-orbit matrix operator $\kappa = -\beta(\sigma L + 1)$ commute with the Dirac Hamiltonian, where L is the total orbital angular momentum operator. The spinor wave functions can be classify following the radial quantum number n and the spin-orbit quantum number κ and can be expressed according to the Pauli-Dirac representation [14, 15, 16].

$$\psi_{n\kappa}(r) = \begin{pmatrix} f_{n\kappa}(r) \\ g_{n\kappa}(r) \end{pmatrix} = \frac{1}{r}\begin{pmatrix} F_{n\kappa}(\vec{r})Y^{\ell}_{jm\kappa}(\theta,\varphi) \\ iG_{n\kappa}(\vec{r})Y^{\bar{\ell}}_{jm(-\kappa)}(\theta,\varphi) \end{pmatrix} \quad (7)$$

where the upper and lower spinor components $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ are the real square-integral radial wave functions. $Y^{\ell}_{jmk}(\theta,\phi)$ and $Y^{\bar{\ell}}_{jm-k}(\theta,\phi)$ are the spin spherical harmonic functions coupled to the total angular momentum j and its projection m on the z axis for $\kappa(\kappa+1) = \ell(\ell+1)$ and $\kappa(\kappa-1) = i(\ell+1)$. The quantum number κ is related to the quantum number ℓ for spin and Pseudospin symmetries as

$$\kappa = \begin{cases} -(\ell+1) = -(j+\frac{1}{2}), (s_{1/2}, p_{3/2}, etc), j = \ell + \frac{1}{2}, \\ \qquad\qquad\qquad\qquad \text{aligned spin } (\kappa < 0) \\ +\ell(j+\frac{1}{2}), (p_{1/2}, d_{3/2}, etc), j = \ell - \frac{1}{2}, \\ \qquad\qquad\qquad\qquad \text{unaligned spin } (\kappa > 0) \end{cases} \quad (8)$$

The quasi-degenerate doublet structure can be expressed in terms of pseudospin angular momentum $\tilde{s} = 1/2$ and pseudo-orbital angular momentum $\tilde{\ell}$ which is defined as

$$\kappa = \begin{cases} -\tilde{\ell} = (-j+\frac{1}{2}), (s_{1/2}, p_{3/2}, etc), j = \tilde{\ell} - \frac{1}{2}, \\ \qquad\qquad\qquad\qquad \text{aligned spin } (\kappa < 0) \\ +(\tilde{\ell}+1) = (j+\frac{1}{2}), (d_{3/2}, f_{5/2}, etc), j = \tilde{\ell} + \frac{1}{2}, \\ \qquad\qquad\qquad\qquad \text{unaligned spin } (\kappa > 0) \end{cases} \quad (9)$$

where $\kappa = \pm1, \pm2, \dots$. Upon direct substitution of equation (7) into equation (4), we can obtain two radial coupled Dirac equation for the two symmetry components as follows:

$$\left(\frac{d}{dr} + \frac{\kappa}{r}\right)F_{n\kappa}(r) = \left[MC^2 + E_{n\kappa} - \Delta(r)\right]G_{n\kappa}(r) \quad (10)$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r}\right)G_{n\kappa}(r) = \left[MC^2 - E_{n\kappa} + \sum(r)\right]F_{n\kappa}(r). \quad (11)$$

For the spin symmetry, $\Delta(r) = C_s$ = constant. Then, we obtain a second-order differential equation for upper-spinor component as

$$\left[-\frac{d^2}{dr^2} + \frac{\kappa(\kappa+1)}{r^2} + \frac{1}{\hbar^2C^2}(MC^2 + E_{n\kappa} - C_s)\sum(r)\right]F_{n\kappa}(r) = \frac{1}{\hbar^2C^2}\left[(E^2_{nk} - M^2C^4 + C_s)\left(MC^2 - E_{n\kappa}\right)\right]F_{n\kappa}(r), \quad (12)$$

and the lower-spinor component is given by

$$G_{n\kappa}(r) = \frac{1}{MC^2 + E_{n\kappa} - C_s}\left(\frac{d}{dr} + \frac{\kappa}{r}\right)F_{n\kappa}(r) \quad (13)$$

It is only the real positive energy states that exist when $C_s = 0$. However, under the pseudospin symmetry, $\Sigma(r) = C_p$ = constant, one can have from equation (10) a second-order differential equation for the lower-spinor component as [14, 15]

$$\left[-\frac{d^2}{dr^2} + \frac{\kappa(\kappa-1)}{r^2} - \frac{1}{\hbar^2C^2}(MC^2 - E_{n\kappa} + C_{ps})\Delta(r)\right]G_{n\kappa}(r) = \frac{1}{\hbar^2C^2}\left[E^2_{n\kappa} - M^2C^4 - C_{ps})(MC^2 - E_{n\kappa})\right]G_{n\kappa}(r), \quad (14)$$

and the upper-spinor component $F_{n\kappa}(r)$ as

$$F_{n\kappa}(r) = \frac{1}{MC^2 - E_{n\kappa} + C_{ps}}\left(\frac{d}{dr} + \frac{\kappa}{r}\right)G_{n\kappa}(r) \quad (15)$$

It is only real negative energy states that exist when $C_p = 0$. If we now include tensor interaction, then we obtain an equation in each case for both spin and pseudospin symmetries as

follows

$$\left[\frac{d^2}{dr^2}-\frac{\kappa(\kappa+1)}{r^2}+\frac{2\kappa}{r}U(r)-\frac{dU(r)}{dr}-U^2(r)+\frac{\frac{d\Delta(r)}{dr}}{M+E_{n\kappa}-\Delta(r)}\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)\right]F_{n\kappa}(r)=\left[(M+E_{n\kappa}-\Delta(r))\left(M-E_{n\kappa}+\sum(r)\right)\right]F_{n\kappa}(r) \quad (16)$$

$$\left[\frac{d^2}{dr^2}-\frac{\kappa(\kappa-1)}{r^2}+\frac{2\kappa}{r}U(r)+\frac{dU(r)}{dr}-U^2(r)+\frac{\frac{d\sum(r)}{dr}}{M-E_{n\kappa}+\sum(r)}\left(\frac{d}{dr}-\frac{\kappa}{r}+U(r)\right)\right]G_{n\kappa}(r)=\left[(M+E_{n\kappa}-\Delta(r))\left(M-E_{n\kappa}+\sum(r)\right)\right]G_{n\kappa}(r). \quad (17)$$

2.1. Parametric Nikiforov-Uvarov Method (PNUM)

The PNUM is one of the analytical techniques of mathematical physics that solves second-order differential equations in quantum mechanics. This method has a general form of the Schrödinger-like equation [17, 18, 19, 20]

$$\left[\frac{d^2}{dr^2}+\frac{c_1-c_2s}{s(1-c_3s)}\frac{d}{ds}+\frac{-\xi_1s^2+\xi_2s-\xi_3}{s^2(1-c_3s)^2}\right]\psi(s)=0 \quad (18)$$

According to the PNUM, the eigenvalues and eigenfunctions can be obtain following the condition [17, 18, 19, 20, 21]

$$c_2n-(2n+1)c_5+\left[n^2-n+2c_8\right]c_3+\sqrt{c_8}\left(\sqrt{c_9}+2nc_3+c_3\right)+\sqrt{c_9}\left(2n+1+\sqrt{c_8}\right)=-c_7 \quad (19)$$

$$\psi(s)=s^{c_{12}}(1-s)^{-c_{12}-\frac{c_{13}}{c_3}}P_n^{(c_{10}-1,\frac{c_{11}}{c_3}c_{10}-1)}(1-2c_3s) \quad (20)$$

The values of the parametric constants in equations (19) and (20) are obtained as follows:

$$\begin{aligned}
&c_4=0.5(1-c_1), c_5=0.5(c_2-2c_3), c_6=c_5^2+\xi_1,\\
&c_7=2c_4c_5-\xi_2, c_8=c_4^2+\xi_3, c_9=c_3(c_7+c_3c_8)+c_6,\\
&c_{10}=1+2\left(c_4+\sqrt{c_8}\right), c_{11}=c_2-2c_5+2c_3\sqrt{c_8}+2\sqrt{c_9},\\
&c_{12}=c_4+\sqrt{c_8}, c_{13}=c_5-c_3\sqrt{c_8}-\sqrt{c_9}.
\end{aligned} \quad (21)$$

3. Solutions of Dirac Equation

3.1. SS Limit

The SS limit occurs when $d\Delta(r)/dr=0$ and $\Delta(r)=C_z$ with $\Sigma(r)=V(r,E)$. Plugging equation (1) and equation (3) into equation (16), we have

$$\frac{d^2F_{n\kappa}(r)}{dr^2}+\left[\frac{\chi_s^1}{r^2}+\chi_s^2r^2+\chi_s^3\right]F_{n\kappa}(r)=0 \quad (22)$$

where

$$\chi_s^1=H_1(1+b_1E_{n\kappa})-\lambda_1(1+a_1E_{n\kappa})\beta_s+2\kappa H_1(1+b_1E_{n\kappa})-H_1^2(1+b_1E_{n\kappa})^2-\kappa(\kappa+1), \quad (23)$$

$$\chi_s^2=-\lambda_2(1+a_2E_{n\kappa})\beta_s-H_2^2(1+b_2E_{n\kappa})^2, \quad (24)$$

$$\chi_s^3=H_2(1+b_2E_{n\kappa})\left[2\kappa-1-2H_1(1+b_1E_{n\kappa})\right]-\beta_s\left[\lambda_3(1+a_3E_{n\kappa})+M-E_{n\kappa}\right], \quad (25)$$

$$\beta_s=M+E_{n\kappa}-C_s. \quad (26)$$

Using a transformation of the form $y=r^2$ in equation (22), we obtain

$$\frac{d^2F_{n\kappa}(y)}{dy^2}+\frac{1}{2y}\frac{dF_{n\kappa}(y)}{dy}+\frac{\chi_s^2y^2+\chi_s^3y+\chi_s^1}{y^2}F_{n\kappa}(y)=0. \quad (27)$$

Comparing equation (27) with (18), we obtain the values of the parametric constants in equation (21) as follows

$$\begin{aligned}
&c_1=0.5, c_2=c_3=0, c_4=0.25, c_5=0, c_6=-0.25\chi_s^2,\\
&c_7=-0.24\chi_s^3, c_8=0.25(0.25-\chi_s^1), c_9=-0.25\chi_s^2,\\
&c_{10}=1+\sqrt{0.25-\chi_s^1}, c_{11}=\sqrt{-\chi_s^2},\\
&c_{12}=0.5\left(0.5+\sqrt{(0.25-\chi_s^1)}\right), c_{13}=-\sqrt{-0.25\chi_s^2}.
\end{aligned} \quad (28)$$

Plugging equation (28) into equation (19) and equation (20), respectively, gives

$$\frac{\lambda_1(1+a_1E_{n\kappa})\beta_s}{4}+\frac{H_1(1+b_1E_{n\kappa})}{2}\left[\frac{H_1(1+b_1E_{n\kappa})-1}{2}-\kappa\right]+\left(\frac{1+2\kappa}{4}\right)^2+\left[\frac{\frac{\beta_s[\lambda_3(1+a_3E_{n\kappa})+M-E_{n\kappa}]}{4}+\frac{H_2(1+b_2E_{n\kappa})}{2}\left[H_1(1+b_1E_{n\kappa})-\kappa-\frac{1}{2}\right]+\aleph_s}{1+2n+\sqrt{H_2^2(1+b_2E_{n\kappa})^2+\lambda_2(1+a_2E_{n\kappa})\beta_s}}\right]^2=0. \quad (29)$$

$$F_{n\kappa}(y)=y^{0.5(0.5+\sqrt{0.25-\chi_s^1}}e^{-\sqrt{-0.25\chi_s^2}y}L_n^{\sqrt{0.25-\chi_s^1}}\left(\sqrt{-\chi_s^2}y\right) \quad (30)$$

$$\aleph_s=\left(n+\frac{1}{2}\right)\sqrt{H_2^2(1+b_2E_{n\kappa})^2+\lambda_2(1+a_2E_{n\kappa})\beta_s}. \quad (31)$$

3.2. PS limit

The pseudospin symmetry limit occurs when $d\Sigma(r)/dr=0$ and $\Sigma(r)=C_p$. In this symmetry limit, the potential is taken as $\Delta(r)=V(r,E)$. Now, substituting equations (1) and (3) into equation (17) and by using the same transformation as before,

we have the parametric constants as

$$c_1 = 0.5, c_2 = c_3 = 0, c_4 = 0.25, c_5 = 0,$$
$$c_6 = 0.25\left[H_2^2(1+b_2E_{n\kappa})^2 - \lambda_2(1+a_2E_{n\kappa})\beta_p\right],$$
$$c_7 = 0.5H_2(1+b_2E_{n\kappa})\left[H_1(1+b_1E_{n\kappa}) - 0.5 - \kappa\right]$$
$$+ 0.25\left[M + E_{n\kappa} - \lambda_3(1+a_3E_{n\kappa})\right]\beta_p,$$
$$c_8 = 0.5H_1(1+b_1E_{n\kappa})\left[0.5H_1(1+b_1E_{n\kappa}) + 0.5 - \kappa\right] +$$
$$(0.25 - 0.5\kappa)^2 - \lambda_1(1+a_1E_{n\kappa})\beta_p,$$
$$c_9 = 0.25\left[H_2^2(1+b_2E_{n\kappa})^2 - \lambda_2(1+a_2E_{n\kappa})\beta_p\right],$$
$$c_{10} = 1+$$
$$\sqrt{1 + H_1(1+b_1E_{n\kappa})\left[H_1(b^E) + 1 - 2\kappa\right] + \kappa(\kappa-1) - \lambda_1(a^E)\beta_p},$$
$$c_{11} = \sqrt{H_2^2(1+b_2E_{n\kappa})^2 - \lambda_2(1+a_2E_{n\kappa})\beta_p},$$
$$c_{12} = 0.25+$$
$$\sqrt{0.5H_1(b^E)\left[0.5H_1(b^E) + 0.5 - \kappa\right] + (0.25 - 0.5\kappa)^2 - \lambda_1(a^E)\beta_p},$$
$$c_{13} = -0.5\sqrt{H_2^2(1+b_2E_{n\kappa})^2 - \lambda_2(1+a_2E_{n\kappa})\beta_p}, \quad (32)$$

where $b^E = 1 + b_1E_{n\kappa}$, $a^E = 1 + a_1E_{n\kappa}$

Substituting equation (32) into equations (19) and (20), the energy for PS limit and its corresponding wave function are given as

$$\frac{H_1(1+b_1E_{n\kappa})}{2}\left[\frac{H_1(1+b_1E_{n\kappa})+1}{2} - \kappa\right] + \left(\frac{1-2\kappa}{4}\right)^2 -$$
$$\frac{\lambda_1(1+a_1E_{n\kappa})\beta_s}{4} +$$
$$\left[\frac{\frac{\beta_p[M-\lambda_3(1+a_3E_{n\kappa})+E_{n\kappa}]}{4} + \frac{H_2(1+b_2E_{n\kappa})}{2}\left[H_1(1+b_1E_{n\kappa}) - \kappa - \frac{1}{2}\right] + \aleph_p}{1 + 2n + \sqrt{H_2^2(1+b_2E_{n\kappa})^2 - \lambda_2(1+a_2E_{n\kappa})\beta_p}}\right]^2$$
$$= 0. \quad (33)$$

$$F_{n\kappa}(y) = y^{0.25+\eta_{p1}}e^{0.5\eta_{p2}y}L_n^{\eta_{p3}}(\eta_4 y). \quad (34)$$

$$\aleph_p = \left(n + \frac{1}{2}\right)\sqrt{H_2^2(1+b_2E_{n\kappa})^2 - \lambda_2(1+a_2E_{n\kappa})\beta_p} \quad (35)$$

$$\eta_{p1} = \sqrt{0.5H_1(b^E)\left[0.5H_1(b^E) + 0.5 - \kappa\right] + \Gamma_1}, \quad (36)$$

where $\Gamma_1 = (0.25 - 0.5\kappa)^2 - \lambda_1(1+a_1E_{n\kappa})\beta_p$.

$$\eta_{p2} = \sqrt{H_2^2(1+b_2E_{n\kappa})^2 - \lambda_2(1+a_2E_{n\kappa})\beta_p}, \quad (37)$$

$$\eta_{p3} = \sqrt{1 + H_1(b^E)\left[H_1(b^E) + 1 - 2\kappa\right] + \kappa(\kappa-1) - \Gamma_2}, \quad (38)$$

where $\Gamma_2 = \lambda_1(1+a_1E_{n\kappa})\beta_p$.

$$\eta_{p4} = \sqrt{H_2^2(1+b_2E_{n\kappa})^2 - \lambda_2(1+a_2E_{n\kappa})\beta_p} \quad (39)$$

3.3. Solutions of the SS and PS via Supersymmetric Approach

In this section, we obtain the solutions of the spin and pseudospin symmetry limits for Kratzer energy-dependent potential via SUSYQM. This method involves the proposition of superpotential function which is the solution of the non-linear Riccati equation.

3.3.1. Solution of the SS limit

To obtain the solution of the spin symmetry limit of the Dirac equation with Kratzer energy-dependent potential, we substitute equations (2) and (3) into equation (16) to have a second-order differential equation of the form

$$\frac{d^2F_{n\kappa}(r)}{dr^2} =$$
$$\left[\frac{\chi_1^s}{r^2} - \frac{\lambda_5(1+a_5E_{n\kappa})\beta_s}{r} + H_2^2(1+b_2E_{n\kappa})^2r^2 + \chi_2^s\right]F_{n\kappa}(r) \quad (40)$$

where we have defined the following for mathematical simplicity

$$\chi_1^s = \kappa(\kappa+1) + \lambda_4(1+a_4E_{n\kappa})\beta_s +$$
$$H_1(1+b_1E_{n\kappa})\left[H_1(1+b_1E_{n\kappa}) - 2\kappa - 1\right] \quad (41)$$

$$\chi_2^s = H_2(1+b_2E_{n\kappa})\left[2H_1(1+b_1E_{n\kappa}) - 2\kappa + 1\right] +$$
$$(M - E_{n\kappa})\beta_s \quad (42)$$

For a non-energy-dependent potential in the absence of tensor interaction, the energy eigenvalues in equation (40) purely depends on the quantum number n and the spin-orbit coupling term κ. This is physically related to energy as $E_{n\kappa} = (n, \kappa(\kappa+1))$. This shows that for $\kappa \neq 0$, the states are degenerate. To solve equation (40) using SUSY approach [22, 23, 24, 25], we can write

$$F_{0\kappa}(r) = exp\left(-\int W(r)dr\right), \quad (43)$$

where $W(r)$ is a superpotential which determines the solution of equation (40). To proceed to the next level, it is necessary to propose a superpotential function [22, 23, 24, 25, 26]. In this work, our superpotential function is proposed as

$$W(r) = \delta_0 - \delta_1 r^{-1} \quad (44)$$

where δ_0 and δ_1 are two different constants. For equation (44) to determine the solution of equation (40), the following condition must be satisfied

$$W^2(r) - \frac{dW(r)}{dr} = \frac{\chi_1^s}{r^2} - \frac{\lambda_5(1+a_5E_{n\kappa})\beta_s}{r} +$$
$$H_2^2(1+b_2E_{n\kappa})^2r^2 + \chi_2^s \quad (45)$$

Substituting equation (44) into equation (45), we easily determine the values of the two constants in equation (44) as

$$\delta_0^2 = \chi_2^s \quad (46)$$

$$\delta_1 = \frac{1}{2} \pm \frac{\sqrt{1 + 4\left[\chi_1^s + H_2^2(1+b_2E_{n\kappa})^2\right]}}{2} \quad (47)$$

$$\delta_0 = \frac{\lambda_5(1 + a_5E_{n\kappa})\beta_s\delta_1^{-1}}{2} \tag{48}$$

To test the correctness of the superpotential function, we construct the partner potentials of the supersymmetric quantum mechanics and examine the shape invariance condition [27, 28, 29]. Our partner potentials are obtain in the following forms

$$V_+(r) = W^2(r) + \frac{dW(r)}{dr} = \delta_0^2 - \frac{2\delta_0\delta_1}{r} + \frac{\delta_1(\delta_1 - 1)}{r^2} \tag{49}$$

$$V_-(r) = W^2(r) - \frac{dW(r)}{dr} = \delta_0^2 - \frac{2\delta_0\delta_1}{r} + \frac{\delta_1(\delta_1 + 1)}{r^2}. \tag{50}$$

Equations (49) and (50) satisfied the shape-invariance condition and so, the following relationship exist

$$V_+(r, a_0) = V_-(r, a_1) + R(a_1) \tag{51}$$

In equation (51), $a_0 = \delta_1$ and the Hamiltonian is shape-invariant. However, $a_1 = f(a_0) = a_0 + 1$, which simply means that the potentials $V_\pm(r)$ are the same apart from a constant and the residual term $R(a_1)$ is independent of the variable r. In this case, a_1 is uniquely determined from an old set of parameter a_0. From the mapping, we now established that $\delta_1 \to \delta_1 + 1$ using the negative partner potential. In terms of the newly introduced parameters, we express the residual term as

$$R(a_1) = \frac{\lambda_5^2(1 + a_5E_{n\kappa})^2\beta_s^2}{4a_0^2} - \frac{\lambda_5^2(1 + a_5E_{n\kappa})^2\beta_s^2}{4a_1^2} \tag{52}$$

$$R(a_2) = \frac{\lambda_5^2(1 + a_5E_{n\kappa})^2\beta_s^2}{4a_1^2} - \frac{\lambda_5^2(1 + a_5E_{n\kappa})^2\beta_s^2}{4a_2^2} \tag{53}$$

$$R(a_3) = \frac{\lambda_5^2(1 + a_5E_{n\kappa})^2\beta_s^2}{4a_2^2} - \frac{\lambda_5^2(1 + a_5E_{n\kappa})^2\beta_s^2}{4a_3^2} \tag{54}$$

The energy of the system is obtained using the above summation given by $\Sigma_{k=1}^{n} R(a_k)$ which is generalized as

$$R(a_n) = \frac{\lambda_5^2(1 + a_5E_{n\kappa})^2\beta_s^2}{4a_{n-1}^2} - \frac{\lambda_5^2(1 + a_5E_{n\kappa})^2\beta_s^2}{4a_n^2} \tag{55}$$

In view of the negative partner potential, the complete energy equation for the spin symmetry is given as

$$H_2(1 + b_2E_{n\kappa})\left[1 - 2\kappa + H_2(1 + b_2E_{n\kappa})\right] + (M - E_{n\kappa})\beta_s = \frac{\lambda_5^2\left[(1 + a_5E_{n\kappa})\beta_s\right]^2}{4(\delta_1 + n)^2} \tag{56}$$

3.3.2. Solution of the PS limit

To obtain the solution of the pseudospin symmetry limit, we substitute equations (2) and (3) into equation (17) to have

$$\frac{d^2G_{n\kappa}(r)}{dr^2} = \frac{\chi_1^p + \lambda_5(1 + a_5E_{n\kappa})\beta_p r + \chi_2^p r^2 + H_2^2(1 + b_2E_{n\kappa})^2 r^4}{r^2} G_{n\kappa}(r) \tag{57}$$

Table 1: Energies in the SS limit for modified Mie-type-constant EDP with $\lambda_1 = \lambda_2 = \lambda_3 = 0.5, a_1 = 3, a_2 = 2, a_3 = 1, b_1 = b_2 = 0.1, M = 5\ fm^{-1}$ and $C_s = 1\ fm^{-1}$.

ℓ	κ	n	(ℓ, j)	$H_{1,2} = 0$	$H_{1,2} = 0.1$	$H_{1,2} = 0.2, 0.1$	$H_{1,2} = 0.1, 0.2$
0	-1	0	$0s_{1/2}$	1.942623900	1.958361278	1.959096533	1.972020529
0	-1	1	$1s_{1/2}$	2.941736416	2.957567872	2.958305747	2.971859998
0	-1	2	$2s_{1/2}$	4.060873315	4.076229954	4.076927679	4.090458326
0	-1	3	$3s_{1/2}$	5.254527544	5.269270999	5.269915044	5.283193841
1	-2	0	$0p_{3/2}$	1.841748059	1.861208075	1.850731135	1.889789645
1	-2	1	$1p_{3/2}$	2.850509762	2.869662294	2.859097611	2.898566721
1	-2	2	$2p_{3/2}$	3.981140297	3.999437112	3.989244626	4.027491284
1	-2	3	$3p_{3/2}$	5.185188130	5.202574765	5.192859711	5.229500168
2	-3	0	$0d_{5/2}$	1.644091503	1.670587775	1.650604087	1.715582858
2	-3	1	$1d_{5/2}$	2.667778395	2.692515093	2.671366252	2.737514242
2	-3	2	$2d_{5/2}$	3.820578636	3.843285779	3.822525569	3.886288212
2	-3	3	$3d_{5/2}$	5.045509610	5.066514491	5.046609652	5.107241634
3	-4	0	$0f_{7/2}$	1.362484903	1.400794769	1.374384444	1.463630855
3	-4	1	$1f_{7/2}$	2.394333292	2.428432855	2.397977358	2.491841820
3	-4	2	$2f_{7/2}$	3.577245062	3.606844862	3.576008806	3.666695125
3	-4	3	$3f_{7/2}$	4.833507250	4.859747029	4.829865691	4.915639885
1	1	0	$0p_{1/2}$	1.841748059	1.852413864	1.877371357	1.836808899
1	1	1	$1p_{1/2}$	2.850509762	2.861449107	2.885908697	2.847162481
1	1	2	$2p_{1/2}$	3.981140297	3.991828450	4.015024293	3.978925935
1	1	3	$3p_{1/2}$	5.185188130	5.195485760	5.217335080	5.183790476
2	2	0	$0d_{3/2}$	1.644091503	1.649346874	1.685727304	1.616913066
2	2	1	$1d_{3/2}$	2.667778395	2.674654649	2.711126809	2.644310749
2	2	2	$2d_{3/2}$	3.820578636	3.827991306	3.862736003	3.800294001
2	2	3	$3d_{3/2}$	5.045509610	5.052997276	5.085717151	5.027654662
3	3	0	$0f_{5/2}$	1.362484903	1.357450308	1.402754356	1.305640324
3	3	1	$1f_{5/2}$	2.394333292	2.393986861	2.442017313	2.344787927
3	3	2	$2f_{5/2}$	3.577245062	3.579492175	3.625897479	3.534961395
3	3	3	$3f_{5/2}$	4.833507250	4.837006867	4.880792058	4.796634088

Table 2: Energies in the PS limit for modified Mie-type-constant EDP with $\lambda_1 = \lambda_3 = 0.5$, $\lambda_2 = -0.5$, $a_1 = -3$, $a_2 = -2$, $a_3 = -1$, $b_1 = b_2 = 0.1$, $M = 5\ fm^{-1}$ and $C_s = 1\ fm^{-1}$.

ℓ	κ	n	(ℓ, j)	$H_{1,2} = 0$	$H_{1,2} = 0.1$	$H_{1,2} = 0.2, 0.1$	$H_{1,2} = 0.1, 0.2$
1	-1	1	$1s_{1/2}$	-1.758132998	-1.769902692	-1.779164454	-1.773124361
2	-2	1	$1p_{3/2}$	-1.891711144	-1.914557485	-1.929357426	-1.923214633
3	-3	1	$1d_{5/2}$	-2.100102133	-2.132843969	-2.153277701	-2.145586292
4	-4	1	$1f_{7/2}$	-2.390428405	-2.431076159	-2.456965418	-2.446048454
1	-1	2	$2s_{1/2}$	-2.424883237	-2.435163923	-2.443110934	-2.438169927
2	-2	2	$2p_{3/2}$	-2.549727011	-2.569554464	-2.582073017	-2.577422600
3	-3	2	$2d_{5/2}$	-2.740445644	-2.768710697	-2.785650672	-2.780425981
4	-4	2	$2f_{7/2}$	-3.000000000	-3.035085686	-3.056131285	-3.049319087
1	2	1	$0d_{3/2}$	-1.758132998	-1.735243670	-1.728251600	-1.719908479
2	3	1	$0f_{5/2}$	-1.891711144	-1.858269052	-1.845486264	-1.837996526
3	4	1	$0g_{7/2}$	-2.100102133	-2.341193985	-2.038745712	-2.316883988
4	5	1	$0h_{9/2}$	-2.390428405	-2.341193985	-2.316253094	-2.316883988
1	2	2	$1d_{3/2}$	-2.424883237	-2.404762492	-2.398620806	-2.391343780
2	3	2	$1f_{5/2}$	-2.549727011	-2.956867588	-2.509496082	-2.502502503
3	4	2	$1g_{7/2}$	-2.740445644	-2.956867588	-2.936575208	-2.682113098
4	5	2	$1h_{9/2}$	-3.000000000	-2.956867588	-2.936575208	-2.934010648

where

$$\chi_1^p = \kappa(\kappa - 1) + H_1(1 + b_1E_{n\kappa})\left[H_1(1 + b_1E_{n\kappa}) - 2\kappa + 1\right] - \lambda_4(1 + a_4E_{n\kappa})\beta_p \tag{58}$$

$$\chi_2^p = H_2(1 + b_2E_{n\kappa})\left[2H_1(1 + b_1E_{n\kappa}) - 2\kappa - 1\right] + (M + E_{n\kappa})(M - E_{n\kappa} + C_p) \tag{59}$$

To avoid repetition of works and algebra, we follow the same steps as in the spin symmetry and obtain the energy equation for the pseudospin spin symmetry as

$$H_2(1 + b_2E_{n\kappa})\left[2H_1(1 + b_1E_{n\kappa}) - 2\kappa - 1\right] + (M + E_{n\kappa})\beta_p = \left[\frac{-\lambda_5(1 + a_5E_{n\kappa})\beta_p}{2(\delta_p + n)}\right]^2 \tag{60}$$

$$\delta_p = \frac{1}{2} + \frac{1}{2}\sqrt{4\chi_1^p + 1 + 4H_2^2(1 + b_2 E_{n\kappa})^2} \tag{61}$$

Table 3: Energies in the SS limit for modified Mie-type-constant EDP with $\lambda_1 = \lambda_2 = \lambda_3 = 0.5, a_1 = 3, a_2 = 2, a_3 = 1, b_1 = b_2 = 0.1, M = 5\ fm^{-1}$ and $C_s = 1\ fm^{-1}$

ℓ	κ	n	(ℓ, j)	$H_{1,2} = 0.2, 0$	$H_{1,2} = 0, 0.2$
0	-1	0	$0s_{1/2}$	1.942079678	1.968171442
0	-1	1	$1s_{1/2}$	2.941066860	2.967805266
0	-1	2	$2s_{1/2}$	4.060044875	4.086322787
0	-1	3	$3s_{1/2}$	5.253529416	5.279016863
1	-2	0	$0p_{3/2}$	1.818436560	1.897264650
1	-2	1	$1p_{3/2}$	2.826966136	2.905851918
1	-2	2	$2p_{3/2}$	3.958335296	4.034259115
1	-2	3	$3p_{3/2}$	5.163329221	5.235688819
2	-3	0	$0d_{5/2}$	1.601252028	1.732796555
2	-3	1	$1d_{5/2}$	2.622584840	2.755428278
2	-3	2	$2d_{5/2}$	3.776272981	3.903614939
2	-3	3	$3d_{5/2}$	5.003004662	5.123603755
3	-4	0	$0f_{7/2}$	1.306444952	1.487803464
3	-4	1	$1f_{7/2}$	2.329887999	2.519191159
3	-4	2	$2f_{7/2}$	3.512241944	3.694086997
3	-4	3	$3f_{7/2}$	4.770643993	4.941943853
1	1	0	$0p_{1/2}$	1.889417309	1.808576037
1	2	1	$1p_{1/2}$	2.897131599	2.819215988
1	1	2	$2p_{1/2}$	4.025207202	3.952158620
1	3	3	$3p_{1/2}$	5.226530876	5.158310400
2	4	0	$0d_{3/2}$	1.713977782	1.577278752
2	2	1	$1d_{3/2}$	2.737952422	2.604208929
2	2	2	$2d_{3/2}$	3.887416000	3.761839331
2	2	3	$3d_{3/2}$	5.108364728	4.991200184
3	3	0	$0f_{5/2}$	1.449429077	1.257493042
3	3	1	$1f_{5/2}$	2.486841044	2.293004999
3	3	2	$2f_{5/2}$	3.666840421	3.484650816
3	3	3	$3f_{5/2}$	4.918103438	4.748960521

Table 4: Energies in the PS limit for modified Mie-type-constant EDP with $\lambda_1 = \lambda_3 = 0.05$, $\lambda_2 = -0.05$, $a_1 = 6$, $a_2 = -4$, $a_3 = 5$, $b_1 = b_2 = 0.1$, $M = 15\ fm^{-1}$ and $C_s = 5\ fm^{-1}$.

ℓ	κ	n	(ℓ, j)	$H_{1,2} = 0.2, 0$	$H_{1,2} = 0, 0.2$
1	-1	1	$1s_{1/2}$	-1.775253455	-1.763815218
2	-2	1	$1p_{3/2}$	-1.920130164	-1.908460034
3	-3	1	$1d_{5/2}$	-2.140140406	-2.125315642
4	-4	1	$1f_{7/2}$	-2.441826999	-2.420424044
1	-1	2	$2s_{1/2}$	-2.439648344	-2.430168824
2	-2	2	$2p_{3/2}$	-2.573814603	-2.564901362
3	-3	2	$2d_{5/2}$	-2.773637600	-3.028396662
4	-4	2	$2f_{7/2}$	-3.041719317	-3.028396662
1	2	1	$0d_{3/2}$	-1.742804990	-1.726917318
2	3	1	$0f_{5/2}$	-1.865069919	-1.850942038
3	4	1	$0g_{7/2}$	-2.061722917	-2.053329000
4	5	1	$0h_{9/2}$	-2.340384372	-2.342315770
1	2	2	$1d_{3/2}$	-2.411514069	-2.397477706
2	3	2	$1f_{5/2}$	-2.526953758	-2.513529136
3	4	2	$1g_{7/2}$	-2.708412767	-2.698017234
4	5	2	$1h_{9/2}$	-2.959189164	-2.954567108

Table 5: Energies in the SS limit for Kratzer EDP with $\lambda_4 = \lambda_5 = 0.5$, $a_1 = 4.0, a_2 = 2$, $b_1 = b_2 = 0.1, M = 10\ fm^{-1}$ and $C_s = 5\ fm^{-1}$

ℓ	κ	n	(ℓ, j)	Kratzer potential		Coulomb potential	
				$H_{1,2} = 0$	$H_{1,2} = 0.1$	$H_{1,2} = 0.1$	$H_{1,2} = 0.1, 0$
0	-1	0	$0s_{1/2}$	0.370204340	0.393187068	1.252040915	1.230390156
0	-1	1	$1s_{1/2}$	0.945739430	0.963640344	2.192321102	2.167520674
0	-1	2	$2s_{1/2}$	1.435407674	1.453860405	2.839097594	2.811476651
0	-1	3	$3s_{1/2}$	1.899432950	1.919827326	3.304673071	3.274484128
1	-2	0	$0p_{3/2}$	0.568272837	0.605461699	2.196388371	2.167520674
1	-2	1	$1p_{3/2}$	1.067819409	1.102125131	2.847354128	2.811476651
1	-2	2	$2p_{3/2}$	1.530395283	1.565774885	3.316370162	3.274484128
1	-2	3	$3p_{3/2}$	1.979996399	2.018417622	3.663421017	3.616377889
2	-3	0	$0d_{5/2}$	1.262754541	0.865450254	2.858804254	2.811476651
2	-3	1	$1d_{5/2}$	0.820311004	1.309314656	3.330517856	3.274484128
2	-3	2	$2d_{5/2}$	1.694588465	1.744582496	3.679846038	3.616377889
2	-3	3	$3d_{5/2}$	2.125227188	2.180498632	3.943595506	3.873809868
3	-4	0	$0f_{7/2}$	1.085509689	1.138674624	3.345044630	3.274484128
3	-4	1	$1f_{7/2}$	1.493098313	1.550825160	3.696546742	3.616377889
3	-4	2	$2f_{7/2}$	1.901400337	1.965254589	3.962133536	3.873809868
3	-4	3	$3f_{7/2}$	2.315910246	2.387891090	4.166042378	4.070830840
1	1	0	$0p_{1/2}$	0.568272837	0.542744504	1.976844362	1.975929806
1	3	1	$1p_{1/2}$	1.067819409	1.048345302	2.667541872	2.668584697
1	1	2	$2p_{1/2}$	1.530395283	1.513218985	3.163567616	3.166216009
1	5	3	$3p_{1/2}$	1.979996399	1.963746819	3.529456299	3.533423105
2	6	0	$0d_{3/2}$	0.820311004	0.786755216	2.654220464	2.668584697
2	2	1	$1d_{3/2}$	1.262754541	1.231133014	3.147935985	3.166216009
2	2	2	$2d_{3/2}$	1.694588465	1.663175926	3.511836761	3.533423105
2	2	3	$3d_{3/2}$	2.125227188	2.092809055	3.785235098	3.809600325
3	3	0	$0f_{5/2}$	1.085509689	1.044886371	3.132908744	3.166216009
3	3	1	$1f_{5/2}$	1.493098313	1.451243785	3.494674460	3.533423105
3	3	2	$2f_{5/2}$	1.901400337	1.857234266	3.766256136	3.809600325
3	3	3	$3f_{5/2}$	2.315910246	2.268233378	3.973416820	4.020621390

4. Discussion

In Table 1, we presented the energy eigenvalues of the spin symmetry for equal and unequal values of H_1 and H_2. For $H_1 = H_2 = 0$, which is the solution without energy-dependent tensor interaction, the following degeneracies were produced: $0p_{3/2} = 0p_{1/2}$, $1p_{3/2}=1p_{1/2}$, $3p_{3/2}=3p_{1/2}$, $0d_{5/2}$, $0d_{3/2}$, $1d_{5/2}=1d_{3/2}$, $2d_{5/2} = 2d_{3/2}$, $3d_{5/2}=3d_{3/2}$, $0f_{7/2}=0f_{5/2}$, $1f_{7/2}=1f_{5/2}$. $2f_{7/2}=2f_{5/2}$ and $3f_{7/2}=3f_{5/2}$/. These degeneracies are the usual degeneracies obtained with non-energy-dependent potentials. However, for $H_1 = H_2 = 0.1$, which is a solution with energy-dependent tensor interaction, there are no degeneracies. This shows that the energy-dependent tensor potential has broken the energy degenerate doublets in the system. For a non-energy-dependent tensor potential, even at $H = 0.5$ and $H = 1$, there are still degenerate doublets. For $H_1 = 0.2$, $H_2 = 0.1$ and $H_1 = 0.1$, $H_2 = 0.2$, there are no degeneracy production. In Table 2, we presented energy eigenvalues of the pseudospin symmetry for equal and unequal values of the tensor strengths i.e. H_1 and H_2. For $H_1 = H_2 = 0$, the following degenerate doublets are obtain: $1s_{1/2} = 0d_{3/2}$, $1p_{3/2} = 0f_{5/2}$, $1d_{5/2} = 0g_{7/2}$, $1f_{7/2} = 0h_{9/2}$, $2s_{1/2} = 1d_{3/2}$, $2p_{3/2} = 1f_{5/2}$, $2d_{5/2} = 1g_{7/2}$ and $2f_{7/2} = 1h_{9/2}$. These degeneracies are also equal to the degeneracies produced for non EDP for non-tensor interaction. For $H_1 = H_2 = 0.1$, there are no degeneracies. Similarly, for $H_1 > H_2$ and $H_1 < H_2$, there are no degenerate doublets. This also shows that the inclusion of the EDT term breaks the whole degeneracies even at small values of the tensor strengths. In

Table 6: Energies in the PS limit for Kratzer EDP with $\lambda_4 = \lambda_5 = 0.5$, $\lambda_2 = 0$, $a_1 = 4.0, a_2 = 2$, $b_1 = b_2 = 0.1$, $M = 10\ fm^{-1}$ and $C_s = 5\ fm^{-1}$

ℓ	κ	n	(ℓ, j)	Kratzer potential		Coulomb potential	
				$H_{1,2} = 0$	$H_{1,2} = 0.1$	$H_{1,2} = 0.1$	$H_{1,2} = 0.1, 0$
1	-1	1	$1s_{1/2}$	-9.096854737	-9.097243710	-3.957853178	-3.956166280
2	-2	1	$1p_{3/2}$	-9.103820973	-9.104900620	-4.705516764	-4.702953988
3	-3	1	$1d_{5/2}$	-9.114041578	-9.115790810	-5.338447790	-5.334597330
4	-4	1	$1f_{7/2}$	-9.127260481	-9.129650770	-5.877946235	-5.872766190
1	-1	2	$2s_{1/2}$	-9.165271209	-9.165625395	-4.704351342	-4.702953988
2	-2	2	$2p_{3/2}$	-9.171400175	-9.172390830	-5.337077070	-5.334597330
3	-3	2	$2d_{5/2}$	-9.180401886	-9.182010810	-5.876538030	-5.872766190
4	-4	2	$2f_{7/2}$	-9.192060938	-9.194263445	-6.338969120	-6.333932324
1	2	1	$0d_{3/2}$	-9.096854737	-9.095134090	-3.852996864	-3.855104284
2	3	1	$0f_{5/2}$	-9.103820973	-9.101415860	-4.624203482	-4.628695316
3	4	1	$0g_{7/2}$	-9.114041578	-9.110975965	-5.272927745	-5.279249573
4	5	1	$0h_{9/2}$	-9.127260481	-9.123565500	-5.823169340	-5.831036969
1	2	2	$1d_{3/2}$	-9.165271209	-9.163683540	-4.625917968	-4.628695316
2	3	2	$1f_{5/2}$	-9.171400175	-9.169181270	-5.274472190	-5.279249573
3	4	2	$1g_{7/2}$	-9.180401886	-9.177572380	-5.824661165	-5.831036969
4	5	2	$1h_{9/2}$	-9.192060938	-9.188647735	-6.294424900	-6.302161924

Note: $H_{i,j} = 0$ means $H_i = H_j = 0$. $H_{i,j} = 0.1, 0.2$ means $H_i = 0.1$, $H_j = 0.2$.

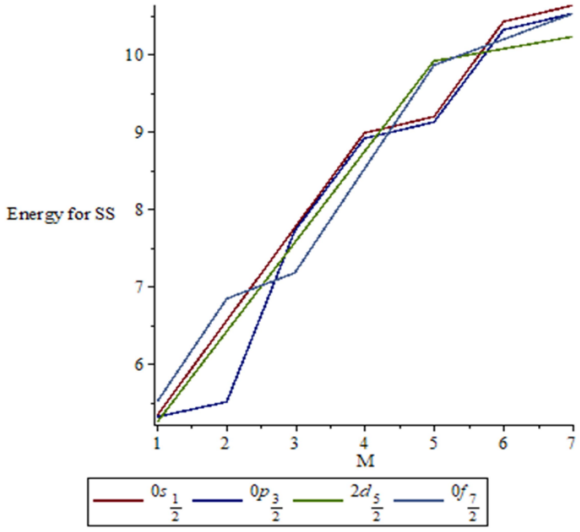

Figure 1: Energies in the SS limit against mass M for modified Mie-type-constant EDP with $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $a_1 = 3$, $a_2 = 2$, $a_3 = 1$, $b_1 = b_2 = 1$, and $C_s = 5\ fm^{-1}$.

Tables 3 and 4, we presented the energy for SS and PS respectively for Coulomb energy-dependent tensor potential ($H_2 = 0$) and constant energy-dependent tensor potential ($H_1 = 0$). In both cases, there are no degeneracies. To check the accuracy and correctness of the energy-dependent tensor potential, we also studied the solutions of the spin and pseudospin symmetries with the same energy-dependent tensor potential with the Kratzer energy-dependent potential. The special cases of this potential was studied numerically. The results of the two symmetries are given in Tables 5 and 6. In Table 5, the energy for spin symmetry is given for both Kratzer and Coulomb energy-dependent potentials. For Kratzer energy-dependent potential, the degeneracies obtained in Table 1 were equal obtained. For Coulomb energy-dependent potential ($\lambda_4 = 0$), it was considered for Coulomb-constant energy-dependent tensor potential and Coulomb energy-dependent tensor potential ($H_2 = 0$). For $H_1 = H_2 = 0.1$, the same energy degeneracies obtained

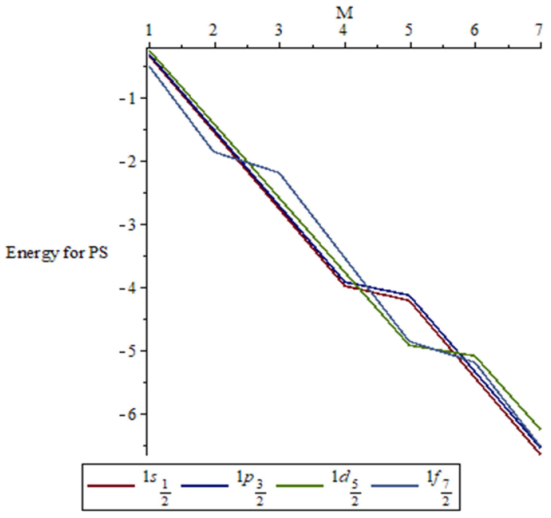

Figure 2: Energies in the PS limit against mass M for modified Mie-type-constant EDP with $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $a_1 = 3$, $a_2 = 2$, $a_3 = 1$, $b_1 = b_2 = 1$, and $C_s = 5\ fm^{-1}$.

in Table 1 were also obtained, but for $H_2 = 0$, which reduces the Coulomb-constant energy-dependent tensor potential to Coulomb energy-dependent tensor potential, a new set of energy degeneracies were formed. The degeneracies formed are $1s_{1/2} = 0p_{3/2}$, $2s_{1/2} = 1p_{3/2} = 0d_{5/2}$, $3s_{1/2} = 2p_{3/2} = 1d_{5/2} = 0f_{7/2}$, $3p_{3/2} = 2d_{5/2} = 1f\ _{7/2}$, $3d_{5/2} = 2f_{7/2}$, $1p_{1/2} = 0d_{3/2}$, $2p_{1/2} = 1d_{3/2} = 0f_{5/2}$, $3p_{1/2} = 2d_{3/2} = 1f_{5/2}$ and $3d_{3/2} = 2f_{5/2}$. These are new degeneracies different from the degeneracies obtained with ordinary Coulomb tensor potential. For the pseudospin symmetry in Table 6, the following degeneracies were formed with Coulomb energy-dependent tensor potential for Coulomb energy-dependent potential: $1d_{5/2} = 2p_{3/2}$, $1f_{7/2} = 2d_{5/2}$, $0f_{5/2} = 1d_{3/2}$, $0g_{7/2} = 1f_{5/2}$ and $0h_{9/2} = 1g_{7/2}$. For ordinary Coulomb tensor potential with $H = 0.5$ and $H = 1$, the degeneracies formed in each case are different from those formed in the present work. Figure 1 and Figure 2 showed the variation of the energy of SS and PS respectively with the mass M for the modified Mie-type potential. The energy of the SS increases with the mass while that of the PS decreases with the mass.

5. Conclusion

In this work, we have employed two traditional techniques to solve Dirac equation with two energy-dependent potential and an energy-dependent tensor interaction without the use of any approximation scheme to the centrifugal term. The degeneracies formed in our results without the application of tensor potential are exactly the degeneracies formed for non-energy-dependent potential. However, the degeneracies formed when the energy-dependent tensor potential was applied differ from the degeneracies formed with non-energy-dependent tensor potential. In the case of Coulomb energy-dependent potential with Coulomb energy-dependent tensor potential under spin symmetry, there are four degenerate doublets, three degenerate doublets and two degenerate doublets which cannot be formed in

the case of non-energy-dependent tensor potentials. For the combination of Coulomb-constant energy-dependent tensor potential, a small value of the tensor strengths can easily break the energy doublets which is not possible for ordinary tensor potential.

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