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Original Research

Portfolio Strategy for an Investor with Logarithm Utility and Stochastic Interest Rate under Constant Elasticity of Variance Model

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Abstract

This paper is aim at maximizing the expected utility of an investor's terminal wealth; to achieve this, we study the optimal portfolio strategy for an investor with logarithm utility function under constant elasticity of variance (CEV) model in the presence of stochastic interest rate. A portfolio comprising of a risk free asset and a risky asset is considered where the risk free interest rate follows the Cox- Ingersoll-Ross (CIR) model and the risky asset is modelled by CEV. Using power transformation, change of Variable and asymptotic expansion technique, an explicit solution of the optimal portfolio strategy and the Value function are obtained. Furthermore, numerical simulations are presented to study the effect of some parameters on the optimal portfolio strategy under stochastic interest rate.

Keywords:

Keywords: Stochastic interest rate, optimal portfolio strategy, asymptotic technique, constant elasticity of variance, logarithm utility

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1. Introduction

In the study of optimal portfolio strategy in a financial market, volatility plays a vital role in influencing the behaviour of the risky assets due to its fluctuating nature as a result of different information available in the financial market. For an investor to make relatively right choice when investing in risky assets, there is need to consider stochastic volatility models and not constant volatility in order to understand the fluctuating nature of the risky assets. One of such stochastic volatility model is the CEV model.

The CEV model was developed by [1] and is an extension geometric Brownian motion (GBM). According to [2], the model is capable of capturing the implied volatility skew. A lot of researchers such as ([3],[4]) studied utility maximization under constant elasticity model in defined contribution (DC) pension scheme. In [5], optimal investment and reinsurance problem of utility maximization under CEV model was studied. The authors in [2] studied optimal investment problem with taxes, dividend and transaction cost using CEV model and logarithm utility function. The optimal portfolio strategy with multiple

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contributors in a DC pension fund using Legendre transformation method was studied by [6]. The authors in [7] solved the optimal portfolio problem with default risk and refund of premium clause in a DC plan; in their work, the stock market price followed the CEV model. In [8], the effect of additional voluntary contribution on the investment strategies under CEV model; they used power transformation method in solving their problem. The study of optimal portfolio with stochastic interest have been investigated by many authors though the risky assets were modelled by GBMs; they include [9], who studied optimal portfolio management with stochastic interest rate for a protected case of DC fund. In ([10]-[11]), stochastic interest rate model was used to obtain optimal portfolio allocation in a DC plan. Also, the authors in ([12]-[13]) considered investment strategy with interest rate of Vasicek type while the authors in ([14]-[16]), studied the optimal portfolio problem when the interest rate is of affine interest.

From the available literatures, most of the authors that worked on optimal portfolio strategies under CEV model considered cases where the risk free interest rate is constant except for [17] who studied the optimal portfolio strategies under CEV model with stochastic interest rate. They pointed out that one of the main reasons why other authors could not combined CEV model and stochastic interest in finding the optimal portfolio strategies was in the difficulty to find a closed form solution of the optimal portfolio strategies analytically. Also they pointed out that in financial market, interest rate is not constant rather a fluctuating processes and that the interest rate volatility presents another source of risks in financial market; In other words, when this risk is not taken into consideration, we are actually undermining the risk generated by this interest rate which is crucial in affecting the prices of various assets available in financial market. In their work, they used the exponential utility to optimize the expected utility of an insurer with exponential utility function and used the Legendry transformation method and asymptotic expansion method to obtain solutions for the optimal portfolio strategies.

In this work, we maximize the expected utility of an insurer's wealth by studying the optimal portfolio strategies of an insurer with logarithm utility function whose risky asset is model by CEV model and the risk free interest is stochastic and follows the CIR model. Furthermore, we use the power transformation, change of variable and asymptotic approach to derive an asymptotic solution of the optimal portfolio strategies and value function. Also some numerical simulations to explain our results are given. The main difference between our work and that of [17] is that we consider an investor with logarithm utility instead of exponential utility and apply power transformation method and change of variable method instead of Legendre transformation method.

2. Preliminaries

For a financial market with portfolio comprising of a risk free asset (treasury security) and a risky asset (marketable security) which is open continuously for a period of $T > 0$ representing the expiring date of the investment. Let (Ω, F, P) be a probability space which is complete, Ω and P are real space probability measure respectively, $\{B_r(t), B_s(t) : t \geq 0\}$ is a set of Brownian motions and F the filtration representing information produced by the Brownian motions.

Let the risk free asset price $C(t)$ at time t be given as

$$\frac{dC(t)}{C(t)} = r(t)\,dt, \quad C(0) > 0 \tag{1}$$

where $r(t)$ is the short interest rate and follows the (CIR) model whose dynamics is

$$\begin{cases} dr(t) = (b - cr(t))\,dt - a_1\sqrt{r(t)}dB_r(t) \\ r(0) = r_0 > 0 \end{cases} \tag{2}$$

and b , c and a_1 are positive numbers such that the following condition holds $a_1^2 < 2b$ (Feller's Condition). [17]

Let $S(t)$ be the price of the marketable security whose price process follows the CEV model which is an extension of the GMB. The CEV as earlier stated has the capability to capture the volatility skew of the marketable security unlike the GBM whose volatility is constant. From the work of [4, 6, 8, 17], the dynamic of the price process of the marketable security is given by the stochastic differential equation when $t \geq 0$ as follows

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma S^{\beta}(t)\,dB_s(t) \tag{3}$$

where μ , σ and β are positive constant and represent the instantaneous expected rate of return, instantaneous volatility and elasticity parameter respectively see [2−4]. Also, $B_r(t)$ and $B_s(t)$ are assumed to be correlated instantaneous correlation coefficient $\rho = 1$ such that $dB_r(t)\,dB_s(t) = dt$ see [17].

Note, when $\beta = 0$ in equation (3), the model in (3) reduce to that of GMB see [2].

3. Wealth Formulations and Methodology

Let $Z(t)$ be the investor's wealth at time t. Also, let φ and φ_1 be the proportions of the investor's wealth to be invested in risky asset and risk-free asset respectively such that $\varphi_1 = 1-\varphi$. Since the investor's total wealth is summation of his investments in the two assets hence the differential form of the investor's total wealth is

$$dZ(t) = Z(t)\left(\varphi_1 \frac{dC(t)}{C(t)} + \varphi \frac{dS(t)}{S(t)}\right) \tag{4}$$

substituting equations (1) and (3) into (4), we have

$$\begin{cases} dZ(t) = Z(t)((\varphi(\mu - r) + r)dt + \varphi\sigma S^{\beta}(t)\, dB_s(t)) \\ Z(0) = Z_0 \end{cases} \tag{5}$$

Next, let the investor's utility at any given state Z at time t be given as

$$J_{\varphi}(t, r, s_2, z) = E_{\varphi}\left[U(Z(T)) \quad \begin{matrix} r(t) = r, S(t) = s, \\ Z(t) = z \end{matrix} \right] \tag{6}$$

With boundary condition $J(t, r, s_2, z) = U(z)$

where r is the risk free interest rate and z is the wealth. The objective here is to determine the optimal portfolio strategies and the optimal value function of the investor given as

φ^* and $J(t, r, s_2, z) = \sup_{\varphi} J_{\varphi}(t, r, s_2, z)$ such that

$$J_{\varphi^*}(t, r, s_2, z) = J(t, r, s_2, z) \tag{7}$$

The value function $J_{\varphi^*}(t, r, s_2, z)$ can be considered as a kind of utility function.

From the maximum principle and Ito's lemma, the Hamilton Jacobi Bellman (HJB) equation which is a nonlinear partial differential equation associated with (6) is obtained by maximizing the expected utility $J_{\varphi}(t, r, s_2, z)$ subject to the investor's wealth in as follows

$$\left\{ \begin{matrix} J_t + \mu s J_s + \frac{1}{2}\sigma^2 s^{2\beta+2} J_{ss} \\ + rzJ_z + (b - cr) J_r + \frac{1}{2} r a_1^2 \\ +\sigma a_1\sqrt{r} s^{\beta+1} J_{sr} + sup \left(\begin{matrix} \frac{1}{2}\varphi^2\sigma^2 z^2 s^{2\beta} J_{zz} \\ +\varphi \begin{bmatrix} (\mu - r) zJ_z \\ +\sigma^2 z s^{2\beta+1} J_{zs} \\ +\sigma a_1\sqrt{r} z s^{\beta} J_{zr} \end{bmatrix} \end{matrix} \right) \end{matrix} \right\} = 0 \tag{8}$$

Differentiating (8) with respect to φ , we obtain the first order maximizing condition as

$$\left(\begin{matrix} \varphi\sigma^2 z^2 s^{2\beta} J_{zz} + (\mu - r) zJ_z + \sigma^2 z s^{2\beta+1} J_{zs} \\ +\sigma a_1\sqrt{r} z s^{\beta} J_{zr} \end{matrix} \right) = 0 \tag{9}$$

Solving (9) for φ we have

$$\varphi^* = -\frac{\left[(\mu - r) J_z + \sigma^2 s^{2\beta+1} J_{zs} + \sigma a_1\sqrt{r} s^{\beta} J_{zr} \right]}{z\sigma^2 s^{2\beta} J_{zz}} \tag{10}$$

Where φ^* is the optimal portfolio strategy of the risky asset. Substituting (10) into (8), we have

$$\left[\begin{matrix} J_t + \mu s J_s + \frac{1}{2}\sigma^2 s^{2\beta+2} J_{ss} + rzJ_z + (b - cr) J_r \\ +\frac{1}{2} r a_1^2 J_{rr} + \sigma a_1\sqrt{r} s^{\beta+1} J_{sr} \\ +\frac{1}{2}\frac{\left[(\mu - r)J_z + \sigma^2 s^{2\beta+1} J_{zs} + \sigma a_1\sqrt{r} s^{\beta} J_{zr} \right]^2}{\sigma^2 s^{2\beta} J_{zz}} \end{matrix} \right] = 0 \tag{11}$$

4. Optimal Portfolio Strategies for an investor with Logarithm Utility

In this section, we consider an investor with utility function which exhibit constant relative risk aversion (CRRA) different from the one in [17] where the investor exhibits constant absolute risk aversion (CARA). Since our interest here is to determine the optimal portfolio strategies for the investor with CRRA utility, we choose the logarithm utility function similar to the one in [2].

From [2, 14], The logarithm utility function is given as

$$U(z) = lnz \tag{12}$$

Next, we conjecture a solution to (11) similar to the one in [2] with the form:

$$\begin{cases} J(t, r, s, z) = w(t, r, s) + v(t, r, s) \ln z \\ w(T, r, s) = 0, \; v(T, r, s) = 1, \end{cases} \tag{13}$$

$$\left\{ \begin{matrix} J_t = w_t + v_t \ln z, \quad J_s = w_s + v_s \ln z, \quad J_{ss} = w_{ss} + v_{ss} \ln z, \\ J_r = w_r + v_r \ln z, \; J_{rr} = w_{rr} + v_{rr} \ln z, J_{sr} = w_{sr} + v_{sr} \ln z \\ J_{sr} = w_{sr} + v_{sr} \ln z, \; J_{zz} = -\frac{v}{z^2}, \quad J_{zr} = \frac{v_r}{z}, \quad J_{zs} = \frac{v_s}{z} \end{matrix} \right\} \tag{14}$$

Substituting (14) into (11), we have

$$\left(\begin{matrix} \begin{bmatrix} v_t + \mu s v_s + \frac{1}{2}\sigma^2 s^{2\beta+2} v_{ss} \\ + (b - cr) v_r + \frac{1}{2} r a_1^2 v_{rr} \\ +\sigma a_1\sqrt{r} s^{\beta+1} v_{sr} \end{bmatrix} \ln z \\ + \begin{bmatrix} w_t + \mu s w_s + \frac{1}{2}\sigma^2 s^{2\beta+2} w_{ss} \\ + (b - cr) v_r + rv + \frac{1}{2} r a_1^2 w_{rr} \\ +\sigma a_1\sqrt{r} s^{\beta+1} w_{sr} \\ +\frac{1}{2}\frac{\left[(\mu - r)v + \sigma^2 s^{2\beta+1} v_s + \sigma a_1\sqrt{r} s^{\beta} v_r \right]^2}{\sigma^2 s^{2\beta} v} \end{bmatrix} \end{matrix} \right) = 0 \tag{15}$$

Splitting (15) we have

$$\left(\begin{matrix} v_t + \mu s v_s + \frac{1}{2}\sigma^2 s^{2\beta+2} v_{ss} + (b - cr) v_r \\ +\frac{1}{2} r a_1^2 v_{rr} + \sigma a_1\sqrt{r} s^{\beta+1} v_{sr} \end{matrix} \right) = 0 \tag{16}$$

$$\left[\begin{matrix} w_t + \mu s w_s + \frac{1}{2}\sigma^2 s^{2\beta+2} w_{ss} + (b - cr) v_r \\ +rv + \frac{1}{2} r a_1^2 w_{rr} + \sigma a_1\sqrt{r} s^{\beta+1} w_{sr} \\ +\frac{1}{2}\frac{\left[(\mu - r)v + \sigma^2 s^{2\beta+1} v_s + \sigma a_1\sqrt{r} s^{\beta} v_r \right]^2}{\sigma^2 s^{2\beta} v} \end{matrix} \right] = 0 \tag{17}$$

Taking the boundary condition $v(T, r, s) = 1$ into consideration, we solve (16) as follows using power transformation and change variable approach.

Proposition 4.1

The solution of equation (16) is given as

$$v(t, r, s) = h(t, r, q) = 1$$

where

$$h(t, r, q) = h_1(t, r, q) + \sqrt{\alpha} h_2(t, r, q) + \alpha h_3(t, r, q)$$

and

$h_1(t, r, q) = 1, \; h_2(t, r, q) = 0, \; and \; h_3(t, r, q) = 0$

Proof

Assume

$$\left\{ \begin{array}{c} v(t,r,s) = h(t,r,q), \quad q = s^{-2\beta} \\ h(T,r,s) = 1 \end{array} \right\} \tag{18}$$

Then

$$\left(\begin{array}{c} v_t = h_t, \quad v_s = -2\beta s^{-2\beta-1} h_q, \\ v_{ss} = 2\beta(2\beta+1) s^{-2\beta-2} h_q + 4\beta^2 s^{-4\beta-2} h_{qq}, \\ v_r = h_r, \quad v_{rr} = h_{rr}, \quad v_{rs} = -2\beta s^{-2\beta-1} h_{rq} \end{array} \right) \tag{19}$$

Substituting (19) into (16), we have

$$\left[\begin{array}{c} h_t - 2\mu\beta q h_q + \sigma^2\beta(2\beta+1) h_q + 2\beta^2\sigma^2 q h_{qq} \\ +(b-cr) h_r + \frac{1}{2} r a_1^2 h_{rr} - 2\beta\sigma a_1 \sqrt{r}\sqrt{q} h_{rq} \end{array} \right] = 0 \tag{20}$$

We can rewrite (20) as

$$(A + B + C) h = 0 \tag{21}$$

Where

$$A = \left[(b-cr) h_r + \frac{1}{2} r a_1^2 h_{rr}\right] h \tag{22}$$

$$B = \left[h_t + \beta\left(\sigma^2(2\beta+1) - 2\mu q\right) h_q + 2\beta^2\sigma^2 q h_{qq}\right] h \tag{23}$$

$$C = \left[-2\beta\sigma a_1 \sqrt{r}\sqrt{q} h_{rq}\right] h \tag{24}$$

Next we follow the approach in [17] by applying the asymptotic expansion method to solve the problem in (21).

Assume that the volatility follows a slow fluctuating process, we attempt to determine an asymptotic solution for (21) by using the slow-fluctuating process r_α to replace r in (2), such that $0 < \alpha \ll 1$ is a small positive integer:

$$dr_\alpha(t) = (b - cr_\alpha(t))\, dt - a_1\sqrt{r_\alpha(t)} dB_r(t) \tag{25}$$

Substituting (25) into (21) and also replacing $b - cr$ by $\alpha(b-cr)$ and $\sqrt{r}$ by $\sqrt{\alpha}\sqrt{r}$, we will have

$$\left(\alpha A + B + \sqrt{\alpha} C\right) h_\alpha = 0 \tag{26}$$

Next, we conjecture a solution for (26) as follows

$$h_\alpha(t,r,q) = h_1(t,r,q) + \sqrt{\alpha} h_2(t,r,q) + \alpha h_3(t,r,q) \tag{27}$$

Substituting (27) into (26)

$$\left(\alpha A + B + \sqrt{\alpha} C\right)\left(\begin{array}{c} h_1(t,r,q) + \sqrt{\alpha} h_2(t,r,q) \\ +\alpha h_3(t,r,q) \end{array} \right) = 0$$

Simplifying the above equation, we arrive at

$$\left(\begin{array}{c} Bh_1(t,r,q) + [Bh_2(t,r,q) + Ch_1(t,r,q)]\sqrt{\alpha} \\ +[Ah_1(t,r,q) + Bh_3(t,r,q) + Ch_2(t,r,q)]\,\alpha \end{array} \right) = 0$$

This implies that

$$\left\{ \begin{array}{l} Bh_1(t,r,q) = 0 \\ h_1(T,r,s) = 1 \end{array} \right. \tag{28}$$

$$\left\{ \begin{array}{l} Bh_2(t,r,q) + Ch_1(t,r,q) = 0 \\ h_1(T,r,s) = 1, \ h_2(T,r,s) = 0 \end{array} \right. \tag{29}$$

$$\left\{ \begin{array}{l} Ah_1(t,r,q) + Bh_3(t,r,q) + Ch_2(t,r,q) \\ h_1(T,r,s) = 1, \ \mathrm{h}_2(T,r,s) = 0 \ h_3(T,r,s) = 0 \end{array} \right. \tag{30}$$

To obtain the solution of (27), we solve (28), (29) and (30). Recall from (22), (23) and (24), equation (28), (29) and (30) can be expressed as

$$\left\{ \begin{array}{c} h_{1t} + \beta\left(\sigma^2(2\beta+1) - 2\mu q\right) h_{1q} + 2\beta^2\sigma^2 q h_{1qq} = 0 \\ h_1(T,r,q) = 1 \end{array} \right. \tag{31}$$

$$\left\{ \begin{array}{c} h_{2t} + \beta\left(\sigma^2(2\beta+1) - 2\mu q\right) h_{2q} \\ +2\beta^2\sigma^2 q h_{2qq} - 2\beta\sigma a_1\sqrt{r}\sqrt{q} h_{1rq} = 0 \\ h_2(T,r,q) = 0 \end{array} \right. \tag{32}$$

$$\left\{ \begin{array}{c} h_{3t} + \beta\left(\sigma^2(2\beta+1) - 2\mu q\right) h_{3q} + 2\beta^2\sigma^2 q h_{3qq} \\ +(b-cr) h_{1r} + \frac{1}{2} r a_1^2 h_{1rr} - 2\beta\sigma a_1\sqrt{r}\sqrt{q} h_{2rq} \\ h_3(T,r,q) = 0 \end{array} \right. \tag{33}$$

Let

$$h_1(t,r,q) = D_0(t,r) + qD_1(t,r) \tag{34}$$

And

$$\left(h_{1t} = D_{0t} + qD_{1t}, h_{1q} = D_1, h_{1qq} = 0\right) \tag{35}$$

Substituting (35) in (31), we have

$$D_{0t} + \beta\sigma^2(2\beta+1) D_1 = 0, \quad D_0(T,r) = 1 \tag{36}$$

$$D_{1t} - 2\mu\beta D_1 = 0, \quad D_1(T,r) = 0 \tag{37}$$

Solving (37), we have

$$D_1(t,r) = 0 \tag{38}$$

Putting (38) into (36) solving it, we have

$$D_0(t,r) = 1 \tag{39}$$

Hence from (34)

$$h_1(t,r,q) = 1 \tag{40}$$

Next, we solve (32), by assuming a solution of the form

$$h_2(t,r,q) = \left(\begin{array}{c} D_2(t,r) + q^{\frac{1}{2}} D_3(t,r) + qD_4(t,r) \\ +q^{\frac{3}{2}} D_5(t,r) \end{array} \right) \tag{41}$$

And

$$\left(\begin{array}{c} h_{2t} = D_{2t} + q^{\frac{1}{2}} D_{3t} + q D_{4t} + q^{\frac{3}{2}} D_{5t}\,, \\ h_{2q} = \frac{1}{2} q^{-\frac{1}{2}} D_3 + D_4 + \frac{3}{2} q^{\frac{1}{2}} D_5 \\ h_{2qq} = -\frac{1}{4} q^{-\frac{3}{2}} D_3 + \frac{3}{4} q^{-\frac{1}{2}} D_5, \quad h_{1rq} = 0 \end{array}\right) \tag{42}$$

Substituting (42) into (32), we have

$$D_{2t} + \beta\sigma^2 (2\beta + 1) D_4 = 0, \quad D_2 (T, r) = 0 \tag{43}$$

$$D_{4t} - 2\mu\beta D_4 = 0, \quad D_4 (T, r) = 0 \tag{44}$$

$$D_{5t} - 3\mu\beta D_5 = 0, \quad D_5 (T, r) = 0 \tag{45}$$

$$D_{3t} - \mu\beta D_3 + \frac{3}{2}\beta\sigma^2 (3\beta + 1) D_5 = 0, \quad D_3 (T, r) = 0 \tag{46}$$

Solving (43), (44), (45) and (46) with their boundary conditions, we have

$$D_2 (t, r) = 0, \; D_3 (t, r) = 0, \; D_4 (t, r) = 0, D_5 (t, r) = 0$$

hence from (41)

$$h_2 (t, r, q) = 0 \tag{47}$$

Next , we attempt to solve (33), by assuming a solution of the form

$$h_3 (t, r, q) = \left(\begin{array}{c} D_6 (t, r) + q^{\frac{1}{2}} D_7 (t, r) + q D_8 (t, r) \\ + q^{\frac{3}{2}} D_9 (t, r) + q^2 D_9 (t, r) \end{array}\right) \tag{48}$$

$$\left(\begin{array}{c} h_{3t} = D_{6t} + q^{\frac{1}{2}} D_{7t} + q D_{8t} + q^{\frac{3}{2}} D_{9t} + q^2 D_{10t}\,, \\ h_{3q} = \frac{1}{2} q^{-\frac{1}{2}} D_7 + D_8 + \frac{3}{2} q^{\frac{1}{2}} D_9 + 2q D_{10} \\ h_{2qq} = -\frac{1}{4} q^{-\frac{3}{2}} D_7 + \frac{3}{4} q^{-\frac{1}{2}} D_9 + 2 D_{10} h_{1r} = 0, \\ h_{1rr} = 0, h_{2rq} = 0 \end{array}\right) \tag{49}$$

Substituting into (33), we have

$$D_{6t} + \beta\sigma^2 (2\beta + 1) D_8 = 0, \quad D_6 (T, r) = 0 \tag{50}$$

$$D_{7t} - \mu\beta D_7 + \frac{3}{2}\beta\sigma^2 (3\beta + 1) D_9 = 0, \quad D_7 (T, r) = 0 \tag{51}$$

$$D_{8t} - 2\mu\beta D_8 + 2\beta\sigma^2 (4\beta + 1) D_{10} = 0, \quad D_8 (T, r) = 0 \tag{52}$$

$$D_{9t} - 3\mu\beta D_9 = 0, \quad D_9 (T, r) = 0 \tag{53}$$

$$D_{10t} - 4\mu\beta D_{10} = 0, \quad D_{10} (T, r) = 0 \tag{54}$$

Solving (50), (51), (52), (53) and (54), we obtain

$$D_6 (t, r) = 0, D_7 (t, r) = 0, \; D_8 (t, r) = 0,$$
$$D_9 (t, r) = 0, D_{10} (t, r) = 0$$

Hence from (48)

$$h_3 (t, r, q) \tag{55}$$

Therefore, from (27), we have

$$h_\alpha (t, r, q) = h_1 (t, r, q) + \sqrt{\alpha} h_2 (t, r, q) + \alpha h_3 (t, r, q) = 1$$

Hence from (18),

$$v (t, r, s) = 1$$

Proposition 4.2

The solution of equation (17) is given as

$$w (t, r, s) = f_1 (t, r, q) + \sqrt{\alpha} f_2 (t, r, q) + \alpha f_3 (t, r, q)$$

Where

$$\left(\begin{array}{c} f_1 (t, r, q) = \left[\frac{(2\beta+1)(\mu-r)^2}{8\beta\mu} + r\right](T - t) \\ + \left(q \frac{(\mu-r)^2}{4\mu\beta\sigma^2} - \frac{(2\beta+1)(\mu-r)^2}{8\beta\mu^2}\right)\left[1 - e^{2\mu\beta(t-T)}\right] \\ f_2 (t, r, q) = K_2 (t, r) + q^{\frac{1}{2}} K_3 (t, r) + q K_4 (t, r) \\ f_3 (t, r, q) = K_6 (t, r) + q^{\frac{1}{2}} K_7 (t, r) + q K_8 (t, r) \end{array}\right)$$

and

$$\left(\begin{array}{c} K_2 (t, r) = \left[\frac{(2\beta+1)(\mu-r)^2}{8\beta\mu} + r\right](T - t) \\ - \frac{(2\beta+1)(\mu-r)^2}{8\beta\mu^2}\left[1 - e^{2\mu\beta(t-T)}\right] \\ K_3 (t, r) = \frac{a_1\sqrt{r}(\mu-r)}{\sigma\beta\mu^2}\left[1 - 2e^{\mu\beta(t-T)} + e^{2\mu\beta(t-T)}\right] \\ K_4 (t, r) = \frac{(\mu-r)^2}{4\mu\beta\sigma^2}\left[1 - e^{2\mu\beta(t-T)}\right] \\ K_6 (t, r) = \left(\begin{array}{c} \left[\begin{array}{c} \frac{(2\beta+1)(\mu-r)(b-cr)\left[1-e^{2\mu\beta(t-T)}\right]}{8\beta\sigma^2\mu^2} \\ + \frac{(2\beta+1)(\mu-r)^2}{4\sigma^2\mu} + r \\ \frac{r(2\beta+1)(\mu-r)(b-cr)}{4\beta\mu^2} \\ + \frac{ra_1^2(2\beta+1)(\mu-2r)}{8\beta\mu^2} + \frac{(\mu-3r)a_1^2}{2\mu^2} \end{array}\right](T - t) \\ + \left[\begin{array}{c} \frac{r(2\beta+1)(\mu-r)(b-cr)}{4\mu} - \frac{(b-cr)}{2} \\ + \frac{ra_1^2(2\beta+1)(\mu-2r)}{8\mu} \end{array}\right]\left(t^2 - T^2\right) \\ \left[\begin{array}{c} \frac{r(2\beta+1)(\mu-r)(b-cr)}{8\beta^2\mu^3} + \\ \frac{(2\beta+1)(\mu-r)\left(ra_1^2+\mu-r\right)}{8\beta\sigma^2\mu^2} \\ + \frac{ra_1^2(2\beta+1)(\mu-2r)}{8\mu} \\ + \frac{(\mu-3r)a_1^2}{4\beta\mu^3} \end{array}\right]\left(1 - e^{2\mu\beta(t-T)}\right) \\ \frac{(\mu-3r)a_1^2}{4\beta\mu^3}\left(1 - e^{\mu\beta(t-T)}\right) \end{array}\right) \\ K_7 (t, r) = \frac{a_1\sqrt{r}(\mu-r)}{\sigma\beta\mu^2}\left[1 - 2e^{\mu\beta(t-T)} + e^{2\mu\beta(t-T)}\right] \\ K_8 (t, r) = \left[\begin{array}{c} \frac{(\mu-r)^2}{4\mu\beta\sigma^2}\left[1 - e^{2\mu\beta(t-T)}\right] \\ + \left(\frac{\begin{array}{c} ra_1^2 \\ -2 (b - cr) (\mu - r) \end{array}}{4\mu\beta\sigma^2}\right)\left(\begin{array}{c} \frac{1}{2\mu\beta}\left[1 - e^{2\mu\beta(t-T)}\right] \\ + (t - T) e^{2\mu\beta(t-T)} \end{array}\right) \end{array}\right] \end{array}\right)$$

Proof

Substitute $(t, r, s) = 1$, $v_s = 0$, $v_r = 0$ into (17), we have

$$\left[\begin{array}{c} w_t + \mu s w_s + \frac{1}{2}\sigma^2 s^{2\beta+2} w_{ss} + (b - cr) w_r + r + \frac{1}{2} r a_1^2 w_{rr} \\ + \sigma a_1 \sqrt{r} s^{\beta+1} w_{sr} + \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2 s^{2\beta}} \end{array}\right] = 0 \tag{56}$$

$$w (T, r, s) = 0$$

Assume

$$\left\{\begin{array}{c} w (t, r, s) = f (t, r, q), \qquad q = s^{-2\beta} \\ f (T, r, q) = 0 \end{array}\right. \tag{57}$$

Then

$$\left\{\begin{array}{c} w_t = f_t, \;\; w_s = -2\beta s^{-2\beta-1} f_q\,, \\ w_{ss} = 2\beta (2\beta + 1) s^{-2\beta-2} f_q + 4\beta^2 s^{-4\beta-2} f_{qq}, \\ w_r = f_r, \;\; w_{rr} = f_{rr}, \;\; w_{rs} = -2\beta s^{-2\beta-1} f_{rq} \end{array}\right. \tag{58}$$

Substituting (58) into (56), we have

$$\begin{bmatrix} f_t - 2\mu\beta q f_q + \sigma^2\beta(2\beta+1) f_q \\ +\frac{1}{2}\frac{(\mu-r)^2 q}{\sigma^2} + r + 2\beta^2\sigma^2 q f_{qq} + (b-cr) f_r \\ +\frac{1}{2} r a_1^2 f_{rr} - 2\beta\sigma a_1\sqrt{r}\sqrt{q} f_{rq} \end{bmatrix} = 0 \tag{59}$$

We can rewrite (59) as

$$(E+F+G) f = 0 \tag{60}$$

Where

$$E = \left[(b-cr) f_r + \frac{1}{2} r a_1^2 f_{rr}\right] f \tag{61}$$

$$F = \begin{bmatrix} f_t + \beta\left(\sigma^2(2\beta+1) - 2\mu q\right) f_q \\ +2\beta^2\sigma^2 q f_{qq} + \frac{1}{2}\frac{(\mu-r)^2 q}{\sigma^2} + r \end{bmatrix} f \tag{62}$$

$$G = \left[-2\beta\sigma a_1\sqrt{r}\sqrt{q} f_{rq}\right] f \tag{63}$$

Similarly, we apply the same approach used in solving (21) to determine the solution of (60) as follows

$$dr_\alpha(t) = (b - cr_\alpha(t))\,dt - a_1\sqrt{r_\alpha(t)}\,dB_r(t) \tag{64}$$

Substituting (64) into (60) and also replacing $b - cr$ by $\alpha(b-cr)$ and $\sqrt{r}$ by $\sqrt{\alpha}\sqrt{r}$, we will have

$$\left(\alpha E + F + \sqrt{\alpha} G\right) f_\alpha = 0 \tag{65}$$

Next, we conjecture a solution for (65) as follows

$$f_\alpha(t,r,q) = f_1(t,r,q) + \sqrt{\alpha} f_2(t,r,q) + \alpha f_3(t,r,q) \tag{66}$$

Substituting (66) into (65) and simplifying it, we have

$$\begin{gathered} F f_1(t,r,q) + \left[F f_2(t,r,q) + G f_1(t,r,q)\right]\sqrt{\alpha} \\ + \left[E f_1(t,r,q) + F f_3(t,r,q) + G f_2(t,r,q)\right]\alpha = 0 \end{gathered}$$

This implies that

$$\begin{cases} F f_1(t,r,q) = 0 \\ f_1(T,r,s) = 0 \end{cases} \tag{67}$$

$$\begin{cases} F f_2(t,r,q) + G f_1(t,r,q) = 0 \\ f_1(T,r,s) = f_2(T,r,s) = 0 \end{cases} \tag{68}$$

$$\begin{cases} E f_1(t,r,q) + F f_3(t,r,q) + G f_2(t,r,q) \\ f_1(T,r,s) = f_2(T,r,s) = f_3(T,r,s) = 0 \end{cases} \tag{69}$$

To obtain the solution of (66), we solve (67), (68) and (69).

Recall from (61), (62) and (63), equation (67), (68) and (69) can be expressed as

$$\begin{cases} f_{1t} + \beta\left(\sigma^2(2\beta+1) - 2\mu q\right) f_{1q} + 2\beta^2\sigma^2 q f_{1qq} \\ +\frac{1}{2}\frac{(\mu-r)^2 q}{\sigma^2} + r = 0 \\ f_1(T,r,q) = 1 \end{cases} \tag{70}$$

$$\begin{cases} f_{2t} + \beta\left(\sigma^2(2\beta+1) - 2\mu q\right) f_{2q} + 2\beta^2\sigma^2 q f_{2qq} \\ +\frac{1}{2}\frac{(\mu-r)^2 q}{\sigma^2} + r - 2\beta\sigma a_1\sqrt{r}\sqrt{q} f_{1rq} = 0 \\ f_2(T,r,q) = 0 \end{cases} \tag{71}$$

$$\begin{cases} f_{3t} + \beta\left(\sigma^2(2\beta+1) - 2\mu q\right) f_{3q} + 2\beta^2\sigma^2 q f_{3qq} \\ +\frac{1}{2}\frac{(\mu-r)^2 q}{\sigma^2} + r \\ +(b-cr) f_{1r} + \frac{1}{2} r a_1^2 f_{1rr} - 2\beta\sigma a_1\sqrt{r}\sqrt{q} f_{2rq} \\ f_3(T,r,q) = 0 \end{cases} \tag{72}$$

Let

$$f_1(t,r,q) = K_0(t,r) + qK_1(t,r) \tag{73}$$

and

$$\left(f_{1t} = K_{0t} + qK_{1t},\ f_{1q} = K_1,\ f_{1qq} = 0\right) \tag{74}$$

Substituting (74) in (70), we have

$$K_{0t} + \beta\sigma^2(2\beta+1) K_1 + r = 0, \quad K_0(T,r) = 0 \tag{75}$$

$$K_{1t} - 2\mu\beta K_1 + \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2} = 0, \quad K_1(T,r) = 0 \tag{76}$$

Solving (76), we have

$$K_1(t,r) = \frac{(\mu-r)^2}{4\mu\beta\sigma^2}\left[1 - e^{2\mu\beta(t-T)}\right] \tag{77}$$

Putting (77) into (75) solving it, we have

$$K_0(t,r) = \begin{pmatrix} \left[\frac{(2\beta+1)(\mu-r)^2}{8\beta\mu} + r\right](T-t) \\ -\frac{(2\beta+1)(\mu-r)^2}{8\beta\mu^2}\left[1 - e^{2\mu\beta(t-T)}\right] \end{pmatrix} \tag{78}$$

Hence from (34)

$$f_1(t,r,q) = \begin{pmatrix} \left[\frac{(2\beta+1)(\mu-r)^2}{8\beta\mu} + r\right](T-t) \\ +\left(q\frac{(\mu-r)^2}{4\mu\beta\sigma^2} - \frac{(2\beta+1)(\mu-r)^2}{8\beta\mu^2}\right)\left[1 - e^{2\mu\beta(t-T)}\right] \end{pmatrix} \tag{79}$$

Next, we solve (71), by assuming a solution of the form

$$f_2(t,r,q) = K_2(t,r) + q^{\frac{1}{2}} K_3(t,r) + qK_4(t,r) + q^{\frac{3}{2}} K_5(t,r) \tag{80}$$

and

$$\begin{pmatrix} f_{2t} = K_{2t} + q^{\frac{1}{2}} K_{3t} + qK_{4t} + q^{\frac{3}{2}} K_{5t}, \\ f_{2q} = \frac{1}{2} q^{-\frac{1}{2}} K_3 + K_4 + \frac{3}{2} q^{\frac{1}{2}} K_5 \\ f_{2qq} = -\frac{1}{4} q^{-\frac{3}{2}} K_3 + \frac{3}{4} q^{-\frac{1}{2}} K_5, \\ K_{1r} = \frac{(\mu-r)}{2\mu\beta\sigma^2}\left[e^{2\mu\beta(t-T)} - 1\right] \end{pmatrix} \tag{81}$$

Substituting (81) into (71), we have

$$\begin{pmatrix} K_{2t} + \beta\sigma^2(2\beta+1) K_4 + r = 0, K_2(T,r) = 0 \\ \begin{pmatrix} K_{3t} - \mu\beta K_3 + \frac{3}{2}\beta\sigma^2(3\beta+1) K_5 \\ -2\beta\sigma a_1\sqrt{r} K_{1r} = 0 \end{pmatrix}, K_3(T,r) = 0 \\ K_{4t} - 2\mu\beta K_4 + \frac{(\mu-r)^2}{2\sigma^2} = 0, K_4(T,r) = 0 \\ K_{5t} - 3\mu\beta K_5 = 0, K_5(T,r) = 0 \end{pmatrix} \tag{82}$$

Solving (82) with their boundary conditions, we have

$$\left(\begin{array}{c} K_2(t,r) = \left(\begin{array}{c} \left[\frac{(2\beta+1)(\mu-r)^2}{8\beta\mu} + r\right](T-t) \\ -\frac{(2\beta+1)(\mu-r)^2}{8\beta\mu^2}\left[1-e^{2\mu\beta(t-T)}\right] \end{array}\right) \\ K_3(t,r) = \frac{a_1\sqrt{r}(\mu-r)}{\sigma\beta\mu^2}\left[1-2e^{\mu\beta(t-T)}+e^{2\mu\beta(t-T)}\right] \\ K_4(t,r) = \frac{(\mu-r)^2}{4\mu\beta\sigma^2}\left[1-e^{2\mu\beta(t-T)}\right] \\ K_5(t,r) = 0 \end{array}\right) \tag{83}$$

Hence from (80)

$$f_2(t,r,q) = \left(\begin{array}{c} K_2(t,r) + q^{\frac{1}{2}}K_3(t,r) + qK_4(t,r) \\ +q^{\frac{3}{2}}K_5(t,r) \end{array}\right) \tag{84}$$

Where K_2, K_3, K_4, and K_5 are given in equation (83)

Next, we attempt to solve (72), by assuming a solution of the form

$$f_3(t,r,q) = \left(\begin{array}{c} K_6(t,r) + q^{\frac{1}{2}}K_7(t,r) + qK_8(t,r) \\ +q^{\frac{3}{2}}K_9(t,r) + q^2K_{10}(t,r) \end{array}\right) \tag{85}$$

$$\left(\begin{array}{c} f_{3t} = K_{6t} + q^{\frac{1}{2}}K_{7t} + qK_{8t} + q^{\frac{3}{2}}K_{9t} + q^2K_{10t}, \\ f_{3q} = \frac{1}{2}q^{-\frac{1}{2}}K_7 + K_8 + \frac{3}{2}q^{\frac{1}{2}}K_9 + 2qK_{10} \\ f_{3qq} = -\frac{1}{4}q^{-\frac{3}{2}}K_7 + \frac{3}{4}q^{-\frac{1}{2}}K_9 + 2K_{10}, \\ f_{1r} = K_{0r} + qK_{1r},\ f_{1rr} = K_{0rr} + qK_{1rr} \\ f_{2rq} = \frac{1}{2}q^{-\frac{1}{2}}K_{3r} + K_{4r} + \frac{3}{2}q^{\frac{1}{2}}K_{5r} \end{array}\right) \tag{86}$$

Substituting into (72), we have

$$\left(\begin{array}{c} \left(\begin{array}{c} K_{6t} + \beta\sigma^2(2\beta+1)K_8 + r + (b-cr)K_{0r} \\ -\beta\sigma a_1\sqrt{r}K_{3r} + \frac{1}{2}ra_1^2K_{0rr} \end{array}\right) = 0 \\ K_6(T,r) = 0 \\ K_{7t} - \mu\beta K_7 + \frac{3}{2}\beta\sigma^2(3\beta+1)K_9 - 2\beta\sigma a_1\sqrt{r}K_{4r} = 0, \\ K_7(T,r) = 0 \\ \left(\begin{array}{c} K_{8t} + 2\beta\sigma^2(4\beta+1)K_{10} + (b-cr)K_{1r} \\ -2\mu\beta K_8 - 3\beta\sigma a_1\sqrt{r}K_{9r} + \frac{1}{2}ra_1^2K_{1rr} + \frac{(\mu-r)^2}{2\sigma^2} \end{array}\right) = 0 \\ K_8(T,r) = 0 \\ K_{9t} - 3\mu\beta K_9 = 0, K_9(T,r) = 0 \\ K_{10t} - 4\mu\beta K_{10} = 0, K_{10}(T,r) = 0 \end{array}\right) \tag{87}$$

Solving (87), we obtain

$$\left(\begin{array}{c} K_6(t,r) = \left(\begin{array}{c} \left[\begin{array}{c} \frac{(2\beta+1)(\mu-r)(b-cr)\left[1-e^{2\mu\beta(t-T)}\right]}{8\beta\sigma^2\mu^2} \\ +\frac{r(2\beta+1)(\mu-r)(b-cr)}{4\beta\mu^2} \\ +\frac{(2\beta+1)(\mu-r)^2}{4\sigma^2\mu} + r \\ +\frac{ra_1^2(2\beta+1)(\mu-2r)}{8\beta\mu^2} + \frac{(\mu-3r)a_1^2}{2\mu^2} \end{array}\right](T-t) \\ +\left[\begin{array}{c} \frac{r(2\beta+1)(\mu-r)(b-cr)}{4\mu} - \frac{(b-cr)}{2} \\ +\frac{ra_1^2(2\beta+1)(\mu-2r)}{8\mu} \end{array}\right]\left(t^2-T^2\right) \\ \left[\begin{array}{c} \frac{r(2\beta+1)(\mu-r)(b-cr)}{8\beta^2\mu^3} \\ +\frac{(2\beta+1)(\mu-r)\left(ra_1^2+\mu-r\right)}{8\beta\sigma^2\mu^2} \\ +\frac{ra_1^2(2\beta+1)(\mu-2r)}{8\mu} \\ +\frac{(\mu-3r)a_1^2}{4\beta\mu^3} \end{array}\right]\left(1-e^{2\mu\beta(t-T)}\right) \\ \frac{(\mu-3r)a_1^2}{4\beta\mu^3}\left(1-e^{\mu\beta(t-T)}\right) \end{array}\right) \\ K_7(t,r) = \frac{a_1\sqrt{r}(\mu-r)}{\sigma\beta\mu^2}\left[1-2e^{\mu\beta(t-T)}+e^{2\mu\beta(t-T)}\right] \\ K_8(t,r) = \left[\begin{array}{c} \frac{(\mu-r)^2}{4\mu\beta\sigma^2}\left[1-e^{2\mu\beta(t-T)}\right] \\ +\frac{ra_1^2-2(b-cr)(\mu-r)}{4\mu\beta\sigma^2}\left(\begin{array}{c} \frac{1}{2\mu\beta}\left[1-e^{2\mu\beta(t-T)}\right] \\ +(t-T)e^{2\mu\beta(t-T)} \end{array}\right) \end{array}\right] \\ K_9(t,r) = 0 \\ K_{10}(t,r) = 0 \end{array}\right) \tag{88}$$

Hence from (85)

$$f_3(t,r,q) = \left(\begin{array}{c} K_6(t,r) + q^{\frac{1}{2}}K_7(t,r) + qK_8(t,r) \\ +q^{\frac{3}{2}}K_9(t,r) + q^2K_{10}(t,r) \end{array}\right) \tag{89}$$

With K_6, K_7, K_8, K_9, K_{10} given in (88)

Therefore, from (66), we have

$$f_\alpha(t,r,q) = f_1(t,r,q) + \sqrt{\alpha}f_2(t,r,q) + \alpha f_3(t,r,q)$$

With $f_1(t,r,q)$, $f_2(t,r,q)$ and $f_3(t,r,q)$ given in equation (79), (84) and (89) respectively.

Hence from (57),

$$w(t,r,s) = f_1(t,r,q) + \sqrt{\alpha}f_2(t,r,q) + \alpha f_3(t,r,q)$$

Proposition 4.3

The optimal portfolio strategies and optimal value function are given as

$$\varphi^* = \frac{(\mu-r)}{\sigma^2 s^{2\beta}} \tag{90}$$

$$\varphi_1^* = \frac{(\mu-r)}{\sigma^2 s^{2\beta}} \tag{91}$$

And

$$J(t,r,s,z) = \left(\begin{array}{c} \ln z + f_1(t,r,q) + \sqrt{\alpha}f_2(t,r,q) \\ +\alpha f_3(t,r,q) \end{array}\right) \tag{92}$$

where

$$\left(\begin{array}{c} f_1(t,r,q) = \left(\begin{array}{c} \left[\frac{(2\beta+1)(\mu-r)^2}{8\beta\mu} + r\right](T-t) \\ +\left(\begin{array}{c} q\frac{(\mu-r)^2}{4\mu\beta\sigma^2} \\ -\frac{(2\beta+1)(\mu-r)^2}{8\beta\mu^2} \end{array}\right)\left[1-e^{2\mu\beta(t-T)}\right] \end{array}\right) \\ f_2(t,r,q) = K_2(t,r) + q^{\frac{1}{2}}K_3(t,r) + qK_4(t,r) \\ f_3(t,r,q) = K_6(t,r) + q^{\frac{1}{2}}K_7(t,r) + qK_8(t,r) \end{array}\right)$$

and

$$
\left(\begin{array}{c}
K_2(t,r) = \left(\begin{array}{c} \left[\frac{(2\beta+1)(\mu-r)^2}{8\beta\mu} + r\right](T-t) \\ -\frac{(2\beta+1)(\mu-r)^2}{8\beta\mu^2}\left[1 - e^{2\mu\beta(t-T)}\right] \end{array}\right) \\
K_3(t,r) = \left(\frac{a_1\sqrt{r}(\mu-r)}{\sigma\beta\mu^2}\left[\begin{array}{c} 1 - 2e^{\mu\beta(t-T)} \\ +e^{2\mu\beta(t-T)} \end{array}\right]\right) \\
K_4(t,r) = \frac{(\mu-r)^2}{4\mu\beta\sigma^2}\left[1 - e^{2\mu\beta(t-T)}\right] \\
K_6(t,r) = \left(\begin{array}{c}
\left[\begin{array}{c} \frac{(2\beta+1)(\mu-r)(b-cr)\left[1-e^{2\mu\beta(t-T)}\right]}{8\beta\sigma^2\mu^2} \\ + \frac{(2\beta+1)(\mu-r)^2}{4\sigma^2\mu} + r \\ \frac{r(2\beta+1)(\mu-r)(b-cr)}{4\beta\mu^2} \\ + \frac{ra_1^2(2\beta+1)(\mu-2r)}{8\beta\mu^2} + \frac{(\mu-3r)a_1^2}{2\mu^2} \end{array}\right](T-t) \\
+\left[\begin{array}{c} \frac{r(2\beta+1)(\mu-r)(b-cr)}{4\mu} \\ -\frac{(b-cr)}{2} + \frac{ra_1^2(2\beta+1)(\mu-2r)}{8\mu} \end{array}\right]\left(t^2 - T^2\right) \\
\left[\begin{array}{c} \frac{r(2\beta+1)(\mu-r)(b-cr)}{8\beta^2\mu^3} \\ + \frac{(2\beta+1)(\mu-r)\left(ra_1^2+\mu-r\right)}{8\beta\sigma^2\mu^2} \\ + \frac{ra_1^2(2\beta+1)(\mu-2r)}{8\mu} + \frac{(\mu-3r)a_1^2}{4\beta\mu^3} \end{array}\right]\left(1 - e^{2\mu\beta(t-T)}\right) \\
\frac{(\mu-3r)a_1^2}{4\beta\mu^3}\left(1 - e^{\mu\beta(t-T)}\right)
\end{array}\right) \\
K_7(t,r) = \frac{a_1\sqrt{r}(\mu-r)}{\sigma\beta\mu^2}\left[1 - 2e^{\mu\beta(t-T)} + e^{2\mu\beta(t-T)}\right] \\
K_8(t,r) = \left[\begin{array}{c} \frac{(\mu-r)^2}{4\mu\beta\sigma^2}\left[1 - e^{2\mu\beta(t-T)}\right] \\ + \frac{ra_1^2 - 2(b-cr)(\mu-r)}{4\mu\beta\sigma^2}\left(\begin{array}{c} \frac{1}{2\mu\beta}\left[1 - e^{2\mu\beta(t-T)}\right] \\ + (t-T)\,e^{2\mu\beta(t-T)} \end{array}\right) \end{array}\right]
\end{array}\right)
$$

Proof

1. recall that equation (11) is given as
   $\varphi^* = -\frac{\left[(\mu-r)J_z + \sigma^2 s^{2\beta+1} J_{zs} + \sigma a_1 \sqrt{r} s^{\beta} J_{zr}\right]}{z\sigma^2 s^{2\beta} J_{zz}}$.
   From proposition 1 and equation (14)
   $v(t,r,q) = 1 \; J_z = \frac{1}{z}, J_{zz} = -\frac{1}{z^2} \;, J_{zr} = \frac{v_r}{z} = 0 \;, J_{zs} = \frac{v_s}{z} = 0$
   Substituting the above equation into (11), we have

$$
\varphi^* = -\frac{\left[(\mu-r)\frac{1}{z}\right]}{z\sigma^2 s^{2\beta}\left(-\frac{1}{z^2}\right)} = \frac{(\mu-r)\frac{1}{z}}{\sigma^2 s^{2\beta}\frac{1}{z}} = \frac{(\mu-r)}{\sigma^2 s^{2\beta}}
$$

2. recall that from equation (13),

$J(t,r,s,z) = w(t,r,s) + v(t,r,s)\ln z$
From proposition 1 and 2, the proof is completed.

Remark 4.1

From proposition 4.3, we observed that our result is similar to the one in [3] but the difference between our result and theirs is that their interest rate is a constant while ours is stochastic.

5. Numerical Simulations

Here, the numerical simulations are used to study the effect of parameters on the portfolio strategies under logarithm utility. To achieve this, the following data will be use unless stated otherwise: $\sigma = 1$, $\beta = -1$, $\mu = 0.4$, $r(0) = 0.05$, $S(0) = 1.5$, $T = 3$

6. Discussion

Figure 1, present a simulation of optimal portfolio strategies against time. The graph shows that the investor will invest more in marketable security and gradually increases investment in treasury security to balance with the marketable security and as expiry date draws closer; there is a continuous decrease in investment in risky asset and a continuous increase in that of risk free asset. This is so because investors like avoiding risk toward the end of investment especially for highly risky assets.

Figure 2, shows a plot of optimal portfolio strategies against time with different values of the elasticity parameter β. The graph shows that as the elasticity parameter decreases, the insurer is more scared to invest in marketable security as expiration date approaches. Furthermore, we observed a very sharp decline when $\beta = -2$, showing how volatile the risky asset can be hence discouraging for investors with high risk aversion coefficient but when $\beta = 0$, the decline is almost unnoticeable, showing that the risky asset is not volatile. This is shows that geometric Brownian motions i.e when $\beta = 0$ may lead an investor astray while taking investment decisions.

Figure 3, present a simulation of optimal portfolio strategies against time with different values of μ; the graph shows that as μ increases, the investment strategy for the marketable security decreases continuously while that of treasury security increases continuously with time. The reason being the interest rate is not constant. Furthermore, we observe that as expiration date of investment draws closer, the risk-free interest may increase faster than μ , thereby making $\mu - r < 0$. Hence a drastic decrease in optimal portfolio strategy of marketable security.

Figure 4, shows a simulation of the optimal portfolio strategies against time with different values of σ, it is observed that as σ increases the optimal investment in marketable security decreases while that of risk-free asset increases. Recall that σ is the instantaneous volatility representing the risk coefficient of marketable security. Therefore, for risk averse investors, bigger σ , implies less investment in marketable security.

7. Conclusion

This paper considers optimal portfolio strategy for an insurer with logarithm utility using CEV to model the risky asset in the presence of stochastic interest rate. Investment in treasury security and marketable security were considered such that the risk free interest rate follows the CIR model. The asymptotic solutions of the the optimal portfolio strategies and it value function was found using power transformation, change of Variable and asymptotic expansion technique. Furthermore, we present some numerical simulations to study the effect of some parameters on the optimal portfolio strategy under stochastic interest rate.

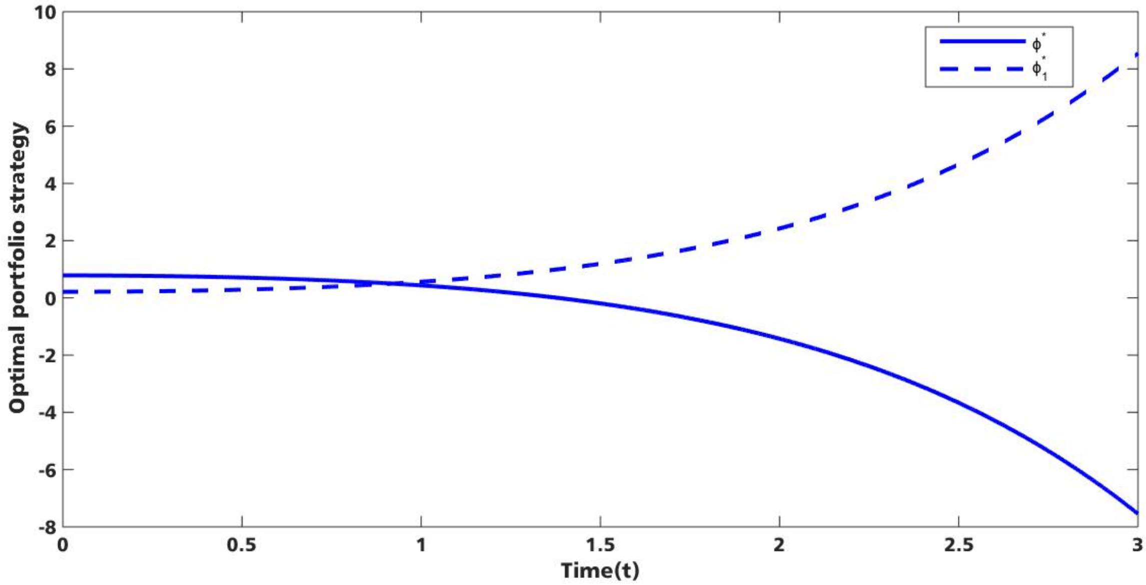

Figure 1. Time evolution of optimal portfolio strategies.

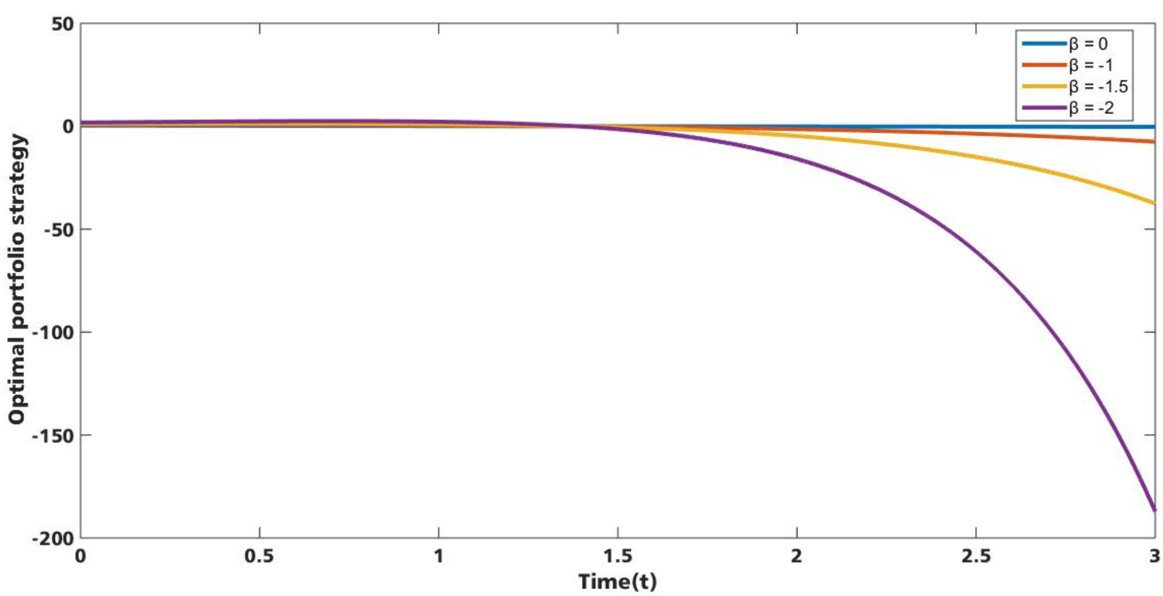

Figure 2. Time evolution of optimal portfolio strategy with different β.

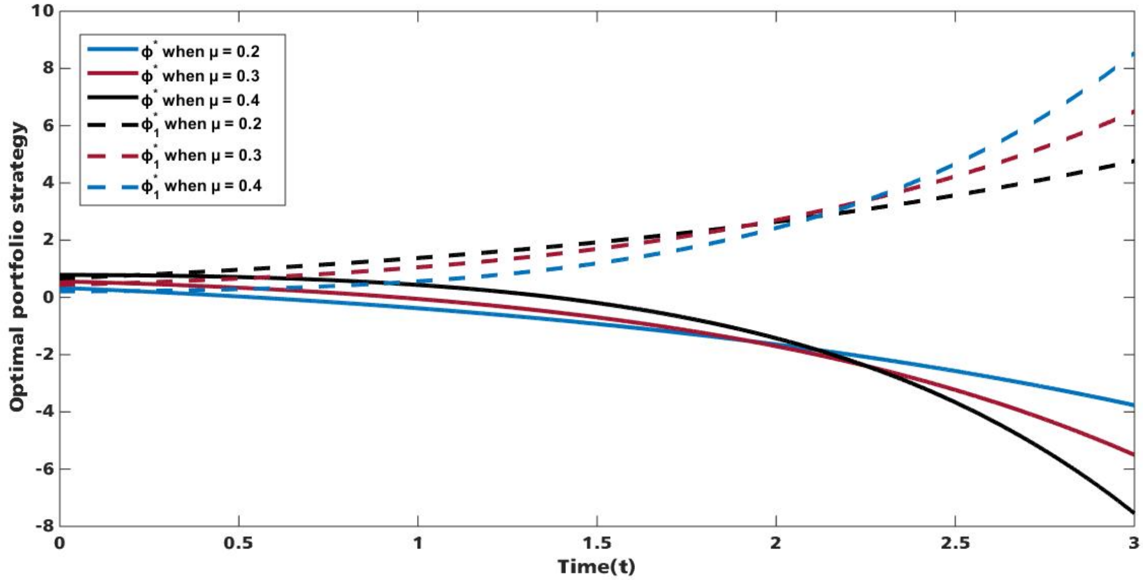

Figure 3. Time evolution of optimal portfolio strategy with different μ.

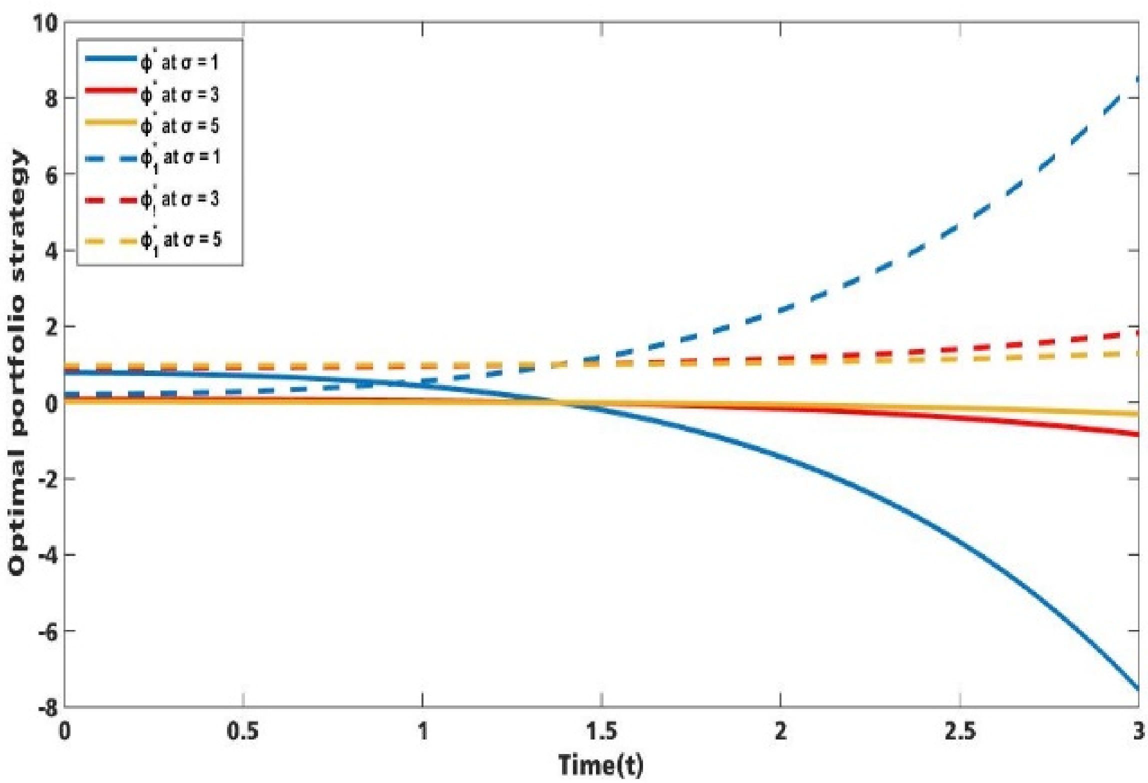

Figure 4. Time evolution of optimal portfolio strategy with different σ.

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