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A Review on Quadrant Interlocking Factorization: WZ and WH Factorization

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Abstract

Quadrant Interlocking Factorization (QIF), an alternative to LU factorization, is suitable for factorizing invertible matrix A such that $det(A) \neq 0$. The QIF, with its application in parallel computing, is the most efficient matrix factorization technique yet underutilized. The two forms of QIF among others, which are not only similar in algorithm but also in computation, are WZ factorization and WH factorization yet differs in applications and properties. This review discusses both the old form of QIF, called WZ factorization, and the latest form of QIF, called WH factorization, with an example and open questions to further the studies between the two factorization techniques.

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1. Introduction

Over a century, factorization of a square matrix has been the research interest of matrix theorists [1]. Many factorization techniques were deployed such as LU, QR and Cholesky factorization. LU factorization is a well-known method with high accuracy deployed in enigma machine during World War I. LU factorization was introduced to solve square system of linear equations by inverting a matrix with underlying Gaussian elimination procedure [2]. This feature combined with the low computational complexity and partial pivoting techniques makes LU-factorization extremely efficient. Without a proper ordering in the matrix, LU factorization may fail to occur. The flaw can be removed by reordering the rows of B so that the first element of the permuted matrix is nonzero [3]. Thus, a proper permutation in rows or columns is sufficient for the LU factorization to be numerically stable [4, 5]. LU factorization is known to be implemented in LAPACK library to exploit the standard software library architectures [6]. To improve the efficiency of computation during factorization, an alternative technique to LU factorization was developed, named Quadrant Interlocking Factorization - QIF [7]. Factorization of matrix B is difficult to compute and applying different optimization techniques couple with parallelism of contemporary computers makes QIF factorization extremely efficient and suitable for parallel computing. Among others factorization techniques such as LU, QR and Cholesky decomposition, QIF proves to be the best factorization algorithm in terms of efficiency, parallelization and accuracy [8, 9].

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2. List of abbreviations and symbols

For the readers to understand this review paper, the symbols and abbreviation used in the article are given in the sections as follows:

2.1. List of abbreviations

QIF	Quadrant Interlocking Factorization
LU	Lower and Upper triangular matrices
Q	Orthogonal matrix
LAPACK	Linear Algebra Package
PIE	Parallel Implicit Elimination
GE	Gaussian elimination
OpenMP	Open Multi-Processing
GPU	Graphics processing unit
CUDA	Compute Unified Device Architecture)
EDK HW/SW	Embedded Development Kit hardware/software
P	Permutation matrix
AMD	Advanced Micro Devices
Intel	Integrated electronics.
MATLAB	Matrix laboratory
GGH	Goldreich-Goldwasser-Halevi encryption scheme

2.2. List of symbols

$\mathbb{R}$	The set of real numbers
$Det(B)$	Determinant of matrix B
$\geq$	greater than or equal to
$\leq$	less than or equal to
$\neq$	Not equal to
B^T	Transpose of matrix B
$\sum_{i=1}^{n} X_i$	The finite sum of spaces
$\prod_{i=1}^{n} X_i$	The finite product of spaces
$\lfloor B \rfloor$	Floor of B
$\lceil B \rceil$	Ceiling of B
$\|\|B\|\|$	Norm of matrix B
H	Hourglass matrix
$H_{T(nz)}$	Total number of nonzero entries in H
$H_{T(z)}$	Total number of zero entries in H
f_m	Filanz submatrix of H
$\lim_{n\to\infty} f(n)$	Limit of a function as n tends to infinity
λ	Lambda
G	Graph
V	Vertex
E	Edge
$\mathscr{E}_A(G)$	Energy of a graph
$\|B\|$	Modulus of B
F_2	Friendship graph
$\mathscr{G}$	Mixed hourglass graph
$M(\mathscr{G})$	Mixed hourglass-adjacency matrix
$\mathscr{E}_M(\mathscr{G})$	Mixed energy of a mixed hourglass graph
A_k	Number of arcs in $\mathscr{G}$.

3. Quadrant interlocking factorization

Quadrant interlocking factorization (*QIF*) or butterfly factorization of nonsingular matrix B was coined by Evans and Hatzopoulos [7]. In 1979, Evans and Hatzopoulos [10, 11] gave details of the factorization as an alternative to *LU* factorization and the avoidance of breakdown of *QIF* algorithm. The factorization breaks up matrices to structural forms which are then regrouped and solved as sub-blocks [12]. *QIF* is known for the adaptability of its direct method to solve systems of linear equations. Thus, the factorization gives rise to the use of Parallel implicit elimination (*PIE*) for the solution of linear system to simultaneously compute two matrix elements (two columns at a time) for parallel implementation, unlike Gaussian elimination (*GE*) which computes one column at a time [13]. The stability of *QIF* comes from the centro-nonsingular matrix (central submatrices are nonsingular) which is far reliable than any other type of factorization [8].

3.1. WZ factorization

W-matrix (bow-tie matrix) and Z-matrix exists together in WZ factorization of nonsingular matrix B [14, 15]. Z-matrix and W-matrix are well-known as interlocking quadrant factors of B having butterfly shape of the form

$$W = \begin{bmatrix} 1 & & & & & & & \circ \\ \bullet & 1 & & & & & & \bullet \\ \bullet & \circ & 1 & & & & \circ & \bullet \\ \bullet & \circ & \circ & 1 & & \circ & \circ & \bullet \\ \bullet & \circ & \bullet & & 1 & \bullet & \circ & \bullet \\ \bullet & \bullet & & & & 1 & \bullet & \bullet \\ \bullet & & & & & & 1 & \bullet \\ \circ & & & & & & & 1 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \circ & \circ & \circ & \circ & \circ & \bullet & \\ & & \circ & \circ & \circ & \bullet & & \\ & & & \circ & \bullet & & & \\ & & & \bullet & \circ & & & \\ & & \bullet & \circ & \circ & \circ & & \\ & \bullet & \circ & \circ & \circ & \circ & \circ & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix}$$

such that

$$B = WZ. \tag{1}$$

Z-matrix and W-matrix of order $n\,(n \geq 3)$ are generally defined as [10, 16]

$$Z = \begin{cases} (\underbrace{0,\dots,0}_{i-1}, z_{i,i},\dots,z_{i,n-i+1},0,\dots,0)^T, & i = 1,\dots,\lfloor\frac{(n+1)}{2}\rfloor; \\ (\underbrace{0,\dots,0}_{n-i}, z_{i,n-i+1},\dots,z_{i,i},0,\dots,0)^T, & i = \lfloor\frac{(n+1)}{2}\rfloor+1,\dots,n \end{cases} \tag{2}$$

$$W = \begin{cases} (1, \underbrace{0, ..., 0}_{n-1}); \\ (w_{i,1}, ..., w_{i,i-1}, 1, \underbrace{0, ..., 0}_{n-2i+1}, w_{i,n-i+2}, ..., w_{i,n}), \quad i = 2, ..., \lfloor \frac{(n+1)}{2} \rfloor; \\ (w_{i,1}, ..., w_{i,n-i}, \underbrace{0, ..., 0}_{2i-n-1}, 1, w_{i,i+1}, ..., w_{i,n}), \quad i = \lfloor \frac{(n+1)}{2} \rfloor + 1, ..., n-1; \\ (\underbrace{0, ..., 0}_{n-1}, 1). \end{cases} \tag{3}$$

In WZ factorization, there are $\sum_{k=1}^{\lfloor \frac{n}{2}-1 \rfloor} (n-2k)$ of 2×2 linear systems to be solved which account for the elements in W-matrix and Z-matrix [17]. The direct method to solve the linear systems of QIF under the nonsingularity constraint presumed for their determinants solely depends on a conventional method called Cramer's rule. The unique solution $(x_1, x_2, ..., x_n)$ provided by Cramer's rule [18] to the system

$$Bx_i = c, \tag{4}$$

is

$$x_i = \frac{det(B_{i|c})}{det(B)}, \tag{5}$$

where, $det(B) \neq 0,\ B = (b_{i,j})\ 1 \leq i, j \leq n,\ x_i = (x_1, ..., x_n)^T,\ c = (c_1, ..., c_n)^T;\ \ x, c \in \mathbb{R}^n,\ B \in \mathbb{R}^{n \times n}$. $B_{i|c}$ is the matrix obtained from B by substituting the vector column of c to the ith column of B, for $i = 1, 2, ..., n$.

3.1.1. WZ factorization algorithm

For the WZ factorization, matrix B is nonsingular. However, if central matrix of B is singular then interchange columns or rows of the matrix by suitable permutation to avoid breakdown of the factorization method, else the factorization breakdown. The establishment of elements in W-matrix (with 1's in its main diagonal and 0's in the antidiagonal), column ith and $(n-1)$th are obtained by solving simultaneous equation via Cramer's rule which requires matrix B to be successfully updated and this update changes matrix B to Z-matrix [19, 20]. The matrix update of WZ factorization indicates the most time consuming part of the factorization. The steps to obtain Z-matrix is as follows:

Step 1: Let $B^{(0)} = Z^{(0)}$for initial update and obtain the first and last rows of Z-matrix as $b^{(0)}_{1,1} = z^{(0)}_{1,1},\ b^{(0)}_{1,i} = z^{(0)}_{1,i},\ b^{(0)}_{1,n} = z^{(0)}_{1,n},\ b^{(0)}_{n,1} = z^{(0)}_{n,1},\ b^{(0)}_{n,i} = z^{(0)}_{n,i},\ b^{(0)}_{n,n} = z^{(0)}_{n,n}$, where $i = 2, ..., n-1$. Now, we compute $w^{(1)}_{i,1}$ and $w^{(1)}_{i,n}$ from $(n-2)$ sets of 2×2 linear system in Equation (6) of matrix B using Cramer's rule

$$\begin{cases} z^{(0)}_{1,1} w^{(1)}_{i,1} + z^{(0)}_{n,1} w^{(1)}_{i,n} = -z^{(0)}_{i,1} \\ z^{(0)}_{1,n} w^{(1)}_{i,1} + z^{(0)}_{n,n} w^{(1)}_{i,n} = -z^{(0)}_{i,n}. \end{cases} \tag{6}$$

The values of $w^{(1)}_{i,1}$ and $w^{(1)}_{i,n}$ are put in matrix form as:

$$W^{(1)} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ w^{(1)}_{2,1} & 1 & \ddots & \vdots & w^{(1)}_{2,n} \\ \vdots & 0 & \ddots & 0 & \vdots \\ w^{(1)}_{n-1,1} & \vdots & \ddots & 1 & w^{(1)}_{n-1,n} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Step 2: We update matrix B (let $B^{(1)} = Z^{(1)}$ for the first update) and compute:

$$Z^{(1)} = W^{(1)} Z^0. \tag{7}$$

We, therefore, proceed analogously for the inner square matrices of $Z^{(1)}$ of size $(n-2)$ and so on.

Step 3: Next, we compute $w_{i,k}^{(k)}$ and $w_{i,n-k+1}^{(k)}$ from Equation (8) by solving its 2×2 linear equations using Cramer's rule, where $k = 1,2,...,\frac{n}{2}-1$; $i = k+1,...,n-k$.

$$\begin{cases} z_{k,k}^{(k-1)} w_{i,k}^{(k)} + z_{n-k+1,k}^{(k-1)} w_{i,n-k+1}^{(k)} = -z_{i,k}^{(k-1)} \\ z_{k,n-k+1}^{(k-1)} w_{ik}^{(k)} + z_{n-k+1,n-k+1}^{(k-1)} w_{i,n-k+1}^{(k)} = -z_{i,n-k+1}^{(k-1)}. \end{cases} \tag{8}$$

Then, we put the values of $w_{i,k}^{(k)}$ and $w_{i,n-k+1}^{(k)}$ in a matrix form as:

$$W^{(k)} = \begin{bmatrix} 1 & & & & \\ w_{k+1,k}^{(k)} & \ddots & & & w_{k+1,n-k+1}^{(k)} \\ \vdots & & \ddots & & \vdots \\ w_{n-1,k}^{(k)} & & & \ddots & w_{n-k,n-k+1}^{(k)} \\ & & & & 1 \end{bmatrix}.$$

Step 4: We further compute for kth such successful steps as:

$$Z^{(k)} = W^{(k)} Z^{(k-1)}. \tag{9}$$

To arrive at the Z-matrix, we let $Z^{(k)} = Z$. Thus,

$$Z = \begin{bmatrix} z_{1,1}^{(0)} & z_{1,2}^{(0)} & \cdots & \cdots & \cdots & z_{1,n-1}^{(0)} & z_{1,n}^{(0)} \\ 0 & \ddots & \vdots & \cdots & \vdots & \iddots & 0 \\ 0 & 0 & z_{k,k}^{(k-1)} & & z_{k,n+1-k}^{(k-1)} & 0 & 0 \\ \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & z_{n+1-k,k}^{(k-1)} & & z_{n+1-k,n+1-k}^{(k-1)} & 0 & 0 \\ 0 & \iddots & \vdots & \cdots & \vdots & \ddots & 0 \\ z_{n,1}^{(0)} & z_{n,2}^{(0)} & \cdots & \cdots & \cdots & z_{n,n-1}^{(0)} & z_{n,n}^{(0)} \end{bmatrix}.$$

A complete one-stage in WZ factorization is when $Z^{(k-1)}$ is computed. However, the factorization requires $\lfloor \frac{(n-1)}{2} \rfloor$ stages to compute all the elements of the matrix W and Z [10]. After the algorithm of WZ factorization is established, the following theorems were put forward:

Theorem 3.1. *Factorization Theorem [8]. Let $B \in R^{n\times n}$ be a nonsingular matrix that has a unique QIF factorization, then $B = WZ$ if and only if the submatrices of B are invertible.*

Theorem 3.2. *[8]. If $B \in R^{n\times n}$ is nonsingular matrix, then there exist a row permutation matrix P for QIF to be carried out with pivoting such that $PB = WZ$.*

Corollary 3.3. *[8]. Every symmetric positive definite and strictly diagonally dominant matrix has a QIF.*

Theorem 3.4. *[21]. Let B be nonsingular tri-diagonal diagonally dominant, then its factored Z-matrix from QIF factorization is also tri-diagonal diagonally dominant.*

Due to its uniqueness, WZ factorization exists for every nonsingular matrix often with pivoting [22, 8]. Pivoting results in swapping rows or columns in a matrix or by multiplying the matrix with permutation matrix which improves the numerical stability of WZ factorization [1]. WZ factorization will not fail without pivoting if the matrix is real symmetric, positive definite or diagonally dominant, see [23, 8, 24]. The matrix norm of WZ factorization is the Frobenius norm given as [25]

$$\|B - WZ\|_F = \sqrt{\left(\sum_{i=1}^{n}\sum_{j=1}^{n} |b_{i,j} - w_{i,j} z_{i,j}|\right)}. \tag{10}$$

The numerical accuracy $\left(-log_{10}\frac{\|B-WZ\|}{n\cdot\|B\|}\right)$ of WZ factorization based on the relative error depends on the matrix norms [16]. Thus, the efficiency of WZ factorization depends on an efficacious use of the memory echelon (processors array). If there is no sufficient fast memory, then the processor will create waiting time for the data and thereby reducing its efficiency [26].

3.1.2. Importance of WZ factorization

WZ factorization has been applied in modeling of Markov chains aside its parallelization usage [27]. WZ factorization offers parallelization in solving both sparse and dense linear system to enhance performance using OpenMP, GPU, CUDA or EDK HW/SW codesign architecture [28, 29, 12]. Then, Yalamov [24] presented that WZ factorization is faster on computer with a parallel architecture than any other matrix factorization methods. The factorization has been applied in scientific computing - especially in science and engineering, see [30, 31, 27, 32, 33]. The block WZ factorization is discussed in [34, 15, 23, 26] where the Z-matrix is divided into r^2 block each of the size $s \times s$ and $n = r \times s$.

Theorem 3.5. *[26] The block WZ factorization exists if the matrix B has a strict dominant diagonal.*

Z-matrix of even order (also applicable to odd order) can be partitioned to form structured Z_{system} of 2×2 triangular block matrices which is defined as

$$Z_{system} = \begin{cases} \begin{bmatrix} Z_{1,1} & Z_{1,2} \\ Z_{2,1} & Z_{2,2} \end{bmatrix} & \text{if } n \text{ is even;} \\ \begin{bmatrix} Z_{1,1} & x_1 & Z_{1,2} \\ 0 & x & 0 \\ Z_{2,1} & x_2 & z_{2,2} \end{bmatrix} & \text{if } n \text{ is odd.} \end{cases} \tag{11}$$

Each block contains $\frac{n}{2} \times \frac{n}{2}$ block size if n is even dimension while additional column vector, $\tilde{x}$, position at $\frac{n+1}{2}$th column of the matrix if n is odd dimension [26]. This column vector, $\tilde{x}$, can be further partitioned into x_1, x and x_2. Then, the Schur complement of a matrix block in Equation (11) is defined as follows

$$Z_{system}/Z_{1,1} = Z_{2,2} - Z_{2,1}Z_{1,1}^{-1}Z_{1,2}. \tag{12}$$

Theorem 3.6. *[26] If the Z_{system} and the matrix $Z_{1,1}$ are invertible then the matrix $\left(Z_{2,2} - Z_{2,1}Z_{1,1}^{-1}Z_{1,2}\right)$ is a lower triangular invertible matrix.*

3.1.3. WZ factorization versus LU factorization

WZ factorization proves to be better on Intel processors than on AMD processors [16]. Even though WZ factorization and *LU factorization* have similar computational complexity, the WZ factorization still shown to be better than LU factorization (except block LU factorization) irrespective of the version of MATLAB or the number of processors used [35, 16]. However, for a uniprocessor, WZ factorization does not exhibit any advantage over LU factorization since every step performed is in serial [12]. While LU factorization performs elimination in serial with $n-1$ steps, WZ factorization executes components in parallel with $\frac{n}{2}$ steps if n is even or $\frac{n-1}{2}$ steps if n is odd. WZ factorization simultaneously computes two matrix elements (two columns at a time), unlike LU factorization which computes one column at a time [13, 12, 36]. Unlike WZ factorization, LU factorization is not unique but block LU factorization with higher diagonal blocks gives similar analytic result as WZ factorization [37]. For larger matrices, the stimulation time on the multiprocessor show that the WZ factorization is faster than LU factorization appears to be 20% for all values of processor [38, 39].

Aside not being better than WZ factorization on parallel computer or mesh multiprocessors, LU factorization does not account for the property of centrosymmetric matrix in its factors [40]. For sparse matrices, LU and WZ factorization generate approximately similar number of non-zero elements [41]. Though, LU factorization relies on leading principal submatrices $\left([b_{ij}]_{i,k=1}^{k}, l = 1, ..., n\right)$ whereas WZ factorization relies on nonsingular central submatrices $\left([b_{jk}]_{j,k=l}^{n+1-l}, l = 1, ..., \left[\frac{n+1}{2}\right]\right)$ [15]. Based on the form of matrices, incomplete WZ preconditioning gives better results than the incomplete LU factorization [26]. Although some parts of the equation in LU factorization that consist many loops can be parallelized, the WZ factorization is uniquely known for its ability to offer parallelization. Even if WZ factorization and LU factorization are both implemented on OpenMP, the WZ factorization performs better in execution time than LU factorization when the number of thread increases [42, 43].

3.2. WH factorization

Demeure [44] first posited the term hourglass matrix in describing a matrix derived from factorizing a square matrix via quadrant interlocking factorization or bowtie-hourglass factorization. It was further elucidated that hourglass matrix is synonymous to Z-matrix which can be partitioned into blocks structured Z-system [15, 28]. Unfortunately, there are changes in structure of Z-matrix from WZ factorization which depend on the type of matrix (Toeplitz, Hankel, Hermitian, centrosymmetric, diagonally dominant or tridiagonal matrix) being factorized. Nevertheless, Z-matrix may not always imply hourglass matrix nor their applications are always indistinguishable. Consequently, the notion of sameness between hourglass matrix and Z-matrix was gradually dropped over time without a cogent reason. Recently, Babarinsa and Kamarulhaili [45] gave meticulous details of hourglass matrix, denoted as (H-matrix), and its quadrant interlocking factorization by restricting the computed entries of the factorization to be nonzero in accordance with the shape of hourglass device.

Definition 3.1. *[45] Let H be an hourglass matrix of order $n\,(n \geq 3)$, then hourglass matrix is defined as*

$$H = \begin{cases} h_{i,j} & 1 \leq i \leq \lfloor \frac{(n+1)}{2} \rfloor, \; i \leq j \leq n+1-i; \\ h_{i,j} & \lceil \frac{(n+2)}{2} \rceil \leq i \leq n, \; n+1-i \leq j \leq i; \\ 0 & otherwise, \end{cases} \tag{13}$$

where $h_{i,j} \in \mathbb{R}$ are strictly nonzero elements.

In other words, hourglass matrix is a nonsingular matrix of order $n\,(n \geq 3)$ with nonzero entries from the ith to the $(n-i+$

1) element of the ith and $(n-i+1)$ row of the matrix, otherwise 0's, for $i=1,2,...,\lfloor\frac{n+1}{2}\rfloor$. To buttress the shape of hourglass matrix, Figure 1 illustrates the structural comparison between the hourglass device and hourglass matrix with nonzero elements denoted with black dots, otherwise 0's.

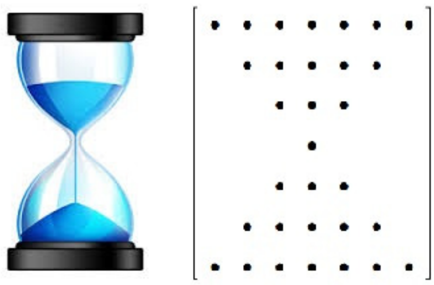

Figure 1. Structural comparison between hourglass device and hourglass matrix.

The factorization algorithm to obtain hourglass matrix is known as WH factorization. Like the factorization of Z-matrix, the factorization of hourglass matrix requires W-matrix (see the form of W-matrix in equation 3) to be computed during the factorization process. During WH factorization of nonsingular matrix B, H-matrix exists together with W-matrix, such that

$$B = WH. \tag{14}$$

The matrix norm, numerical accuracy and efficiency of WH factorization algorithm is similar to WZ factorization. The factorization of hourglass matrix and Z-matrix are quite similar yet the factorization for hourglass matrix restricts the computed entries to be nonzero at every stage during the factorization. However, the QIF of hourglass matrix specifies the number of times row-interchange can be done at each stage of the factorization if the computed entries yield zero, else the factorization breakdown.

3.2.1. WH factorization algorithm

The sequential steps for the factorization are as follows [45]:

Step 1: Let $B=H^{(0)}$ for initial update and check if the first row $\left(h^{(0)}_{1,j}\right)$ and last row $\left(h^{(0)}_{n,j}\right)$ of $H^{(0)}$ contains zero. If $h^{(0)}_{1,j}=0$ or $h^{(0)}_{n,j}=0$, then use suitable row-interchange in $H^{(0)}$, where $j=1,2,...,n$. Then, we compute $w^{(1)}_{i,1}$ and $w^{(1)}_{i,n}$ in Equation (15) from matrix $H^{(0)}$ by solving 2×2 system of linear equations via Equation 5

$$\begin{cases} h^{(0)}_{1,1}w^{(1)}_{i,1}+h^{(0)}_{n,1}w^{(1)}_{i,n}=-h^{(0)}_{i,1}; \\ h^{(0)}_{1,n}w^{(1)}_{i,1}+h^{(0)}_{n,n}w^{(1)}_{i,n}=-h^{(0)}_{i,n}, \end{cases} \tag{15}$$

Whenever $h^{(0)}_{n,n}h^{(0)}_{1,1}-h^{(0)}_{1,n}h^{(0)}_{n,1}=0$ use suitable row-interchange to avoid factorization breakdown. Then the values of $w^{(1)}_{i,1}$ and $w^{(1)}_{i,n}$ can be written in W-matrix as:

$$W^{(1)}=\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ w^{(1)}_{2,1} & 1 & \cdots & \iddots & w^{(1)}_{2,n} \\ \vdots & 0 & \ddots & 0 & \vdots \\ w^{(1)}_{n-1,1} & \iddots & \cdots & 1 & w^{(1)}_{n-1,n} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Step 2: We, therefore, update matrix $H^{(0)}$ to $H^{(1)}$ for the first update by evaluating its entries as

$$h^{(1)}_{i,j}=h^{(0)}_{i,j}+w^{(1)}_{i,1}h^{(0)}_{1,j}+w^{(1)}_{i,n}h^{(0)}_{n,j}, \tag{16}$$

If one of the computed entry $h^{(1)}_{2,j}=0$ or $h^{(1)}_{n-1,j}=0$ in Equation (16), then apply row-interchange in $H^{(1)}$ at $h^{(1)}_{i,j}$ for $i=2,...,n-1$ and $j=1,...,n$ in no more than $(n-4)$ times, else the factorization breakdown.

Step 3: Now, we compute $w^{(k)}_{i,k}$ and $w^{(k)}_{i,n-k+1}$ from matrix $H^{(k-1)}$ by solving 2×2 linear systems in Equation (17) to generalize for every update of $H^{(k)}$ and proceed similarly for the inner square matrices of size $(n-2k)$ and so on. That is,

$$\begin{cases} h^{(k-1)}_{k,k}w^{(k)}_{i,k}+h^{(k-1)}_{n-k+1,k}w^{(k)}_{i,n-k+1}=-h^{(k-1)}_{i,k}; \\ h^{(k-1)}_{k,n-k+1}w^{(k)}_{i,k}+h^{(k-1)}_{n-k+1,n-k+1}w^{(k)}_{i,n-k+1}=-h^{(k-1)}_{i,n-k+1}, \end{cases} \tag{17}$$

where $k=1,2,...,\lfloor\frac{n-1}{2}\rfloor$; $i=k+1,...,n-k$. Whenever $h^{(k-1)}_{n-k+1,n-k+1}h^{(k-1)}_{k,k}-h^{(k-1)}_{n-k+1,k}h^{(k-1)}_{k,n-k+1}=0$ use suitable row-interchange to avoid factorization breakdown. Then, we put the values $w^{(k)}_{i,k}$ and $w^{(k)}_{i,n-k+1}$ in a W-matrix of the form

$$W^{(k)}=\begin{bmatrix} 1 & & & & & 0 \\ & \ddots & & & \iddots & \\ & & 1 & 0 & & \\ & & w^{(k)}_{k+1,k} \ \ddots & \iddots \ w^{(k)}_{k+1,n-k+1} & & \\ & & \vdots \quad \ddots & \vdots & & \\ & & w^{(k)}_{n-1,k} \ \iddots & \ddots \ w^{(k)}_{n-k,n-k+1} & & \\ & & 0 & 1 & & \\ & \iddots & & & \ddots & \\ 0 & & & & & 1 \end{bmatrix}.$$

Step 4: We finally compute for kth steps of $h^{(k)}_{i,j}$ as:

$$h^{(k)}_{i,j}=h^{(k-1)}_{i,j}+w^{(k)}_{i,k}h^{(k-1)}_{k,j}+w^{(k)}_{i,n-k+1}h^{(k-1)}_{n-k+1,j}, \tag{18}$$

where $j=k+1,...,n-k$. From Equation (18), if one of the computed entries is zero, then apply possible row-interchange in no more than $(n-2k)$ times in $H^{(k-1)}$ and re-factorize, else the factorization breakdown to produce H^k (H-matrix). After the successful kth steps we get hourglass matrix ($H^{(k)}=H$) of the form:

$$H=\begin{bmatrix}
h^{(0)}_{1,1} & h^{(0)}_{1,2} & h^{(0)}_{1,3} & \cdots & \cdots & \cdots & \cdots & \cdots & h^{(0)}_{1,n-2} & h^{(0)}_{1,n-1} & h^{(0)}_{1,n}\\
0 & h^{(1)}_{2,2} & h^{(1)}_{2,3} & \cdots & \cdots & \cdots & \cdots & \cdots & h^{(1)}_{2,n-2} & h^{(1)}_{2,n-1} & 0\\
0 & 0 & h^{(2)}_{3,3} & \cdots & \cdots & \cdots & \cdots & \cdots & h^{(2)}_{3,n-2} & 0 & 0\\
\vdots & 0 & 0 & \ddots & \vdots & \vdots & \vdots & \iddots & 0 & 0 & \vdots\\
\vdots & \vdots & \vdots & & h^{(k-1)}_{k,k} & \cdots & h^{(k-1)}_{k,n-k+1} & & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \vdots & & \vdots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & & h^{(k-1)}_{n-k+1,k} & \cdots & h^{(k-1)}_{n-k+1,n-k+1} & & \vdots & \vdots & \vdots\\
\vdots & 0 & 0 & \iddots & \vdots & \vdots & \vdots & \ddots & 0 & 0 & \vdots\\
0 & 0 & h^{(2)}_{n-2,3} & \cdots & \cdots & \cdots & \cdots & \cdots & h^{(2)}_{n-2,n-2} & 0 & 0\\
0 & h^{(1)}_{n-1,2} & h^{(1)}_{n-1,3} & \cdots & \cdots & \cdots & \cdots & \cdots & h^{(1)}_{n-1,n-2} & h^{(1)}_{n-1,n-1} & 0\\
h^{(0)}_{n,1} & h^{(0)}_{n,2} & h^{(0)}_{n,3} & \cdots & \cdots & \cdots & \cdots & \cdots & h^{(0)}_{n,n-2} & h^{(0)}_{n,n-1} & h^{(0)}_{n,n}
\end{bmatrix}.$$

If there exists row-interchange at any stage k that yields permutation matrix P, then

$$H=\left(W^{(k-1)}P^{(k-1)}W^{(k-2)}P^{(k-2)}\ldots W^{(2)}P^{(2)}W^{(1)}P^{(1)}\right)^{-1}B. \tag{19}$$

Recall that $k=1,2,...,\lfloor\frac{n-1}{2}\rfloor$ and that there are $\lfloor\frac{n-1}{2}\rfloor$ stages in the factorization. From every successful loops $i,j=k+1,k+2,...,n-k$ for each stage, there are $(n-2k)$ simultaneous equations each to be solved in $(n-2k)$ times during the factorization using Cramer's rule. To avoid breakdown at its filanz submatrices (see Definition 3.3), there must be row-interchange at every stage of the factorization process. This ensures that the 2×2 submatrix has the least condition number adopting any matrix norm. The overall time of WH factorization algorithm may increase if there is row-interchange at every stage of the factorization process due to the moving and sorting of data in and out of the processor.

3.2.2. On hourglass matrix

Proposition 3.7. *[45] Let H, $H_{T(nz)}$ and $H_{T(z)}$ be an hourglass matrix of order $n\,(n\geq 3)$, the total number of nonzero entries, and the total number of zero entries in hourglass matrix respectively. Then,*

$$H_{T(nz)}=\frac{n^2+2n-\left|(n+1)\bmod 2-1\right|}{2} \tag{20}$$

and

$$H_{T(z)}=\frac{n^2-2n+\left|(n+1)\bmod 2-1\right|}{2}. \tag{21}$$

Definition 3.2. *[45] Filanz submatrix, denoted as $f_m^{1\leq i\leq\lceil\frac{n-1}{2}\rceil}$, is a 2×2 non-singular matrix obtained by taking the first and the last nonzero elements of the ith and $(n+1-i)$th row of H-matrix given as*

$$f_m^{1\leq i\leq\lceil\frac{n-1}{2}\rceil}=\begin{bmatrix} h^{(i-1)}_{i,i} & h^{(i-1)}_{i,n+1-i}\\ h^{(i-1)}_{n+1-i,i} & h^{(i-1)}_{n+1-i,n+1-i}\end{bmatrix}_{1\leq i\leq\lceil\frac{n-1}{2}\rceil} \tag{22}$$

Definition 3.3. *[45] Epicenter element, denoted as $h_{\frac{n+1}{2},\frac{n+1}{2}}$ is the nonzero element located at the intersection of $(\frac{n+1}{2})$ row and $(\frac{n+1}{2})$ column of hourglass matrix of odd order.*

Proposition 3.8. *[45] Let $det(H)$ be the determinant of hourglass matrix of order $(n\geq 3)$ Then,*

$$det\,(H)=\begin{cases}\displaystyle\prod_{i=1}^{\lceil\frac{n-1}{2}\rceil}\begin{vmatrix} h^{(i-1)}_{i,i} & h^{(i-1)}_{i,n+1-i}\\ h^{(i-1)}_{n+1-i,i} & h^{(i-1)}_{n+1-i,n+1-i}\end{vmatrix} & \text{if } n \text{ is even;}\\[2ex] \displaystyle h_{\left(\frac{n+1}{2},\frac{n+1}{2}\right)}\prod_{i=1}^{\lceil\frac{n-1}{2}\rceil}\begin{vmatrix} h^{(i-1)}_{i,i} & h^{(i-1)}_{i,n+1-i}\\ h^{(i-1)}_{n+1-i,i} & h^{(i-1)}_{n+1-i,n+1-i}\end{vmatrix} & \text{if } n \text{ is odd.}\end{cases} \tag{23}$$

The determinant of matrix B can be computed as

$$det(B)=det\left(W^{(k-1)}\cdot P^{(k-1)}\cdot\ldots\cdot W^{(2)}\cdot P^{(2)}\cdot W^{(1)}\cdot P^{(1)}\cdot\right)^{-1}H. \tag{24}$$

where

$$det\left(W^{(k-1)}\cdot\ldots\cdot W^{(2)}\cdot W^{(1)}\right)^{-1}=1$$

and

$$det\left(P^{(k-1)}\cdot\ldots\cdot P^{(2)}\cdot P^{(1)}\right)^{-1}=(-1)^{p_n}.$$

But

$$(-1)^{p_n}=\begin{cases}1 & \text{if even number of rows are interchanged,}\\ -1 & \text{if odd number of rows are interchanged.}\end{cases}$$

Therefore,

$$det(B)=(-1)^{p_n}det\,(H) \tag{25}$$

where p_n is the total number of permutation matrix (successful row interchange) occurs in the factorization.

Partitioning of hourglass matrix of order n $(n>3)$ into 2×2 block triangular matrices with each block containing $\lfloor\frac{n}{2}\rfloor\times\lfloor\frac{n}{2}\rfloor$ matrices is called H_{system}. That is

$$H_{system}=\begin{cases}\begin{bmatrix}H_{1,1} & H_{1,2}\\ H_{2,1} & H_{2,2}\end{bmatrix} & \text{if } n \text{ is even;}\\ \begin{bmatrix}H_{1,1} & x_1 & H_{1,2}\\ 0 & x & 0\\ H_{2,1} & x_2 & H_{2,2}\end{bmatrix} & \text{if } n \text{ is odd.}\end{cases} \tag{26}$$

Where

$$H_{1,2}=\begin{cases}h_{ij}, & 1\le i\le\lceil\frac{n-1}{2}\rceil,\ \lfloor\frac{n+3}{2}\rfloor\le j\le n+1-i;\\ 0, & otherwise.\end{cases}\qquad H_{2,2}=\begin{cases}h_{ij}, & \lfloor\frac{n+3}{2}\rfloor\le i\le n,\ n+1-i\le j\le\lceil\frac{n-1}{2}\rceil;\\ 0, & otherwise.\end{cases}$$

$$H_{1,1}=\begin{cases}h_{ij}, & 1\le i\le\lceil\frac{n-1}{2}\rceil,\ i\le j\le\lceil\frac{n-1}{2}\rceil;\\ 0, & otherwise.\end{cases}\qquad H_{2,1}=\begin{cases}h_{ij}, & \lfloor\frac{n+3}{2}\rfloor\le i\le n,\ \lfloor\frac{n+3}{2}\rfloor\le j\le i;\\ 0, & otherwise.\end{cases}$$

$$x_1=\left\{h_{ij},\quad 1\le i\le\frac{n-1}{2},\ j=\frac{n+1}{2}\ .\right.$$

$$x_2=\left\{h_{ij},\quad \frac{n+3}{2}\le i\le n,\ j=\frac{n+1}{2}\ .\right.$$

$$x=\left\{h_{ij},\quad i=\frac{n+1}{2},\ j=i\ .\right.$$

Theorem 3.9. *[46] Schur complement exists in H_{system} only if H-matrix is nonsingular.*

3.3. Comparison between WZ factorization (or Z-matrix) and WH factorization (or H-matrix)

The WZ factorization is possible provided the submatrices of the nonsingular matrix are invertible, while WH factorization does not only depend on the invertibility of the submatrices but also that the elements in the first row and in the last row of its submatrix are nonzero. If assume the entries $h_{i,j}$ is analogous to $z_{i,j}$, then Z-matrix will imply hourglass matrix provided that the computed $z_{i,j}^{(k-1)}$ and $z_{n,j}^{(k-1)}$ are strictly nonzero, for $k=1,2,...,\lfloor\frac{n-1}{2}\rfloor$. However, the entries of Z-matrix are unbound to be nonzero. Then it is obvious that it will no longer be an hourglass matrix if one of its strictly nonzero elements is replaced with zero. In general, every H-matrix is a Z-matrix but the converse is not true, depicted in Figure 2.

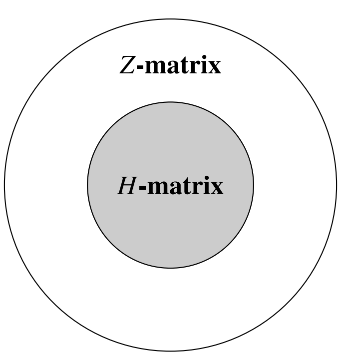

Figure 2. H-matrix as a subset of Z-matrix.

The WZ factorization exists for every nonsingular matrix often with pivoting while WH factorization may fail to exist even if the matrix is nonsingular. Unlike the factorization of Z-matrix, the factorization of an hourglass matrix from a nonsingular matrix may not necessarily be from a symmetric positive definite or diagonally dominant matrix but definitely not from a tridiagonal matrix. WZ factorization may work with large dimension sparse matrices whereas WH factorization will not. In H_{system}, each block has specific number of zero and nonzero entries, unlike Z_{system}.

3.3.1. Unique properties of hourglass matrix

The entries in H-matrix are linearly independent which make the matrix nonsingular. The transpose of hourglass matrix does not retain the shape of the matrix but rather form a bowtie matrix or butterfly matrix. Inverse and nth root of hourglass matrix is again hourglass matrix. The minimum order of hourglass matrix is 3 and the matrix cannot be a symmetric. Regardless of order of hourglass matrix, the total number of zero entries is even. Besides, hourglass matrix has minimum matrix density of 0.5 as $\lim_{n\to\infty} \frac{\frac{n^2+2n-\left|(n+1) \bmod 2-1\right|}{2}}{n^2}$.

3.3.2. Application of WH factorization and hourglass matrix

WH factorization has great tendency in usage than WZ factorization. First, though with little evidence it has been proposed that the linearly independent columns of hourglass matrix forming the basis vectors of a lattice will make it suitable for lattice-base cryptography, especially in GGH - Goldreich-Goldwasser-Halevi encryption scheme, see [47, 46]. The usage of hourglass matrix is expected to be able to reduce the size of bases. This reduction will allow the GGH Scheme to be implemented in higher lattice dimension while still being able to be efficient and practical, and the generation of hourglass matrix can be executed in polynomial time. Lastly, unlike Z-matrix, hourglass matrix has be represented as weighted mixed graph called mixed hourglass graph [48].

We know that a simple graph $G = (V,E)$ is an ordered pair consisting of a set of vertices $V = \{v_1, v_2, ..., v_n\}$ and a set of undirected edges $E = \{e_1, e_2, ..., e_n\}$, no loops nor multiple edges permitted [49, 50]. A mixed graph $G = (V,E,A)$ is an ordered triple consisting of a set of vertices $V = v_1, v_2, ..., v_n$, a set of undirected edges $E = \{e_1, e_2, ..., e_n\}$, and a set of directed arcs A [51]. Now, an hourglass graph (butterfly graph) is a planar undirected graph formed by at least two triangles intersecting in a single vertex, especially from 5-vertex graph of two k_3's or from friendship graph F_2, see for examples [52, 53, 54]. However, a mixed hourglass graph is a mixed complete graph coined from the name of its mixed adjacency matrix which is obtained from hourglass matrix [55].

Definition 3.4. *[48] A mixed hourglass graph $\mathscr{G} = (V,E,A)$ is an ordered triple consisting of a set of vertices $V = \{v_1, v_2, ..., v_n\}$, a set of undirected edges $E = \{e_1, e_2, ..., e_n\}$, and a set of directed arcs A.*

Definition 3.5. *[55] A mixed hourglass-adjacency matrix $M(\mathscr{G})$ of a mixed hourglass graph $\mathscr{G}$ is the $n\times n (n \geq 3)$ matrix $M(\mathscr{G})_{n\times n} = (h_{i,j})_{n\times n}$ defined by*

$$M(\mathscr{G}) = \begin{cases} 1 & \text{if } v_i v_j \text{ is an edge ;} \\ -1 & \text{if } v_i, v_j \text{ is an arc;} \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

Lemma 3.10. *[55] For every mixed hourglass graph $\mathscr{G}$ of order n, the number of undirected edges m is*

$$m = \frac{n-\beta}{2}, \tag{28}$$

where $\beta = \left|(n+1) \bmod 2 - 1\right|$.

Theorem 3.11. *[55] Let $\mathscr{E}_M(\mathscr{G})$ be the mixed energy of a mixed hourglass graph $\mathscr{G}$ of order n and $\lambda_i(\mathscr{G}) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ be the mixed eigenvalues of a mixed hourglass-adjacency matrix $M(\mathscr{G})$. Then*

$$\mathscr{E}_M(\mathscr{G}) = \sum_{i=1}^{n} |\lambda_i(\mathscr{G})| = n - \beta, \tag{29}$$

where $\beta = \left|(n+1) \bmod 2 - 1\right|$.

3.4. Numerical example of WZ and WH factorization

Given a dense nonsingular square matrix B of order 6, apply both WZ and WH factorization algorithm on the matrix.

$$B = \begin{bmatrix} 2 & 0 & 2 & 4 & 3 & -1 \\ 5 & 10 & -7 & 8 & 11 & 4 \\ 0 & -12 & 9 & 6 & 18 & 1 \\ -13 & 12 & 8 & -20 & 14 & 17 \\ 3 & 1 & 1 & -1 & 1 & 4 \\ 10 & 6 & 9 & -13 & 10 & 14 \end{bmatrix}.$$

3.5. WZ factorization of matrix B

Step 1: Let $b^{(0)}_{i,j} = z^{(0)}_{i,j}$, for $i, j = 1, ..., 6$. We now compute the set of 2×2 system of linear equations from

$$\begin{cases} z^{(k-1)}_{k,k} w^{(k)}_{i,k} + z^{(k-1)}_{n-k+1,k} w^{(k)}_{i,n-k+1} = -z^{(k-1)}_{i,k} \\ z^{(k-1)}_{k,n-k+1} w^{(k)}_{i,k} + z^{(k-1)}_{n-k+1,n-k+1} w^{(k)}_{i,n-k+1} = -z^{(k-1)}_{i,n-k+1}. \end{cases}$$

For $k = 1, ..., \lfloor \frac{n-1}{2} \rfloor = 2$.
If $k = 1$ then we have

$$\begin{cases} z^{(0)}_{1,1} w^{(1)}_{i,1} + z^{(0)}_{n,1} w^{(1)}_{i,n} = -z^{(0)}_{i,1} \\ z^{(0)}_{1,n} w^{(1)}_{i,1} + z^{(0)}_{n,n} w^{(1)}_{i,n} = -z^{(0)}_{i,n}. \end{cases}$$

Whenever $i = 2$, then

$$\begin{cases} z^{(0)}_{1,1} w^{(1)}_{2,1} + z^{(0)}_{6,1} w^{(1)}_{2,6} = -z^{(0)}_{2,1} \\ z^{(0)}_{1,6} w^{(1)}_{2,1} + z^{(0)}_{6,6} w^{(1)}_{2,6} = -z^{(0)}_{2,6} \end{cases} \Rightarrow$$

$$\begin{cases} 2w^{(1)}_{2,1} + 10w^{(1)}_{2,6} = & -5 \\ -w^{(1)}_{2,1} + 14w^{(1)}_{2,6} = & -4 \end{cases} \Rightarrow \begin{cases} w^{(1)}_{2,1} = & -\frac{15}{19} \\ w^{(1)}_{2,6} = & -\frac{13}{38} \end{cases}$$

Whenever $i = 3$, then

$$\begin{cases} z_{1,1}^{(0)} w_{3,1}^{(1)} + z_{6,1}^{(0)} w_{3,6}^{(1)} = -z_{3,1}^{(0)} \\ z_{1,6}^{(0)} w_{3,1}^{(1)} + z_{6,6}^{(0)} w_{3,6}^{(1)} = -z_{3,6}^{(0)} \end{cases} \Rightarrow$$

$$\begin{cases} 2w_{3,1}^{(1)} + 10w_{3,6}^{(1)} = & 0 \\ -w_{3,1}^{(1)} + 14w_{3,6}^{(1)} = & -1 \end{cases} \Rightarrow \begin{cases} w_{3,1}^{(1)} = & \frac{5}{19} \\ w_{3,6}^{(1)} = & -\frac{1}{19} \end{cases}$$

Whenever $i = 4$, then

$$\begin{cases} z_{1,1}^{(0)} w_{4,1}^{(1)} + z_{6,1}^{(0)} w_{4,6}^{(1)} = -z_{4,1}^{(0)} \\ z_{1,6}^{(0)} w_{4,1}^{(1)} + z_{6,6}^{(0)} w_{4,6}^{(1)} = -z_{4,6}^{(0)} \end{cases} \Rightarrow$$

$$\begin{cases} 2w_{4,1}^{(1)} + 10w_{4,6}^{(1)} & = 13 \\ -w_{4,1}^{(1)} + 14w_{4,6}^{(1)} & = -17 \end{cases} \Rightarrow \begin{cases} w_{4,1}^{(1)} = & \frac{176}{19} \\ w_{4,6}^{(1)} = & -\frac{21}{38} \end{cases}$$

Whenever $i = 5$, then

$$\begin{cases} z_{1,1}^{(0)} w_{5,1}^{(1)} + z_{6,1}^{(0)} w_{5,6}^{(1)} = -z_{5,1}^{(0)} \\ z_{1,6}^{(0)} w_{5,1}^{(1)} + z_{6,6}^{(0)} w_{5,6}^{(1)} = -z_{5,6}^{(0)}. \end{cases} \Rightarrow$$

$$\begin{cases} 2w_{5,1}^{(1)} + 10w_{5,6}^{(1)} = & -3 \\ -w_{5,1}^{(1)} + 14w_{5,6}^{(1)} = & -4. \end{cases} \Rightarrow \begin{cases} w_{5,1}^{(1)} = & -\frac{1}{19} \\ w_{5,6}^{(1)} = & -\frac{11}{38} \end{cases}$$

Therefore, we write the values of $w_{i,1}^{(1)}$ and $w_{i,n}^{(1)}$ in a matrix form as

$$W^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{15}{19} & 1 & 0 & 0 & 0 & -\frac{13}{38} \\ \frac{5}{19} & 0 & 1 & 0 & 0 & -\frac{1}{19} \\ \frac{176}{19} & 0 & 0 & 1 & 0 & -\frac{21}{38} \\ -\frac{1}{19} & 0 & 0 & 0 & 1 & -\frac{11}{38} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 2: We update $z_{i,j}^0$ to $z_{i,j}^1$ by computing the entries as

$$z_{i,j}^{(k)} = z_{i,j}^{(k-1)} + w_{i,k}^{(k)} z_{k,j}^{(k-1)} + w_{i,n-k+1}^{(k)} z_{n-k+1,j}^{(k-1)} \Rightarrow z_{i,j}^{(1)} = z_{i,j}^{(0)} + w_{i,1}^{(1)} z_{1,j}^{(0)} + w_{i,6}^{(1)} z_{6,j}^{(0)}.$$

When $i = 2$ and $j = 2, 3, 4, 5$, so

$$z_{2,2}^{(1)} = z_{2,2}^{(0)} + w_{2,1}^{(1)} z_{1,2}^{(0)} + w_{2,6}^{(1)} z_{6,2}^{(0)} = 10 + \left(-\frac{15}{19}\right)(0) + \left(-\frac{13}{38}\right)(6) = \frac{151}{19}$$

$$z_{2,3}^{(1)} = z_{2,3}^{(0)} + w_{2,1}^{(1)} z_{1,3}^{(0)} + w_{2,6}^{(2)} z_{6,3}^{(0)} = -7 + \left(-\frac{15}{19}\right)(2) + \left(-\frac{13}{38}\right)(9) = -\frac{443}{38}$$

$$z_{2,4}^{(1)} = z_{2,4}^{(0)} + w_{2,1}^{(1)} z_{1,4}^{(0)} + w_{2,6}^{(4)} z_{6,4}^{(0)} = 8 + \left(-\frac{15}{19}\right)(4) + \left(-\frac{13}{38}\right)(-13) = \frac{353}{38}$$

$$z_{2,5}^{(1)} = z_{2,5}^{(0)} + w_{2,1}^{(1)} z_{1,5}^{(0)} + w_{2,6}^{(3)} z_{6,5}^{(0)} = 11 + \left(-\frac{15}{19}\right)(3) + \left(-\frac{13}{38}\right)(10) = \frac{99}{19}$$

When $i = 3$ and $j = 2, 3, 4, 5$, then

$$z_{2,2}^{(1)} = z_{3,2}^{(0)} + w_{3,1}^{(1)} z_{1,2}^{(0)} + w_{3,6}^{(1)} z_{6,2}^{(0)} = -12 - (\tfrac{5}{19})(0) + \left(-\frac{1}{19}\right)(6) = -\frac{234}{19}$$

$$z_{3,3}^{(1)} = z_{3,3}^{(0)} + w_{3,1}^{(1)} z_{1,3}^{(0)} + w_{3,6}^{(1)} z_{6,3}^{(0)} = 9 - (\tfrac{5}{19})(2) + \left(-\frac{1}{19}\right)(9) = \frac{172}{19}$$

$$z_{3,4}^{(1)} = z_{3,4}^{(0)} + w_{3,1}^{(1)} z_{1,4}^{(0)} + w_{3,6}^{(1)} z_{6,4}^{(0)} = 6 - (\tfrac{5}{19})(4) + \left(-\frac{1}{19}\right)(-13) = \frac{147}{19}$$

$$z_{3,5}^{(1)} = z_{3,5}^{(0)} + w_{3,1}^{(1)} z_{1,5}^{(0)} + w_{3,6}^{(1)} z_{6,5}^{(0)} = 18 - (\tfrac{5}{19})(3) + \left(-\frac{1}{19}\right)(10) = \frac{347}{19}$$

When $i = 4$ and $j = 2, 3, 4, 5$, we have

$$z_{4,2}^{(1)} = z_{4,2}^{(0)} + w_{4,1}^{(1)} z_{1,2}^{(0)} + w_{4,6}^{(1)} z_{6,2}^{(0)} = 12 + \left(\frac{176}{19}\right)(0) + \left(-\frac{21}{38}\right)(6) = \frac{165}{19}$$

$$z_{4,3}^{(1)} = z_{4,3}^{(0)} + w_{4,1}^{(1)} z_{1,3}^{(0)} + w_{4,6}^{(1)} z_{6,3}^{(0)} = 8 + \left(\frac{176}{19}\right)(2) + \left(-\frac{21}{38}\right)(9) = \frac{819}{38}$$

$$z_{4,4}^{(1)} = z_{4,4}^{(0)} + w_{4,1}^{(1)} z_{1,4}^{(0)} + w_{4,6}^{(1)} z_{6,4}^{(0)} = -20 + \left(\frac{176}{19}\right)(4) + \left(-\frac{21}{38}\right)(-13) = \frac{921}{38}$$

$$z_{4,5}^{(1)} = z_{4,5}^{(0)} + w_{4,1}^{(1)} z_{1,5}^{(0)} + w_{4,6}^{(1)} z_{6,5}^{(0)} = 14 + \left(\frac{176}{19}\right)(3) + \left(-\frac{21}{38}\right)(10) = \frac{689}{19}$$

When $i = 5$ and $j = 2, 3, 4, 5$, then

$$z_{5,2}^{(1)} = z_{5,2}^{(0)} + w_{5,1}^{(1)} z_{1,2}^{(0)} + w_{5,6}^{(1)} z_{6,2}^{(0)} = 1 + \left(-\frac{1}{19}\right)(0) + \left(-\frac{11}{38}\right)(6) = -\frac{14}{19}$$

$$z_{5,3}^{(1)} = z_{5,3}^{(0)} + w_{5,1}^{(1)} z_{1,3}^{(0)} + w_{5,6}^{(1)} z_{6,3}^{(0)} = 1 + \left(-\frac{1}{19}\right)(2) + \left(-\frac{11}{38}\right)(9) = -\frac{65}{38}$$

$$z_{5,4}^{(1)} = z_{5,4}^{(0)} + w_{5,1}^{(1)} z_{1,4}^{(0)} + w_{5,6}^{(1)} z_{6,4}^{(0)} = -1 + \left(-\frac{1}{19}\right)(4) + \left(-\frac{11}{38}\right)(-13) = \frac{97}{38}$$

$$z_{5,5}^{(1)} = z_{5,5}^{(0)} + w_{5,1}^{(1)} z_{1,5}^{(0)} + w_{5,6}^{(1)} z_{6,5}^{(0)} = 1 + \left(-\frac{1}{19}\right)(3) + \left(-\frac{11}{38}\right)(10) = -\frac{39}{19}$$

Thus,

$$Z^{(1)} = \begin{bmatrix} 2 & 0 & 2 & 4 & 3 & -1 \\ 0 & \frac{151}{19} & \frac{-443}{38} & \frac{353}{38} & \frac{99}{19} & 0 \\ 0 & \frac{-234}{19} & \frac{172}{19} & \frac{147}{19} & \frac{347}{19} & 0 \\ 0 & \frac{165}{19} & \frac{819}{38} & \frac{921}{38} & \frac{689}{19} & 0 \\ 0 & \frac{-14}{19} & \frac{-65}{38} & \frac{97}{38} & \frac{-39}{19} & 0 \\ 10 & 6 & 9 & -13 & 10 & 14 \end{bmatrix}.$$

Step 3: Now, we can compute the next set of 2×2 systems of linear equation from the entries in $z^1_{i,j}$.
Let $k=2$, then

$$\begin{cases} z^{(1)}_{2,2}w^{(2)}_{i,2}+z^{(1)}_{n-1,2}w^{(2)}_{i,n-1}=-z^{(1)}_{i,2}\\ z^{(1)}_{2,n-1}w^{(2)}_{i,2}+z^{(1)}_{n-1,n-1}w^{(2)}_{i,n-1}=-z^{(1)}_{i,n-1}.\end{cases}$$

Whenever $i=3$, then

$$\begin{cases} z^{(1)}_{2,2}w^{(2)}_{3,2}+z^{(1)}_{5,2}w^{(2)}_{3,5}=-z^{(1)}_{3,2}\\ z^{(1)}_{2,5}w^{(2)}_{3,2}+z^{(1)}_{5,5}w^{(2)}_{3,5}=-z^{(1)}_{3,5}\end{cases}\Rightarrow$$

$$\begin{cases}\frac{151}{19}w^{(2)}_{3,2}+(-\frac{14}{19})w^{(2)}_{3,5}= & \frac{234}{19}\\ \frac{99}{19}w^{(2)}_{3,2}+(-\frac{39}{19})w^{(2)}_{3,5}= & -\frac{347}{19}\end{cases}\Rightarrow\begin{cases} w^{(2)}_{3,2}= & \frac{13985}{4503}\\ w^{(2)}_{3,5}= & \frac{75563}{4503}\end{cases}$$

Whenever $i=4$, then

$$\begin{cases} z^{(1)}_{2,2}w^{(2)}_{4,2}+z^{(1)}_{5,2}w^{(2)}_{4,5}=-z^{(1)}_{4,2}\\ z^{(1)}_{2,5}w^{(2)}_{4,2}+z^{(1)}_{5,5}w^{(2)}_{4,5}=-z^{(1)}_{4,5}\end{cases}\Rightarrow$$

$$\begin{cases}\frac{151}{19}w^{(2)}_{4,2}+(-\frac{14}{19})w^{(2)}_{4,5}= & -\frac{165}{19}\\ \frac{99}{19}w^{(2)}_{4,2}+(-\frac{39}{19})w^{(2)}_{4,5}= & -\frac{689}{19}\end{cases}\Rightarrow\begin{cases} w^{(2)}_{4,2}= & \frac{3211}{4503}\\ w^{(2)}_{4,5}= & \frac{87704}{4503}\end{cases}$$

Thus,

$$W^{(2)}=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&\frac{13984}{4503}&1&0&\frac{75563}{4503}&0\\0&\frac{3211}{4503}&0&1&\frac{87704}{4503}&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}$$

Step 4: We then proceed to update $z^1_{i,j}$ to $z^2_{i,j}$ by computing the entries as

$$z^{(k)}_{i,j}=z^{(k-1)}_{i,j}+w^{(k)}_{i,k}z^{(k-1)}_{k,j}+w^{(k)}_{i,n-k+1}z^{(k-1)}_{n-k+1,j}\Rightarrow z^{(2)}_{i,j}=z^{(1)}_{i,j}+w^{(2)}_{i,2}z^{(1)}_{2,j}+w^{(2)}_{i,5}z^{(1)}_{5,j}.$$

When $i=3$ and $j=3,4$, so

$$z^{(2)}_{3,3}=z^{(1)}_{3,3}+w^{(2)}_{3,2}z^{(1)}_{2,3}+w^{(2)}_{3,5}z^{(1)}_{5,3}=\frac{172}{19}+(\frac{13984}{4503})(\frac{-443}{38})+(\frac{75563}{4503})(\frac{-65}{38})=-\frac{9557475}{171114}$$

$$z^{(2)}_{3,4}=z^{(1)}_{3,4}+w^{(2)}_{3,2}z^{(1)}_{2,4}+w^{(2)}_{3,5}z^{(1)}_{5,4}=\frac{147}{19}+(\frac{13984}{4503})(\frac{353}{38})+(\frac{75563}{4503})(\frac{97}{38})=\frac{13589845}{171114}$$

When $i=4$ and $j=3,4$, so

$$z^{(2)}_{4,3}=z^{(1)}_{4,3}+w^{(2)}_{4,2}z^{(1)}_{2,3}+w^{(2)}_{4,5}z^{(1)}_{5,3}=\frac{819}{38}+(\frac{3211}{4503})(\frac{-443}{38})+(\frac{87704}{1171})(\frac{-65}{38})=-\frac{3435276}{171114}$$

$$z^{(2)}_{4,4}=z^{(1)}_{4,4}+w^{(2)}_{4,2}z^{(1)}_{2,4}+w^{(2)}_{4,5}z^{(1)}_{5,4}=\frac{921}{38}+(\frac{3211}{4503})(\frac{353}{38})+(\frac{87704}{4503})(\frac{97}{38})=\frac{13788034}{171114}$$

$$Z=\begin{bmatrix}2&0&2&4&3&-1\\0&\frac{151}{19}&-\frac{413}{38}&\frac{503}{38}&\frac{99}{19}&0\\0&0&-\frac{9557475}{171114}&\frac{13589845}{171114}&0&0\\0&0&-\frac{3435276}{171114}&\frac{13788034}{171114}&0&0\\0&-\frac{14}{19}&-\frac{65}{38}&\frac{97}{38}&-\frac{39}{19}&0\\10&6&9&-13&10&14\end{bmatrix}.$$

Thus,

$$B=\left(W^{(2)}\cdot W^{(1)}\right)^{-1}\cdot Z$$

WH factorization of matrix B

Step 1: We check the first and last row of matrix B before the initial update.

$$b^{(0)}_{1,1}=h^{(0)}_{1,1}=2,\ b^{(0)}_{1,2}=h^{(0)}_{1,2}=0,\ b^{(0)}_{1,3}=h^{(0)}_{1,3}=2,\ b^{(0)}_{1,4}=h^{(0)}_{1,4}=4,\ b^{(0)}_{1,5}=h^{(0)}_{1,5}=3,\ b^{(0)}_{1,6}=h^{(0)}_{1,6}=-1,\ b^{(0)}_{6,1}=h^{(0)}_{6,1}=10,\ b^{(0)}_{6,2}=h^{(0)}_{6,2}=6,\ b^{(0)}_{6,3}=h^{(0)}_{6,3}=9,\ b^{(0)}_{6,4}=h^{(0)}_{6,4}=-13,\ b^{(0)}_{6,5}=h^{(0)}_{6,5}=10,\ b^{(0)}_{6,6}=h^{(0)}_{6,6}=14.$$

Since $h^{(0)}_{1,2}=0$, then we interchange the first row with any other row except the last row. In this case we interchange first row with the fifth row such that the first and last row of the matrix has no zero entry as

$$H^{(0)}=\begin{bmatrix}3&1&1&-1&1&4\\5&10&-7&8&11&4\\0&-12&9&6&18&1\\-13&12&8&-20&14&17\\2&0&2&4&3&-1\\10&6&9&-13&10&14\end{bmatrix}.$$

with $H^{(0)}$ having permutation matrix

$$P^{(1)}=\begin{bmatrix}0&0&0&0&1&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\end{bmatrix}.$$

We begin to compute the set of 2×2 system of linear equations from

$$\begin{cases} h^{(k-1)}_{k,k}w^{(k)}_{i,k}+h^{(k-1)}_{n-k+1,k}w^{(k)}_{i,n-k+1}=-h^{(k-1)}_{i,k}\\ h^{(k-1)}_{k,n-k+1}w^{(k)}_{i,k}+h^{(k-1)}_{n-k+1,n-k+1}w^{(k)}_{i,n-k+1}=-h^{(k-1)}_{i,n-k+1}.\end{cases}$$

For $k=1,...,\lfloor\frac{n-1}{2}\rfloor=2$. Now, let $k=1$ then we have

$$\begin{cases} h^{(0)}_{1,1}w^{(1)}_{i,1}+h^{(0)}_{n,1}w^{(1)}_{i,n}=-h^{(0)}_{i,1}\\ h^{(0)}_{1,n}w^{(1)}_{i,1}+h^{(0)}_{n,n}w^{(1)}_{i,n}=-h^{(0)}_{i,n}.\end{cases}$$

Whenever $i=2$, then

$$\begin{cases} h_{1,1}^{(0)}w_{2,1}^{(1)}+h_{6,1}^{(0)}w_{2,6}^{(1)}=-h_{2,1}^{(0)} \\ h_{1,6}^{(0)}w_{2,1}^{(1)}+h_{6,6}^{(0)}w_{2,6}^{(1)}=-h_{2,6}^{(0)} \end{cases} \Rightarrow$$

$$\begin{cases} 3w_{2,1}^{(1)}+10w_{2,6}^{(1)}= & -5 \\ 4w_{2,1}^{(1)}+14w_{2,6}^{(1)}= & -4 \end{cases} \Rightarrow \begin{cases} w_{2,1}^{(1)}= & -15 \\ w_{2,6}^{(1)}= & 4 \end{cases}$$

Whenever $i=3$, then

$$\begin{cases} h_{1,1}^{(0)}w_{3,1}^{(1)}+h_{6,1}^{(0)}w_{3,6}^{(1)}=-h_{3,1}^{(0)} \\ h_{1,6}^{(0)}w_{3,1}^{(1)}+h_{6,6}^{(0)}w_{3,6}^{(1)}=-h_{3,6}^{(0)} \end{cases} \Rightarrow$$

$$\begin{cases} 3w_{3,1}^{(1)}+10w_{3,6}^{(1)}= & 0 \\ 4w_{3,1}^{(1)}+14w_{3,6}^{(1)}= & -1 \end{cases} \Rightarrow \begin{cases} w_{3,1}^{(1)}=5 \\ w_{3,6}^{(1)}= & -\frac{3}{2} \end{cases}$$

Whenever $i=4$, then

$$\begin{cases} h_{1,1}^{(0)}w_{4,1}^{(1)}+h_{6,1}^{(0)}w_{4,6}^{(1)}=-h_{4,1}^{(0)} \\ h_{1,6}^{(0)}w_{4,1}^{(1)}+h_{6,6}^{(0)}w_{4,6}^{(1)}=-h_{4,6}^{(0)} \end{cases} \Rightarrow$$

$$\begin{cases} 3w_{4,1}^{(1)}+10w_{4,6}^{(1)} & =13 \\ 4w_{4,1}^{(1)}+14w_{4,6}^{(1)} & =-17 \end{cases} \Rightarrow \begin{cases} w_{4,1}^{(1)}= & 176 \\ w_{4,6}^{(1)}= & -\frac{103}{2} \end{cases}$$

Whenever $i=5$, then

$$\begin{cases} h_{1,1}^{(0)}w_{5,1}^{(1)}+h_{6,1}^{(0)}w_{5,6}^{(1)}=-h_{5,1}^{(0)} \\ h_{1,6}^{(0)}w_{5,1}^{(1)}+h_{6,6}^{(0)}w_{5,6}^{(1)}=-h_{5,6}^{(0)}. \end{cases} \Rightarrow$$

$$\begin{cases} 3w_{5,1}^{(1)}+10w_{5,6}^{(1)}= & -2 \\ 4w_{5,1}^{(1)}+14w_{5,6}^{(1)}= & 1. \end{cases} \Rightarrow \begin{cases} w_{5,1}^{(1)}= & -19 \\ w_{5,6}^{(1)}= & \frac{11}{2} \end{cases}$$

Therefore, we write the values of $w_{i,1}^{(1)}$ and $w_{i,n}^{(1)}$ in a matrix form as

$$W^{(1)}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -15 & 1 & 0 & 0 & 0 & 4 \\ 5 & 0 & 1 & 0 & 0 & -\frac{3}{2} \\ 176 & 0 & 0 & 1 & 0 & -\frac{103}{2} \\ -19 & 0 & 0 & 0 & 1 & \frac{11}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 2: We update $h_{i,j}^{0}$ to $h_{i,j}^{1}$ by computing its entries as

$$h_{i,j}^{(k)}=h_{i,j}^{(k-1)}+w_{i,k}^{(k)}h_{k,j}^{(k-1)}+w_{i,n-k+1}^{(k)}h_{n-k+1,j}^{(k-1)} \Rightarrow h_{i,j}^{(1)}=h_{i,j}^{(0)}+w_{i,1}^{(1)}h_{1,j}^{(0)}+w_{i,6}^{(1)}h_{6,j}^{(0)}.$$

When $i=2$ and $j=2,3,4,5$, so

$$h_{2,2}^{(1)}=h_{2,2}^{(0)}+w_{2,1}^{(1)}h_{1,2}^{(0)}+w_{2,6}^{(1)}h_{6,2}^{(0)}=10+(-15)(1)+(4)(6)=19$$

$$h_{2,3}^{(1)}=h_{2,3}^{(0)}+w_{2,1}^{(1)}h_{1,3}^{(0)}+w_{2,6}^{(1)}h_{6,3}^{(0)}=-7+(-15)(1)+(4)(9)=14$$

$$h_{2,4}^{(1)}=h_{2,4}^{(0)}+w_{2,1}^{(1)}h_{1,4}^{(0)}+w_{2,6}^{(1)}h_{6,4}^{(0)}=8+(-15)(-1)+(4)(-13)=29$$

$$h_{2,5}^{(1)}=h_{2,5}^{(0)}+w_{2,1}^{(1)}h_{1,5}^{(0)}+w_{2,6}^{(1)}h_{6,5}^{(0)}=11+(-15)(1)+(4)(10)=36$$

When $i=3$ and $j=2,3,4,5$, then

$$h_{2,2}^{(1)}=h_{3,2}^{(0)}+w_{3,1}^{(1)}h_{1,2}^{(0)}+w_{3,6}^{(1)}h_{6,2}^{(0)}=-12+(5)(1)+(-\tfrac{3}{2})(6)=-16$$

$$h_{3,3}^{(1)}=h_{3,3}^{(0)}+w_{3,1}^{(1)}h_{1,3}^{(0)}+w_{3,6}^{(1)}h_{6,3}^{(0)}=9+(5)(1)+(-\tfrac{3}{2})(9)=\tfrac{1}{2}$$

$$h_{3,4}^{(1)}=h_{3,4}^{(0)}+w_{3,1}^{(1)}h_{1,4}^{(0)}+w_{3,6}^{(1)}h_{6,4}^{(0)}=6+(5)(-1)+(-\tfrac{3}{2})(-13)=\tfrac{41}{2}$$

$$h_{3,5}^{(1)}=h_{3,5}^{(0)}+w_{3,1}^{(1)}h_{1,5}^{(0)}+w_{3,6}^{(1)}h_{6,5}^{(0)}=18+(5)(1)+(-\tfrac{3}{2})(10)=8$$

When $i=4$ and $j=2,3,4,5$, we have

$$h_{4,2}^{(1)}=h_{4,2}^{(0)}+w_{4,1}^{(1)}h_{1,2}^{(0)}+w_{4,6}^{(1)}h_{6,2}^{(0)}=12+(176)(1)+(-\tfrac{103}{2})(6)=-121$$

$$h_{4,3}^{(1)}=h_{4,3}^{(0)}+w_{4,1}^{(1)}h_{1,3}^{(0)}+w_{4,6}^{(1)}h_{6,3}^{(0)}=8+(176)(1)+(-\tfrac{103}{2})(9)=-\tfrac{559}{2}$$

$$h_{4,4}^{(1)}=h_{4,4}^{(0)}+w_{4,1}^{(1)}h_{1,4}^{(0)}+w_{4,6}^{(1)}h_{6,4}^{(0)}=-20+(176)(-1)+(-\tfrac{103}{2})(-13)=\tfrac{1027}{2}$$

$$h_{4,5}^{(1)}=h_{4,5}^{(0)}+w_{4,1}^{(1)}h_{1,5}^{(0)}+w_{4,6}^{(1)}h_{6,5}^{(0)}=14+(176)(1)+(-\tfrac{103}{2})(10)=-325$$

When $i=5$ and $j=2,3,4,5$, then

$$h_{5,2}^{(1)}=h_{5,2}^{(0)}+w_{5,1}^{(1)}h_{1,2}^{(0)}+w_{5,6}^{(1)}h_{6,2}^{(0)}=0+(19)(1)+(\tfrac{11}{2})(6)=52$$

$$h_{5,3}^{(1)}=h_{5,3}^{(0)}+w_{5,1}^{(1)}h_{1,3}^{(0)}+w_{5,6}^{(1)}h_{6,3}^{(0)}=2+(19)(1)+(\tfrac{11}{2})(9)=\tfrac{141}{2}$$

$$h_{5,4}^{(1)}=h_{5,4}^{(0)}+w_{5,1}^{(1)}h_{1,4}^{(0)}+w_{5,6}^{(1)}h_{6,4}^{(0)}=4+(19)(-1)+(\tfrac{11}{2})(-13)=-\tfrac{173}{2}$$

$$h_{5,5}^{(1)}=h_{5,5}^{(0)}+w_{5,1}^{(1)}h_{1,5}^{(0)}+w_{5,6}^{(1)}h_{6,5}^{(0)}=3+(19)(1)+(\tfrac{11}{2})(10)=77$$

In $H^{(1)}$ the entries $h_{2,j}^{(1)}$ and $h_{5,j}^{(1)}$ are nonzero (i.e. $h_{2,2}^{(1)}=19$, $h_{2,3}^{(1)}=14$, $h_{2,4}^{(1)}=-29$, $h_{2,5}^{(1)}=36$, $h_{5,2}^{(1)}=52$, $h_{5,3}^{(1)}=\frac{141}{2}$, $h_{5,4}^{(1)}=-\frac{173}{2}$, $h_{5,5}^{(1)}=77$) for $j=2,3,4,5$. Otherwise apply suitable row interchange in $H^{(0)}$ and re-factorize, else the factorization breakdown.
Step 3: Now, we can compute the next set of 2×2 systems of linear equation from the entries in $h_{i,j}^{1}$.
Let $k=2$, then

$$\begin{cases} h_{2,2}^{(1)}w_{i,2}^{(2)}+h_{n-1,2}^{(1)}w_{i,n-1}^{(2)}=-h_{i,2}^{(1)} \\ h_{2,n-1}^{(1)}w_{i,2}^{(2)}+h_{n-1,n-1}^{(1)}w_{i,n-1}^{(2)}=-h_{i,n-1}^{(1)}. \end{cases}$$

Whenever $i=3$, then

$$\begin{cases} h^{(1)}_{2,2}w^{(2)}_{3,2}+h^{(1)}_{5,2}w^{(2)}_{3,5}=-h^{(1)}_{3,2} \\ h^{(1)}_{2,5}w^{(2)}_{3,2}+h^{(1)}_{5,5}w^{(2)}_{3,5}=-h^{(1)}_{3,5} \end{cases} \Rightarrow$$

$$\begin{cases} 19w^{(2)}_{3,2}+52w^{(2)}_{3,5}= & 16 \\ 36w^{(2)}_{3,2}+77w^{(2)}_{3,5}= & 3-8 \end{cases} \Rightarrow \begin{cases} w^{(2)}_{3,2}= & -\frac{1648}{409} \\ w^{(2)}_{3,5}= & \frac{728}{409} \end{cases}$$

Whenever $i=4$, then

$$\begin{cases} h^{(1)}_{2,2}w^{(2)}_{4,2}+h^{(1)}_{5,2}w^{(2)}_{4,5}=-h^{(1)}_{4,2} \\ h^{(1)}_{2,5}w^{(2)}_{4,2}+h^{(1)}_{5,5}w^{(2)}_{4,5}=-h^{(1)}_{4,5} \end{cases} \Rightarrow$$

$$\begin{cases} 19w^{(2)}_{4,2}+52w^{(2)}_{4,5}= & 121 \\ 36w^{(2)}_{4,2}+77w^{(2)}_{4,5}= & 325 \end{cases} \Rightarrow \begin{cases} w^{(2)}_{4,2}= & \frac{7583}{409} \\ w^{(2)}_{4,5}= & -\frac{1819}{409} \end{cases}$$

Thus,

$$W^{*(2)}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1648}{409} & 1 & 0 & 728409 & 0 \\ 0 & \frac{7583}{409} & 0 & 1 & -\frac{1819}{409} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 4: We then proceed to update the matrix $H^{(1)}$ to $H^{(2)}$ by computing its entries as

$$h^{(k)}_{i,j}=h^{(k-1)}_{i,j}+w^{(k)}_{i,k}h^{(k-1)}_{k,j}+w^{(k)}_{i,n-k+1}h^{(k-1)}_{n-k+1,j} \Rightarrow h^{(2)}_{i,j}= h^{(1)}_{i,j}+w^{(2)}_{i,2}h^{(1)}_{2,j}+w^{(2)}_{i,5}h^{(1)}_{5,j}.$$

When $i=3$ and $j=3,4$, so

$$h^{(2)}_{3,3}=h^{(1)}_{3,3}+w^{(2)}_{3,2}h^{(1)}_{2,3}+w^{(2)}_{3,5}h^{(1)}_{5,3}= \frac{1}{2}+(-\frac{1648}{409})(14)+(\frac{728}{409})(\frac{141}{2})=\frac{56913}{818}$$

$$h^{(2)}_{3,4}=h^{(1)}_{3,4}+w^{(2)}_{3,2}h^{(1)}_{2,4}+w^{(2)}_{3,5}h^{(1)}_{5,4}= \frac{41}{2}+(-\frac{1648}{409})(-29)+(\frac{728}{409})(-\frac{173}{2})=-\frac{13591}{818}$$

When $i=4$ and $j=3,4$, so

$$h^{(2)}_{4,3}=h^{(1)}_{4,3}+w^{(2)}_{4,2}h^{(1)}_{2,3}+w^{(2)}_{4,5}h^{(1)}_{5,3}= -\frac{559}{2}+(\frac{7583}{409})(14)+(-\frac{-1819}{409})(\frac{141}{2})=-\frac{272786}{818}$$

$$h^{(2)}_{4,4}=h^{(1)}_{4,4}+w^{(2)}_{4,2}h^{(1)}_{2,4}+w^{(2)}_{4,5}h^{(1)}_{5,4}= \frac{1027}{2}+(\frac{7583}{409})(-29)+(-\frac{-1819}{409})(-\frac{173}{2})=\frac{294916}{818}$$

$$H=\begin{bmatrix} 3 & 1 & 1 & -1 & 1 & 4 \\ 0 & 19 & 14 & -29 & 36 & 0 \\ 0 & 0 & \frac{56913}{818} & -\frac{13591}{818} & 0 & 0 \\ 0 & 0 & -\frac{272786}{818} & \frac{294916}{818} & 0 & 0 \\ 0 & 52 & \frac{141}{2} & -\frac{173}{2} & 77 & 0 \\ 10 & 6 & 9 & -13 & 10 & 14 \end{bmatrix}.$$

The factorization stops since the entries $h^{(2)}_{3,j}$ and $h^{(2)}_{4,j}$ are nonzero, for $j=3,4$.
To get the matrix B, we express B as

$$B=\left(W^{(2)}\cdot W^{(1)}\cdot P^{(1)}\right)^{-1}\cdot H$$

4. Investigation for further studies on WZ and WH factorization

There are bridging gaps to be uncovered not only between WZ factorization and WH factorization but also within Z-matrix and hourglass matrix. Further research could be carried on the following:

1. If there exists WH factorization for a nonsingular matrix B, then there exists WZ factorization.
2. WZ factorization does not necessarily imply WH factorization.
3. Z_{system} is a matrix group of degree 2 over $\mathbb{R}$ but H_{system} is not.
4. If both H_{system} and $H_{1,1}$ are invertible then $\left(H_{2,2}-H_{2,1}H^{-1}_{1,1}H_{1,2}\right)$ is a lower triangular invertible matrix.
5. If there exists WH factorization for a nonsingular matrix B, then the factorization is unique.
6. Let $\mathscr{E}_M(\mathscr{G})$ be the energy of mixed hourglass graph $\mathscr{G}$ and A_k be the number of arcs in $\mathscr{G}$. Then, does there exist a mixed hourglass graph $\mathscr{G}$ of order n $(n>4)$ satisfying the following?
$$A_k=\mathscr{E}_M(\mathscr{G}).$$

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