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Collocation Method for the Numerical Solution of Multi-Order Fractional Differential Equations

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Abstract

This study presents a collocation approach for the numerical integration of multi-order fractional differential equations with initial conditions in the Caputo sense. The problem was transformed from its integral form into a system of linear algebraic equations. Using matrix inversion, the algebraic equations are solved and their solutions are substituted into the approximate equation to give the numerical results. The effectiveness and precision of the method were illustrated with the use of numerical examples.

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1. Introduction

In the fields of mathematics, physics, chemistry, and engineering, differential and integral equations involving fractions are of the utmost significance. The use of functional equations, such as ordinary and partial differential equations, is typical when applying mathematics to the modeling of problems arising in the real world. In the early 1900s, Italian mathematician Vito Volterra came up with a whole new sort of equation that came to be known as integro-differential equations in order to investigate the phenomenon of population expansion. In these types of equations, one or more derivatives of the function whose value is unknown is placed under the integral sign. Integro-differential equations can be found in a variety of mathematical formulations of physical phenomena. Additionally, these equations can be found in the modeling of certain phenomena in the fields of science and engineering. For instance, the equations of kinetics that support the kinetic theory of rarefied gases, plasma, radiation transmission, and coagulation are some examples. [1]. Some of the numerical solution of fractional differential equations developed in the literature include: Perturbed collocation method [2], Adomian decompositions method by [3-5], Collocation method by [6-9], Chebyshev- Gelerkin method [10], Bernoulli matrix method [11], Differential transform method [12], Pseudospectral method [13], Bernstein Polynomials method [14, 15], the Mellin transform approach [16]. [17] utilized a numerical approach based on the boubaker polynomial to generate approximate numerical solutions to the multi-order fractional differential equations. Their decision was to use an operational matrix for fractional integration based on boubakar polynomi-

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als. Collocation approach for the computational solution of fredholm-volterra fractional order of integro-differential equations was presented by [18]. They solved the problem by first obtaining the linear integral form of it and then transforming it into a system of linear algebraic equations by making use of conventional collocation points.

In this research, the collocation method is utilized to solve multi-order fractional differential equations of the form

$$D^{\beta} y(x) = \sum_{j=0}^{N} q_j(x) D^{\alpha_j} y(x) + h(x) \tag{1}$$

subject to the initial condition

$$y^{(j)}(a_j) = \lambda_j, \quad j = 0, 1, ..., n-1, \; n \in \mathbb{N} \, \beta > \alpha_N, \tag{2}$$

where $y(x)$ is the unknown function, D^{α_j} and D^{β} are the Caputo's derivative, $h(x)$ is the force known -prior. $q_j(x)$ is the known function, a_j and λ_j are known constants.

2. Basic Definitions

In this section, we present certain definitions and fundamental ideas of fractional calculus for the purpose of the formulation of the problem that has been presented.

Definition 2.1: The Caputo derivative with order $\alpha > 0$ of the given function $f(x)$, $x \in (a, b)$ is defined as

$${}_{x}^{C}D_{a}^{\alpha} y(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-s)^{m-\alpha-1} y^{(m)}(s) ds \tag{3}$$

where $m - 1 \leq \alpha \leq m, m \in N, x > 0$

Definition 2.2: Let $(a_n), n \geq 0$ be a sequence of real numbers. The power series in x with coefficients a_n is an expression

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots a_N x^N = \sum_{n=0}^{N} a_n x^n = \phi(x)\, \mathbf{A} \tag{4}$$

where $\phi(x) = [1 \;\; x \;\; x^2 \;\; \cdots \;\; x^N]$, $\mathbf{A} = [a_0 \;\; a_1 \;\; \cdots \;\; a_N]^T$ then $y(x, n) = x^n \mathbf{A}$, $n = 0(1)N$, $n \in Z^+$

Definition 2.3: Standard Collocation Method (SCM). This method is used to determine the desired collocation points within an interval. i.e [a,b] and is given by

$$x_i = a + \frac{(b-a)i}{N}, i = 1, 2, 3, ...N \tag{5}$$

Definition 2.4: Let $y(x)$ be a continuous function, then

$${}_0I_x^{\beta} \left({}_0^C D_x^{\beta} y(x) \right) = y(x) - \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k \tag{6}$$

where $m - 1 < \beta < 1$

Definition 2.5: Let $p(s)$ be an integrable function, then

$${}_0I_x^{\beta}\,(p(s)) = \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} p(s) ds \tag{7}$$

Definition 2.6: The Riemann -Liouville derivative of order $\alpha > 0$ with $n - 1 < \alpha < n$ of the power function $f(t) = t^{p-\alpha}$ is given by

$$D^{\alpha} t^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha} \tag{8}$$

3. Mathematical Background

In this part, we create a collocation approach for numerically solving multi-order fractional differential equations utilizing power series polynomials as the basis function.

Lemma (3.1) (Integral form)

Let $y(x)$ be a solution to (1) subject to (2), the integral form is

$$\begin{aligned} y(x) \;=\; & W(x) + \sum_{j=0}^{N} \frac{1}{\Gamma(m_j - \alpha_j)} \frac{1}{\Gamma(\beta)} \\ & \times \int_0^x (x-s)^{\beta-1} q_j(s) \left[\int_0^s (s-t)^{m_j-\alpha_j-1} y^{(m_j)}(t) dt \right] ds \end{aligned} \tag{9}$$

where

$$W(x) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} h(s) ds$$

Proof. Multiply equation (1) by ${}_0I_x^{\beta}(.)$ gives

$${}_0I_x^{\beta} \left(D^{\beta} y(x) \right) = {}_0I_x^{\beta} \left(\sum_{j=0}^{N} q_j(x) D^{\alpha_j} y(x) \right) + {}_0I_x^{\beta}\,(h(x)) \tag{10}$$

using (6) on equation (10) gives

$$y(x) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + {}_0I_x^{\beta} \left(\sum_{j=0}^{N} q_j(x) D^{\alpha_j} y(x) \right) \tag{11}$$

applying equations (3) and (7) to equation (11) gives

$$\begin{aligned} y(x) = & \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \\ & \times \left(\sum_{j=0}^{N} q_j(x) \frac{1}{\Gamma(m_j-\alpha_j)} \int_0^s (s-t)^{m_j-\alpha_j-1} y^{(m_j)}(t) dt \right) ds \end{aligned} \tag{12}$$

Substituting equation (4) into equation (12) gives

$$\begin{aligned} y(x) = & \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \\ & \times \left(\sum_{j=0}^{N} q_j(x) \frac{1}{\Gamma(m_j-\alpha_j)} \int_0^s (s-t)^{m_j-\alpha_j-1} \frac{d^{m_j}}{dt^{m_j}} (\phi(t)) \; dt\mathbf{A} \right) ds \end{aligned} \tag{13}$$

□

3.1. *Method of Solution*

Collocating at x_i in equation (13) gives

$$\begin{aligned} y(x_i) \;=\; & W(x_i) + \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x_i - s)^{\beta-1} q_j(s) \\ & \times \left(\int_0^s (s-t)^{m_j-\alpha_j-1} \frac{d^{m_j}}{dt^{m_j}} (\phi(t)) \; dt \right) ds\, \mathbf{A} \end{aligned} \tag{14}$$

where

$$W(x_i) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} h(s) ds$$

Simplifying equation (14) gives

$$\phi(x_i)\mathbf{A} = W(x_i) + \left[\begin{array}{c} \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x_i-s)^{\beta-1} q_j(s) \\ \times\left(\int_0^s (s-t)^{m_j-\alpha_j-1} \frac{d^{m_j}}{dt^{m_j}} (\phi(t))\, dt\right) ds \end{array} \right] \mathbf{A} \quad (15)$$

Factorizing the values of **A** from equation (15) gives

$$\left[\begin{array}{c} \phi(x_i) - \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x_i-s)^{\beta-1} q_j(s) \\ \times\left(\int_0^s (s-t)^{m_j-\alpha_j-1} \frac{d^{m_j}}{dt^{m_j}} (\phi(t))\, dt\right) ds \end{array} \right] \mathbf{A} = W(x_i) \quad (16)$$

Equation (16) can be in the form

$$V(x_i)\mathbf{A} = W(x_i) \quad (17)$$

where

$$V(x_i) = \phi(x_i) - \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x_i-s)^{\beta-1} q_j(s) \left(\int_0^s (s-t)^{m_j-\alpha_j-1} \frac{d^{m_j}}{dt^{m_j}} (\phi(t))\; dt\right) ds \quad (18)$$

and

$$\mathbf{A} = [a_0 \quad a_1 \quad \cdots \quad a_N]^T$$

Multiplying both sides of equation (17) by $V^{-1}(x_i)$ gives

$$\mathbf{A} = V^{-1}(x_i) W(x_i) \quad (19)$$

Lemma (3.2)
Let $y(x)$ be approximated by (11) and let

$$L(x) = {}_0I_x^{\beta}\left(\sum_{j=0}^{N} q_j(x) D^{\alpha_j} y(x)\right) \quad (20)$$

If $q_j(s) = s^{p_j}$, then

$$\mathbf{L}(x;n) = \frac{\Gamma(n+1)\Gamma(n-\alpha_j+p_j+1)}{\Gamma(n-\alpha_j+1)\Gamma(\beta+n-\alpha_j+p_j+1)} \times x_i^{\beta+n-\alpha_j+p_j} \mathbf{A} \quad (21)$$

Proof.
Applying equation (3) and (7) into equation (20) gives

$${}_0I_x^{\beta}\left(\sum_{j=0}^{N} q_j(x) D^{\alpha_j} y(x)\right) = \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x-s)^{\beta-1} q_j(s) \left[\int_0^s (s-t)^{m_j-\alpha_j-1} y^{(m_j)}(t) dt\right] ds \quad (22)$$

Substituting (8) into (22) gives

$$= \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^{x} (x-s)^{\beta-1} S^{p_j} \left[\int_0^s (s-t)^{m_j-\alpha_j-1} \left(\frac{\Gamma(n+1)}{\Gamma(n-m_j+1)} t^{n-m_j}\right) dt\right] ds\, \mathbf{A} \quad (23)$$

Let $s-t = (1-v)s$, then $t = vs \implies \frac{dt}{dv} = s \implies dt = sdv$, substituting into (23) gives

$$= \sum_{j=0}^{N} \frac{\Gamma(n+1)}{\Gamma(m_j-\alpha_j)\Gamma(n-m_j+1)} \frac{1}{\Gamma(\beta)} \int_0^{x} (x-s)^{\beta-1} S^{p_j} \left[S^{n-\alpha_j} \int_0^1 (1-v)^{m_j-\alpha_j-1} V^{n-m_j} dt\right] ds\, \mathbf{A} \quad (24)$$

Simplifying (24), we get

$$\mathbf{L}(x;n) = \frac{\Gamma(n+1)\Gamma(n-\alpha_j+p_j+1)}{\Gamma(n-\alpha_j+1)\Gamma(\beta+n-\alpha_j+p_j+1)} x^{\beta+n-\alpha_j+p_j} \mathbf{A} \quad (25)$$

□

Lemma (3.3)
Let $y(t)$ be approximated by (9), let

$$C(x) = {}_0I_x^{\beta}(h(x)) \quad (26)$$

if $h(s) = s^m$, then

$$C(x) = \frac{\Gamma(m+1)}{\Gamma(\beta+m+1)} x^{\beta+m}$$

Proof.
Applying equation (7) into (26) gives

$${}_0I_x^{\beta}(h(x)) = \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} h(s)\, ds$$

Substituting for $h(s)$ gives

$$= \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} s^m ds$$

Let $x-s = (1-u)x,\ s = ux \implies \frac{ds}{du} = x \implies ds = xdu.$

$$C(x) = \frac{\Gamma(m+1)}{\Gamma(\beta+m+1)} x^{\beta+m} \quad (27)$$

□

Lemma (3.4)
Let $y(x)$ be the solution of (1) and (2) then the numerical result gives

$$y(x) = \phi(x_i) V^{-1}(x_i)\, W(x_i) \quad (28)$$

where

$$V(x_i) = \frac{\Gamma(n+1)\Gamma(n-\alpha_j+p_j+1)}{\Gamma(n-\alpha_j+1)\Gamma(\beta+n-\alpha_j+p_j+1)} x_i^{\beta+n-\alpha_j+p_j}$$

and

$$W(x_i) = -\sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x_i^k + \frac{\Gamma(m+1)}{\Gamma(\beta+m+1)} x_i^{\beta+m}$$

Proof.
Approximate solution of equation (17) is

$$y(x) = \phi(x)\,\mathbf{A}$$

From equation (19) $\mathbf{A} = V^{-1}(x_i)\,W(x_i)$
where

$$\begin{aligned} V(x_i) &= \frac{\Gamma(n+1)\Gamma(n-\alpha_j+p_j+1)}{\Gamma(n-\alpha_j+1)\Gamma(\beta+n-\alpha_j+p_j+1)} x_i^{\beta+n-\alpha_j+p_j} \\ &+ \frac{b^{r+n+1}\Gamma(r+1)}{(\sigma+n+1)\,\Gamma(\beta+r+1)} x^{\beta+r} \\ &+ \frac{\Gamma(r+\sigma+n+2)}{(\sigma+n+1)\,\Gamma(\beta+r+\sigma+n+2)} x^{\beta+r+\sigma+n+1} \end{aligned}$$

Substituting for **A** in the approximate solution gives the numerical result

$$y(x) = \phi(x_i)V^{-1}(x_i)\,W(x_i)$$

□

4. Convergence Analysis

In this section, we establish the convergence of the method by substituting the approximate solution into equation (3.0)

$$y_N(x) = W(x) + \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \times \int_0^x (x-s)^{\beta-1} q_j(s) \left[\int_0^s (s-t)^{m_j-\alpha_j-1} y_N^{(m_j)}(t)dt\right] ds \quad (29)$$

Subtracting (9) from (29) gives

$$E_N(x) = y_N(x) - y(x).$$

Hence

$$|E_N(x)| \le \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} q_j(s) \left|\left[\int_0^s (s-t)^{m_j-\alpha_j-1} E_N(t)dt\right] ds\right|$$

Therefore

$$\frac{\|E_N(x_i)\|_\infty}{\|E_N(t)\|_\infty} \le \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x-s)^{\beta-1} \left\|\left[\sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} q_j(s) \left(\int_0^s (s-t)^{m_j-\alpha_j-1} dt\right)\right]\right\| ds$$

5. Numerical Examples

In this section, we considered two numerical examples to evaluate the effectiveness and clarity of the method. A MAPLE 18 program is used to perform the computations. Let $y_n(x)$ and $y(x)$ be the approximate and exact solutions respectively. $\text{Error}_N = |y_n(x) - y(x)|$.

Example 5.1. *[2] Consider multi-order Fractional differential equation .*

$$D^{1.5}y(x) = -x^{-1}D^{0.5}y(x) - x^{0.5}y(x) + f(x)$$

with this condition $y'(0) = y(0) = 0$ and exact solution $y(x) = x^3 - x^2$

$$f(x) = \left[6x\left(\frac{\Gamma(3.5)+\Gamma(2.5)}{\Gamma(2.5)\Gamma(3.5)} + \frac{x^2}{6}\right) - 2\left(\frac{\Gamma(2.5)+\Gamma(1.5)}{\Gamma(1.5)\Gamma(2.5)} + \frac{x^2}{2}\right)\right] x^{0.5}$$

Solution 1. *Comparing with equation (1.1) and Equation (1.2),* $\beta = 1.5, \alpha = 0.5$

Using N = 4 for illustration, and applying equation (6) gives

$$\begin{aligned} y(x) = W(x) &- \frac{1}{\Gamma(1-0.5)} \frac{1}{\Gamma(1.5)} \int_0^x (x-s)^{1.5-1}\, s^{-1} \left[\int_0^s (s-t)^{1-0.5-1} \frac{\Gamma(n+1)}{\Gamma(n-1+1)} t^{n-1} dt\right] ds\,\mathbf{A} \\ &- \frac{1}{\Gamma(1.5)} \int_0^x (x-s)^{1.5-1} s^{0.5}\,(s^n)\,ds\,\mathbf{A} \end{aligned} \quad (30)$$

where

$$W(x) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(1.5)} \int_0^x (x-s)^{1.5-1} f(s)ds$$

Substituting (4) into equation (30) gives

$$\begin{aligned} \phi(x)\,\mathbf{A} = W(x) &- \frac{1}{\Gamma(1-0.5)} \frac{1}{\Gamma(1.5)} \int_0^x (x-s)^{1.5-1}\, s^{-1} \left[\int_0^s (s-t)^{1-0.5-1} \frac{\Gamma(n+1)}{\Gamma(n-1+1)} t^{n-1} dt\right] ds\,\mathbf{A} \\ &- \frac{1}{\Gamma(1.5)} \int_0^x (x-s)^{1.5-1} s^{0.5}\,(s^n)\,ds\,\mathbf{A} \end{aligned} \quad (31)$$

where

$$W(x) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(1.5)} \int_0^x (x-s)^{1.5-1} f(s)ds$$

Equation (31) can be in the form

$$\tau(x)\mathbf{A} = W(x) \quad (32)$$

where

$$\begin{aligned} \tau(x) &= \phi(x) + \frac{1}{\Gamma(1-0.5)} \frac{1}{\Gamma(1.5)} \int_0^x (x-s)^{1.5-1}\, s^{-1} \left[\int_0^s (s-t)^{1-0.5-1} \frac{\Gamma(n+1)}{\Gamma(n-1+1)} t^{n-1} dt\right] ds \\ &+ \frac{1}{\Gamma(1.5)} \int_0^x (x-s)^{1.5-1} s^{0.5}\,(s^n)\,ds \end{aligned}$$

collocating at $x_4 = \begin{bmatrix} \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 \end{bmatrix}$ and substituting the initial conditions gives

$$\tau(x_i)^* \mathbf{A} = f(x_i)^* \tag{33}$$

where

$$\tau_i(x)^* = \begin{bmatrix} 0.0000000000 & 0.0000000000 & 0.0000000000 & 0.0000000000 & 0.0000000000 \\ 0.0470157986 & 0.2239605405 & 0.1797773130 & 0.0750069286 & 0.0274673600 \\ 0.1880631945 & 0.5184447788 & 0.7543711011 & 0.6176863534 & 0.4482932221 \\ 0.4231421877 & 0.9539764130 & 1.8295669120 & 2.1838653930 & 2.3438648990 \\ 0.7522527781 & 1.6010791410 & 3.5816739880 & 5.5056804110 & 7.7368811360 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$f(x)^* = [0.0000000000 \quad -0.1047703844 \quad -0.1366847478 \quad 0.3542984804 \quad 1.9240064220]$$

We now solve for the unknown values **A** making use of matrix inversion results in equation (33):

$$y_4 = \begin{pmatrix} 1.365574320288940 \times 10^{-14} - 2.546407529280260 \times 10^{-12} x \\ -0.999999999275360 x^2 + 0.999999998413614 x^3 + 6.644427230639850 \times 10^{-10} x^4 \end{pmatrix}$$

Example 5.2. *[2] Consider multi-order Fractional differential equation .*

$$D^{1.5} y(x) + \frac{1}{x} D^{0.5} y(x) + x^{\frac{1}{2}} y(x) = +f(x)$$

with this condition $y'(0) = y(0) = 0$ and exact solution $y(x) = -x^3 + x^2$

$$f(x) = \left[2\left(\frac{\Gamma(2.5)+\Gamma(1.5)}{\Gamma(1.5)\Gamma(2.5)} + \frac{x^2}{2} \right) - 6x\left(\frac{\Gamma(3.5)+\Gamma(2.5)}{\Gamma(2.5)\Gamma(3.5)} + \frac{x^2}{9} \right) \right] x^{\frac{1}{2}}$$

Solution 2. *Comparing with equation (1.1) and Equation (1.2),* $\beta = 1.5, \alpha = 0.5$

Using $N = 4$ for illustration,
Applying equation (6) gives

$$\begin{aligned} y(x) = \; & W(x) - \frac{1}{\Gamma(1-0.5)} \frac{1}{\Gamma(1.5)} \int_0^x (x-s)^{1.5-1} s^{-1} \\ & \left[\int_0^s (s-t)^{1-0.5-1} \frac{\Gamma(n+1)}{\Gamma(n-1+1)} t^{n-1} dt \right] ds\, \mathbf{A} \\ & - \frac{1}{\Gamma(1.5)} \int_0^x (x-s)^{1.5-1} s^{0.5} (s^n)\, ds\, \mathbf{A} \end{aligned} \tag{34}$$

Table 1: Exact, approximate and absolute error values for Example 1

x	Exact	Our method$_{N=4}$	error$_{N=4}$	error $_{[2]=4}$
0.0	0.00000000000	1.36557432000e-14	1.3656e-14	5.9232e-13
0.1	-0.900000000e-2	-0.89999999950e-2	5.0000e-12	2.6668e-10
0.2	-0.320000000e-1	-0.31999999980e-1	2.0000e-11	9.6994e-10
0.3	-0.630000000e-1	0.06299999997000	3.0000e-11	1.9604e-09
0.4	-0.960000000e-1	-0.95999999980e-1	2.0000e-11	3.0781e-09
0.5	-0.12500000000	-0.1250000000000	0.0000000	4.1532e-09
0.6	-0.14400000000	-0.1439999999000	1.0000e-10	5.0056e-09
0.7	-0.14700000000	-0.1470000000000	0.0000000	5.4456e-09
0.8	-0.12800000000	-0.1280000001000	1.0000e-10	5.2733e-09
0.9	-0.810000000e-1	-0.81000000160e-1	1.6000e–10	4.2787e-09
1.0	0.00000000000	-2.3555727690e-10	2.3555e-10	2.2421e-09

where

$$W(x) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(1.5)} \int_0^x (x-s)^{1.5-1} f(s) ds$$

Substituting (4) into equation (34) gives

$$\begin{aligned} \phi(x)\, \mathbf{A} = \; & W(x) - \frac{1}{\Gamma(1-0.5)} \frac{1}{\Gamma(1.5)} \int_0^x (x-s)^{1.5-1} s^{-1} \\ & \left[\int_0^s (s-t)^{1-0.5-1} \frac{\Gamma(n+1)}{\Gamma(n-1+1)} t^{n-1} dt \right] ds\, \mathbf{A} \\ & - \frac{1}{\Gamma(1.5)} \int_0^x (x-s)^{1.5-1} s^{0.5} (s^n)\, ds\, \mathbf{A} \end{aligned} \tag{35}$$

where

$$W(x) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(1.5)} \int_0^x (x-s)^{1.5-1} f(s) ds$$

Equation (35) can be in the form

$$\tau(x) \mathbf{A} = W(x) \tag{36}$$

where

$$\begin{aligned} \tau(x) = \; & \phi(x) + \frac{1}{\Gamma(1-0.5)} \frac{1}{\Gamma(1.5)} \int_0^x (x-s)^{1.5-1} \; s^{-1} \\ & \left[\int_0^s (s-t)^{1-0.5-1} \frac{\Gamma(n+1)}{\Gamma(n-1+1)} t^{n-1} dt \right] ds \\ & + \frac{1}{\Gamma(1.5)} \int_0^x (x-s)^{1.5-1} s^{0.5} (s^n)\, ds \end{aligned}$$

Collocating at $x_4 = \begin{bmatrix} \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 \end{bmatrix}$ and substituting the initial conditions gives

$$\tau(x_i)^* \mathbf{A} = f(x_i)^* \tag{37}$$

where

$$\tau_i(x)^* = \begin{bmatrix} 0.5000000000 & 2.3817583340 & 1.9118819450 & 0.7976779168 & 0.2921077680 \\ 0.7071067812 & 1.9493225120 & 2.8363918970 & 2.3224651180 & 1.6855566990 \\ 0.8660254038 & 1.9524590850 & 3.7444893700 & 4.4696155660 & 4.7970791030 \\ 1.0000000000 & 2.1283791670 & 4.7612638900 & 7.3189233350 & 10.2849485700 \\ 1 & 0 & 0 & 0 & 0 \\ 0‘ & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$f(x)^* = \begin{bmatrix} 1.1142040280 & 0.5139267798 & -0.7251261978 & -2.5576594460 & 0 & 0 \end{bmatrix}$$

We now solve for the unknown values **A** making use of matrix inversion results in equation (37);

$$y_4 = \begin{pmatrix} 1.428190898877800 \times 10^{-12} - 2.777014174171200 \times 10^{-10} x \\ +1.000000001121990 x^2 - 1.000000001716120 x^3 + 6.387779194483300 \times 10^{-10} x^4 \end{pmatrix}$$

Table 2: Exact, approximate and absolute error values for Example 2

x	Exact	Our method$_{N=4}$	error$_{N=4}$	error $_{[2]=4}$
0.0	0.00000000000	1.428190899000e-12	1.4281908990000e-12	3.7782e-12
0.1	0.00900000000	0.00899999998200	1.8000000000000e-11	2.4706e-09
0.2	0.03200000000	0.03199999997000	3.0000000000000e-11	1.4306e-08
0.3	0.06300000000	0.06299999997000	3.0000000000000e-11	3.9585e-08
0.4	0.09600000000	0.09599999999000	1.0000000000000e-11	78988e-08
0.5	0.12500000000	0.12499999990000	1.0000000000000e-10	1.2980e-07
0.6	0.14400000000	0.14399999990000	1.0000000000000e-10	1.8590e-07
0.7	0.14700000000	0.14699999980000	2.0000000000000e-10	2.3778e-07
0.8	0.12800000000	0.12799999970000	3.0000000000000e-10	2.7253e-07
0.9	0.08100000000	0.08099999952000	4.8000000000000e-10	2.7386e-07
1.0	0.00000000000	-3.6122208060e-10	3.6122208060000e-10	2.2207e-07

6. Discussion of Results

In this section, we discuss the numerical results obtained by applying the derived numerical method to the solved examples. We observed from the result obtained for example 1 as shown in Table 1 that the approximate solution at N=4 gives $y_4(x) = 1.365574320288940 \times 10^{-14} - 2.546407529280260 \times 10^{-12} x + 1.000000001121990 x^2 - 1.000000001716120 x^3 + 6.387779194483300 \times 10^{-10} x^4$. The numerical result almost converges to the exact solution and produces extremely small errors. This demonstrated that our method outperformed the proposed method by Uwaheren *et al* (2020).

The results of the numerical example 2 in Table 2 shows the approximate solution at N=4 as $y_4(x) = 1.428190898877800 \times 10^{-12} - 2.777014174171200 \times 10^{-10} x + 1.0000001121990 x^2 - 1.0000001716120 x^3 + 6.387779194483300 \times 10^{-10} x^4$. The numerical result converge to the exact solution and give better result than the method proposed by Uwaheren *et al* (2020) at the same value of N. This shows that the numerical method developed is consistent and converges faster.

7. Conclusion

In this paper, a new numerical method was developed for solving multi-order fractional differential equations with initial conditions using collocation method. The numerical method derived is consis- tent, efficient and reliable and easy to compute. Maple code was used to implement the developed method. Solved numerical examples show that the method is reliable and suitable for these kind of problems. We also compare our absolute errors with Uwaheren *et al.* (2020) as shown in Tables 1 and 2. Hence, we safely conclude that our method is preferable to the existing methods.

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