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An Accuracy-preserving Block Hybrid Algorithm for the Integration of Second-order Physical Systems with Oscillatory Solutions

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Abstract

It is a known fact that in most cases, to integrate an oscillatory problem, higher order A-stable methods are often needed. This is because such problems are characterized by stiffness, chaos and damping, thus making them tedious to solve. However, in this research, an accuracy-preserving relatively lower order Block Hybrid Algorithm (BHA) is proposed for solution of second-order physical systems with oscillatory solutions. The sixth order algorithm was derived using interpolation and collocation of power series within a single step interval $[t_n, t_{n+1}]$. In order to circumvent the Dahlquist-barrier and also obtain an accuracy-preserving algorithm, four off-step points were incorporated within the single step interval. A number of special cases of oscillatory problems were solved using the proposed method and the results obtained clearly showed that it outperformed other existing methods we compared our results with even though the BHA is of lower order relative to such methods. Some of the second-order physical systems considered were the Kepler, Bessel and damped problems. Some important properties of the BHA were also analyzed and the results of the analysis showed that it is consistent, zero-stable and convergent.

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1. Introduction

Second-order physical systems with oscillatory solutions find applications in diverse areas of human endeavors like engineering and sciences. Such systems are applied in vibration of mass-spring systems, astrophysics, control theory, mechanics, circuit theory, biology among others [1, 2]. In this research, an accuracy-preserving BHA shall be derived for the direct integration of second order oscillatory physical systems of the form

$$y''(t) = f\,(t, y(t), y'(t)) \tag{1}$$

subject to the initial conditions

$$y(t_0) = y_0,\ y'(t_0) = y_0^{'} \tag{2}$$

on the interval $t \in [t_0, t_N]$, where $f : \Re \times \Re^m \to \Re^m$, N is an integer and m the dimension of equation (1). Equation (1) is assumed to satisfy the uniqueness theorem stated in Theorem 1.1.

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Theorem 1.1 [3]
Let,

$$y^{(n)}(t) = f\left(t,\ y(t),\ y'(t), ...,\ y^{(n-1)}(t)\right),\ y^{(k)}(t_0) = y_k \quad (3)$$

where $k = 0, 1, ..., (n-1), y^{(0)} = y$, $y(t)$ and f are scalars. Let $\mathfrak{R}$ be the region defined by the inequalities $t_0 \le t \le t_0 + a$, $|s_k - y_k| \le b_k$, $k = 0, 1, ..., n-1$, where $y_k \ge 0$ for $k > 0$. Suppose the function $f(t, s_0, s_1, ..., s_{n-1})$ in (3) is non-negative, continuous and non-decreasing in t, and continuous and non-decreasing in s_k for each $k = 0, 1, ..., n-1$ in the region $\mathfrak{R}$. If in addition $f(t, y_0, y_1, ..., y_{n-1}) \neq 0$ in $\mathfrak{R}$ for $t > t_0$, then the initial value problem (3) has at most one solution in $\mathfrak{R}$".

The derivation will be carried out via a continuous scheme based on linear multistep method by incorporating four off-step points. The implementation of the algorithm will be effected in a block-by-block mode, thus making it self-starting (i.e. without the need for predictors).

Equations of the form (1) can be solved by first transforming them into their equivalent system of first order differential equations and then employing an appropriate method [4-6]. However, one of the setbacks of such approach is that some of the vital properties and characteristics of the higher order differential equations are lost in the course of the conversion. Besides, coding such methods are often cumbersome since in most cases subroutines have to be incorporated to provide the starting values.

Some methods have also been proposed in literature for the direct solution of special second order differential equations. Such equations are termed 'special' because they do not depend on y', in order words they are of the form $y'' = f(x, y)$. These methods often require fewer function evaluations and less memory space [7-9].

Over the years, several authors have proposed different methods for the direct solution of oscillatory problems of the form (1). Ref[2] proposed an eleventh order block hybrid method for the direct solution of system of second order differential equations including Hamiltonian systems. The method was derived from continuous scheme via hybrid method approach with several off-step points. The method was implemented in a block manner. Ref[10] formulated a continuous explicit hybrid method for the solution of second order differential equations. The authors interpolated the basis function at both grid and off-grid points while the differential systems were collocated at selected points. The authors further derived starting values of the same order with the methods by adopting Taylor series expansion to circumvent the inherent disadvantage of starting values of lower order. Ref[11] developed an order eight implicit block method for the solution of second order differential equations. The authors adopted the Hermite polynomial as basis function to construct the method that comprises first and second derivatives. The basic properties of the method were analysed and the method was implemented on some linear and nonlinear second order differential equations. Other researchers that also developed direct methods for solving problems of the form (1) are [12-23].

2. Derivation and Implementation of the BHA

2.1. Derivation of the BHA

The accuracy-preserving BHA shall be derived by seeking a continuous approximate solution $Y(t)$ to the second order oscillatory problems of the form (1) on the interval $[t_n,\ t_{n+1}]$. This is expressed compactly in a vector form as,

$$Y(t) = \left[1\ t\ t^2 ...\ \ t^7\right] \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ . \\ . \\ . \\ \sigma_7 \end{bmatrix} \quad (4)$$

where $\sigma_0, \sigma_1, \sigma_2, ..., \sigma_7$are uniquely determined parameters in $\mathfrak{R}^n$. Equation (4) is interpolated at the points $(t_{n+\tau},\ y_{n+\tau})$, $\tau = r, s$ and collocated at the points $(t_{n+\kappa},\ f_{n+\kappa})$, $\kappa = 0, p, q, r, s, 1$, where τ and $\kappa$are the interpolation and collocation points respectively. The off-step points p, q, r, s are taken as $p = 1/5$, $q = 2/5$, $r = 3/5$ and $s = 4/5$. The parameters $\sigma_0, \sigma_1, \sigma_2, ..., \sigma_7$are determined by imposing the conditions

$$Y\left(t_{n+\tau}\right) = y_{n+\tau},\ \ \tau = r, s \quad (5)$$

$$Y''\left(t_{n+\kappa}\right) = f_{n+\kappa},\ \ \kappa = 0, p, q, r, s, 1 \quad (6)$$

to obtain the system of equations

$$\begin{bmatrix} 1 & t_{n+r} & t_{n+r}^2 & t_{n+r}^3 & t_{n+r}^4 & t_{n+r}^5 & t_{n+r}^6 & t_{n+r}^7 \\ 1 & t_{n+s} & t_{n+s}^2 & t_{n+s}^3 & t_{n+s}^4 & t_{n+s}^5 & t_{n+s}^6 & t_{n+s}^7 \\ 0 & 0 & 2 & 6t_n & 12t_n^2 & 20t_n^3 & 30t_n^4 & 42t_n^5 \\ 0 & 0 & 2 & 6t_{n+p} & 12t_{n+p}^2 & 20t_{n+p}^3 & 30t_{n+p}^4 & 42t_{n+p}^5 \\ 0 & 0 & 2 & 6t_{n+q} & 12t_{n+q}^2 & 20t_{n+q}^3 & 30t_{n+q}^4 & 42t_{n+q}^5 \\ 0 & 0 & 2 & 6t_{n+r} & 12t_{n+r}^2 & 20t_{n+r}^3 & 30t_{n+r}^4 & 42t_{n+r}^5 \\ 0 & 0 & 2 & 6t_{n+s} & 12t_{n+s}^2 & 20t_{n+s}^3 & 30t_{n+s}^4 & 42t_{n+s}^5 \\ 0 & 0 & 2 & 6t_{n+1} & 12t_{n+1}^2 & 20t_{n+1}^3 & 30t_{n+1}^4 & 42t_{n+1}^5 \end{bmatrix} \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ . \\ . \\ . \\ \sigma_7 \end{bmatrix} = \begin{bmatrix} y_{n+r} \\ y_{n+s} \\ f_n \\ f_{n+p} \\ f_{n+q} \\ f_{n+r} \\ f_{n+s} \\ f_{n+1} \end{bmatrix} \quad (7)$$

The system (7) is solved using SWP 5.5 for the parameters $\sigma_j^{\prime}s$ and then substituted into equation (4) to get the continuous form

$$Y(t) = \begin{bmatrix} 1 & t & t^2 & \dots & t^7 \end{bmatrix} \begin{bmatrix} 1 & t_{n+r} & t_{n+r}^2 & t_{n+r}^3 & t_{n+r}^4 & t_{n+r}^5 & t_{n+r}^6 & t_{n+r}^7 \\ 1 & t_{n+s} & t_{n+s}^2 & t_{n+s}^3 & t_{n+s}^4 & t_{n+s}^5 & t_{n+s}^6 & t_{n+s}^7 \\ 0 & 0 & 2 & 6t_n & 12t_n^2 & 20t_n^3 & 30t_n^4 & 42t_n^5 \\ 0 & 0 & 2 & 6t_{n+p} & 12t_{n+p}^2 & 20t_{n+p}^3 & 30t_{n+p}^4 & 42t_{n+p}^5 \\ 0 & 0 & 2 & 6t_{n+q} & 12t_{n+q}^2 & 20t_{n+q}^3 & 30t_{n+q}^4 & 42t_{n+q}^5 \\ 0 & 0 & 2 & 6t_{n+r} & 12t_{n+r}^2 & 20t_{n+r}^3 & 30t_{n+r}^4 & 42t_{n+r}^5 \\ 0 & 0 & 2 & 6t_{n+s} & 12t_{n+s}^2 & 20t_{n+s}^3 & 30t_{n+s}^4 & 42t_{n+s}^5 \\ 0 & 0 & 2 & 6t_{n+1} & 12t_{n+1}^2 & 20t_{n+1}^3 & 30t_{n+1}^4 & 42t_{n+1}^5 \end{bmatrix}^{-1} \begin{bmatrix} y_{n+r} \\ y_{n+s} \\ f_n \\ f_{n+p} \\ f_{n+q} \\ f_{n+r} \\ f_{n+s} \\ f_{n+1} \end{bmatrix} \tag{8}$$

On simplification and evaluation of the continuous scheme (8), we obtain

$$Y(t) = \sum_{\tau=r,s} \alpha_\tau(x) y_{n+\tau} + h^2 \left(\sum_{j=0}^{1} \beta_j(t) f_{n+j} + \beta_k(t) f_{n+k} \right), \ k = p, q, r, s \tag{9}$$

where

$$\left.\begin{aligned}
\alpha_r(x) &= -5x + 4 \\
\alpha_s(x) &= -3 + 5x \\
\beta_0(x) &= -\tfrac{625}{1008}x^7 + \tfrac{125}{48}x^6 - \tfrac{425}{96}x^5 + \tfrac{125}{32}x^4 - \tfrac{137}{72}x^3 + \tfrac{1}{2}x^2 - \tfrac{397}{6300}x + \tfrac{1}{375} \\
\beta_p(x) &= \tfrac{3125}{1008}x^7 - \tfrac{875}{72}x^6 + \tfrac{1775}{96}x^5 - \tfrac{1925}{144}x^4 + \tfrac{25}{6}x^3 - \tfrac{14059}{50400}x + \tfrac{127}{3000} \\
\beta_q(x) &= -\tfrac{3125}{504}x^7 + \tfrac{1625}{72}x^6 - \tfrac{1475}{48}x^5 + \tfrac{2675}{144}x^4 - \tfrac{25}{6}x^3 - \tfrac{157}{1260}x + \tfrac{29}{375} \\
\beta_r(x) &= \tfrac{3125}{504}x^7 - \tfrac{125}{6}x^6 + \tfrac{1225}{48}x^5 - \tfrac{325}{24}x^4 + \tfrac{25}{9}x^3 - \tfrac{5813}{25200}x + \tfrac{161}{1500} \\
\beta_s(x) &= -\tfrac{3125}{1008}x^7 + \tfrac{1375}{144}x^6 - \tfrac{1025}{96}x^5 + \tfrac{1525}{288}x^4 - \tfrac{25}{24}x^3 - \tfrac{1}{1575}x + \tfrac{4}{375} \\
\beta_1(x) &= \tfrac{625}{1008}x^7 - \tfrac{125}{72}x^6 + \tfrac{175}{96}x^5 - \tfrac{125}{144}x^4 + \tfrac{1}{6}x^3 - \tfrac{107}{50400}x - \tfrac{1}{3000}
\end{aligned}\right\} \tag{10}$$

and x is expressed as

$$x = \frac{t - t_n}{h} \tag{11}$$

Solving (9) for the independent solution at the grid points gives the method

$$Y_{n+j} = \sum_{i=0}^{1} \frac{(jh)^i}{i\,!} y_n^{(i)} + h^2 \left(\sum_{j=0}^{1} \eta_j(x) f_{n+j} + \eta_k(x) f_{n+k} \right), \ k = p, q, r, s \tag{12}$$

where

$$\left.\begin{aligned}
\eta_0(x) &= -\tfrac{625}{144}x^6 + \tfrac{125}{8}x^5 - \tfrac{2125}{96}x^4 + \tfrac{125}{8}x^3 - \tfrac{137}{24}x^2 + x \\
\eta_p(x) &= \tfrac{3125}{144}x^6 - \tfrac{875}{12}x^5 + \tfrac{8875}{96}x^4 - \tfrac{1925}{36}x^3 + \tfrac{25}{2}x^2 \\
\eta_q(x) &= -\tfrac{3125}{72}x^6 + \tfrac{1625}{12}x^5 - \tfrac{7375}{48}x^4 + \tfrac{2675}{36}x^3 - \tfrac{25}{2}x^2 \\
\eta_r(x) &= \tfrac{3125}{72}x^6 - 125x^5 + \tfrac{6125}{48}x^4 - \tfrac{325}{6}x^3 + \tfrac{25}{3}x^2 \\
\eta_s(x) &= -\tfrac{3125}{144}x^6 + \tfrac{1375}{24}x^5 - \tfrac{5125}{96}x^4 + \tfrac{1525}{72}x^3 - \tfrac{25}{8}x^2 \\
\eta_1(x) &= \tfrac{625}{144}x^6 - \tfrac{125}{12}x^5 + \tfrac{875}{96}x^4 - \tfrac{125}{36}x^3 + \tfrac{1}{2}x^2
\end{aligned}\right\} \tag{13}$$

On the evaluation of (12) at $x = p, q, r, s, 1$, we obtain the discrete accuracy-preserving BHA whose coefficients are presented in Table 1.

Therefore, the accuracy-preserving BHA is given explicitly as,

$$\left.\begin{aligned}
y_{n+\frac{1}{5}} &= y_n + \tfrac{1}{5}hy_n' + h^2\left(\tfrac{1231}{126000}f_n + \tfrac{863}{50400}f_{n+\frac{1}{5}} - \tfrac{761}{63000}f_{n+\frac{2}{5}} + \tfrac{941}{12600}f_{n+\frac{3}{5}} - \tfrac{341}{126000}f_{n+\frac{4}{5}} + \tfrac{107}{252000}f_{n+1}\right) \\
y_{n+\frac{2}{5}} &= y_n + \tfrac{2}{5}hy_n' + h^2\left(\tfrac{71}{3150}f_n + \tfrac{544}{7875}f_{n+\frac{1}{5}} - \tfrac{37}{1575}f_{n+\frac{2}{5}} + \tfrac{136}{7875}f_{n+\frac{3}{5}} - \tfrac{101}{15750}f_{n+\frac{4}{5}} + \tfrac{8}{7875}f_{n+1}\right) \\
y_{n+\frac{3}{5}} &= y_n + \tfrac{3}{5}hy_n' + h^2\left(\tfrac{123}{3500}f_n + \tfrac{3501}{28000}f_{n+\frac{1}{5}} - \tfrac{9}{3500}f_{n+\frac{2}{5}} + \tfrac{87}{2800}f_{n+\frac{3}{5}} - \tfrac{9}{875}f_{n+\frac{4}{5}} + \tfrac{9}{5600}f_{n+1}\right) \\
y_{n+\frac{4}{5}} &= y_n + \tfrac{4}{5}hy_n' + h^2\left(\tfrac{376}{7875}f_n + \tfrac{1424}{7875}f_{n+\frac{1}{5}} + \tfrac{176}{7875}f_{n+\frac{2}{5}} + \tfrac{608}{7875}f_{n+\frac{3}{5}} - \tfrac{16}{1575}f_{n+\frac{4}{5}} + \tfrac{16}{7875}f_{n+1}\right) \\
y_{n+1} &= y_n + hy_n' + h^2\left(\tfrac{61}{1008}f_n + \tfrac{475}{2016}f_{n+\frac{1}{5}} + \tfrac{25}{504}f_{n+\frac{2}{5}} + \tfrac{125}{1008}f_{n+\frac{3}{5}} + \tfrac{25}{1008}f_{n+\frac{4}{5}} + \tfrac{11}{2016}f_{n+1}\right)
\end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned}
y'_{n+\frac{1}{5}} &= y_n' + h\left(\tfrac{19}{288}f_n + \tfrac{1427}{7200}f_{n+\frac{1}{5}} - \tfrac{133}{1200}f_{n+\frac{2}{5}} + \tfrac{241}{3600}f_{n+\frac{3}{5}} - \tfrac{173}{7200}f_{n+\frac{4}{5}} + \tfrac{3}{800}f_{n+1}\right) \\
y'_{n+\frac{2}{5}} &= y_n' + h\left(\tfrac{14}{225}f_n + \tfrac{43}{150}f_{n+\frac{1}{5}} + \tfrac{7}{255}f_{n+\frac{2}{5}} + \tfrac{7}{255}f_{n+\frac{3}{5}} - \tfrac{1}{75}f_{n+\frac{4}{5}} + \tfrac{1}{450}f_{n+1}\right) \\
y'_{n+\frac{3}{5}} &= y_n' + h\left(\tfrac{51}{800}f_n + \tfrac{219}{800}f_{n+\frac{1}{5}} + \tfrac{57}{400}f_{n+\frac{2}{5}} + \tfrac{57}{400}f_{n+\frac{3}{5}} - \tfrac{21}{800}f_{n+\frac{4}{5}} + \tfrac{3}{800}f_{n+1}\right) \\
y'_{n+\frac{4}{5}} &= y_n' + h\left(\tfrac{14}{225}f_n + \tfrac{64}{255}f_{n+\frac{1}{5}} + \tfrac{8}{75}f_{n+\frac{2}{5}} + \tfrac{64}{255}f_{n+\frac{3}{5}} + \tfrac{14}{255}f_{n+\frac{4}{5}} + (0)f_{n+1}\right) \\
y'_{n+1} &= y_n' + h\left(\tfrac{19}{288}f_n + \tfrac{25}{96}f_{n+\frac{1}{5}} + \tfrac{25}{144}f_{n+\frac{2}{5}} + \tfrac{25}{144}f_{n+\frac{3}{5}} + \tfrac{25}{96}f_{n+\frac{4}{5}} + \tfrac{19}{288}f_{n+1}\right)
\end{aligned}\right\} \tag{15}$$

Equations (14) and (15) together form the proposed BHA.

Table 1. Coefficients of the BHA

	y_n	y'_n	f_n	f_{n+p}	f_{n+q}	f_{n+r}	f_{n+s}	f_{n+1}
y_{n+p}	1	$\frac{1}{5}$	$\frac{1231}{126000}$	$\frac{863}{50400}$	$\frac{-761}{63000}$	$\frac{941}{12600}$	$\frac{-341}{126000}$	$\frac{107}{252000}$
y_{n+q}	1	$\frac{2}{5}$	$\frac{71}{3150}$	$\frac{544}{7875}$	$\frac{-37}{1575}$	$\frac{136}{7875}$	$\frac{-101}{15750}$	$\frac{8}{7875}$
y_{n+r}	1	$\frac{3}{5}$	$\frac{123}{3500}$	$\frac{3501}{28000}$	$\frac{-9}{3500}$	$\frac{87}{2800}$	$\frac{-9}{875}$	$\frac{9}{5600}$
y_{n+s}	1	$\frac{4}{5}$	$\frac{376}{7875}$	$\frac{1424}{7875}$	$\frac{176}{7875}$	$\frac{608}{7875}$	$\frac{-16}{1575}$	$\frac{16}{7875}$
y_{n+1}	1	1	$\frac{61}{1008}$	$\frac{475}{2016}$	$\frac{25}{504}$	$\frac{125}{1008}$	$\frac{25}{1008}$	$\frac{11}{2016}$
y'_{n+p}	0	1	$\frac{19}{288}$	$\frac{1427}{7200}$	$\frac{-133}{1200}$	$\frac{241}{3600}$	$\frac{-173}{7200}$	$\frac{3}{800}$
y'_{n+q}	0	1	$\frac{14}{225}$	$\frac{43}{150}$	$\frac{7}{255}$	$\frac{7}{255}$	$\frac{-1}{75}$	$\frac{1}{450}$
y'_{n+r}	0	1	$\frac{51}{800}$	$\frac{219}{800}$	$\frac{57}{400}$	$\frac{57}{400}$	$\frac{-21}{800}$	$\frac{3}{800}$
y'_{n+s}	0	1	$\frac{14}{225}$	$\frac{64}{255}$	$\frac{8}{75}$	$\frac{64}{255}$	$\frac{14}{55}$	0
y'_{n+1}	0	1	$\frac{19}{288}$	$\frac{25}{96}$	$\frac{25}{144}$	$\frac{25}{144}$	$\frac{25}{96}$	$\frac{19}{288}$

2.2. Implementation of the BHA

Unlike the conventional predictor-corrector methods that require starting values, the proposed BHA is self-starting. It is implemented in block mode over a non- overlapping one-step interval $[t_n, t_{n+1}]$, $n = 0, 1, 2, ..., N-1$. Firstly, we set the data inputs namely initial conditions, step size and differential equation to be solved. Thus at $n = 0$, the block method is applied simultaneously on the subinterval $[t_0, t_1]$ to get $\left[y_{\kappa_i}, y'_{\kappa_i}\right]^T$, $i = 1, 2, ...5$. Applying the values of the known previous block, we get the values of the next subinterval $[t_1, t_2]$ and the process goes on in this manner until the final subinterval $[t_{N-1}, t_N]$ is obtained. The code was written in MATLAB 2021a.

3. Properties of the BHA

The properties of the proposed BHA shall be analysed in this section.

3.1. Order and Error Constant of the BHA

Definition 3.1
The linear difference operators associated with the BHA in equations (14) and (15) are defined as

$$\left.\begin{array}{l}
\ell_p\left\{y(t_n);h\right\} = y\left(t_n+ph\right) - \left(\alpha_r(t)y\left(t_n+rh\right) + \alpha_s(t)y\left(t_n+sh\right) + h^2\sum_{j=0}^{1}\left(\beta_j(t)f_{n+j} + \beta_k(t)f_{n+k}\right)\right), k = p,q,r,s \\
\ell_q\left\{y(t_n);h\right\} = y\left(t_n+qh\right) - \left(\alpha_r(t)y\left(t_n+rh\right) + \alpha_s(t)y\left(t_n+sh\right) + h^2\sum_{j=0}^{1}\left(\beta_j(t)f_{n+j} + \beta_k(t)f_{n+k}\right)\right), k = p,q,r,s \\
\ell_r\left\{y(t_n);h\right\} = y\left(t_n+rh\right) - \left(\alpha_r(t)y\left(t_n+rh\right) + \alpha_s(t)y\left(t_n+sh\right) + h^2\sum_{j=0}^{1}\left(\beta_j(t)f_{n+j} + \beta_k(t)f_{n+k}\right)\right), k = p,q,r,s \\
\ell_s\left\{y(t_n);h\right\} = y\left(t_n+sh\right) - \left(\alpha_r(t)y\left(t_n+rh\right) + \alpha_s(t)y\left(t_n+sh\right) + h^2\sum_{j=0}^{1}\left(\beta_j(t)f_{n+j} + \beta_k(t)f_{n+k}\right)\right), k = p,q,r,s \\
\ell_1\left\{y(t_n);h\right\} = y\left(t_n+h\right) - \left(\alpha_r(t)y\left(t_n+rh\right) + \alpha_s(t)y\left(t_n+sh\right) + h^2\sum_{j=0}^{1}\left(\beta_j(t)f_{n+j} + \beta_k(t)f_{n+k}\right)\right), k = p,q,r,s
\end{array}\right\} \tag{16}$$

$$\left.\begin{array}{l}
\ell'_p\left\{y(t_n);h\right\} = hy'\left(t_n+ph\right) - \left(\alpha'_r(t)y\left(t_n+rh\right) + \alpha'_s(t)y\left(t_n+sh\right) + h^2\sum_{j=0}^{1}\left(\beta'_j(t)f_{n+j} + \beta'_k(t)f_{n+k}\right)\right), k = p,q,r,s \\
\ell'_q\left\{y(t_n);h\right\} = hy'\left(t_n+qh\right) - \left(\alpha'_r(t)y\left(t_n+rh\right) + \alpha'_s(t)y\left(t_n+sh\right) + h^2\sum_{j=0}^{1}\left(\beta'_j(t)f_{n+j} + \beta'_k(t)f_{n+k}\right)\right), k = p,q,r,s \\
\ell'_r\left\{y(t_n);h\right\} = hy'\left(t_n+rh\right) - \left(\alpha'_r(t)y\left(t_n+rh\right) + \alpha'_s(t)y\left(t_n+sh\right) + h^2\sum_{j=0}^{1}\left(\beta'_j(t)f_{n+j} + \beta'_k(t)f_{n+k}\right)\right), k = p,q,r,s \\
\ell'_s\left\{y(t_n);h\right\} = hy'\left(t_n+sh\right) - \left(\alpha'_r(t)y\left(t_n+rh\right) + \alpha'_s(t)y\left(t_n+sh\right) + h^2\sum_{j=0}^{1}\left(\beta'_j(t)f_{n+j} + \beta'_k(t)f_{n+k}\right)\right), k = p,q,r,s \\
\ell'_1\left\{y(t_n);h\right\} = hy'\left(t_n+h\right) - \left(\alpha'_r(t)y\left(t_n+rh\right) + \alpha'_s(t)y\left(t_n+sh\right) + h^2\sum_{j=0}^{1}\left(\beta'_j(t)f_{n+j} + \beta'_k(t)f_{n+k}\right)\right), k = p,q,r,s
\end{array}\right\} \tag{17}$$

Suppose $y(t)$is differentiable, the terms ℓ and ℓ' are expanded as Taylor series about the point t_n to get

$$\left.\begin{array}{l}
\ell_k\left\{y(t_n);h\right\} = c_k h^{p+2} y^{(p+2)}\left(t_n\right) + O\left(h^{p+3}\right), \ k = p,q,r,s \\
\ell'_k\left\{y(t_n);h\right\} = c'_k h^{p+2} y^{(p+2)}\left(t_n\right) + O\left(h^{p+3}\right), \ k = p,q,r,s
\end{array}\right\} \tag{18}$$

where p in equation (18) denotes the order of the BHA and ℓ and ℓ' the error constants.

Table 2. Order and error constant of the BHA

$y(t_{n+j})$	Order	Error Constant
$j = 1/5$	6	-2.1058×10^{-8}
$j = 2/5$	6	-5.1471×10^{-8}
$j = 3/5$	6	-8.0571×10^{-8}
$j = 4/5$	6	-1.0836×10^{-7}
j=1	6	-1.4550×10^{-7}

Therefore, the proposed BHA is of order 6.

Definition 3.2 [24]

A linear multistep method for a second order problem is said to be of order pif it satisfies $c_0=c_1=c_2=\ldots=c_p=c_{p+1}=0$, $c_{p+2} \neq 0$, where,

$$\left.\begin{aligned} c_0 &= \textstyle\sum_{j=0}^{k} \alpha_j \\ c_1 &= \textstyle\sum_{j=0}^{k} \left(j\alpha_j - \beta_j\right) \\ &\;\;. \\ &\;\;. \\ &\;\;. \\ c_p &= \textstyle\sum_{j=0}^{k} \left[\frac{1}{p!} j^p \alpha_j - \frac{1}{(p-1)!} j^{p-1} \beta_j\right],\ p = 2,3,...,q+1 \end{aligned}\right\} \tag{19}$$

The parameter $c_{p+2} \neq 0$ *is referred to as the error constant of the method with the local truncation error defined as* $t_{n+k} = c_{p+2}h^{p+2}y^{(p+2)}(t_n) + O\left(h^{p+3}\right)$*".*

The BHA is expanded in Taylor series about the point t_n to obtain

$$\begin{bmatrix} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{5}h\right)^j}{j!} y_n^j - y_n - \frac{1}{5}hy_n' - \frac{1231}{126000}h^2 y_n'' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \left\{ \frac{863}{50400}\left(\frac{1}{5}\right)^j - \frac{761}{63000}\left(\frac{2}{5}\right)^j + \frac{941}{12600}\left(\frac{3}{5}\right)^j - \frac{341}{126000}\left(\frac{4}{5}\right)^j + \frac{107}{252000}(1)^j \right\} \\ \sum_{j=0}^{\infty} \frac{\left(\frac{2}{5}h\right)^j}{j!} y_n^j - y_n - \frac{2}{5}hy_n' - \frac{71}{3150}h^2 y_n'' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \left\{ \frac{544}{7875}\left(\frac{1}{5}\right)^j - \frac{37}{1575}\left(\frac{2}{5}\right)^j + \frac{136}{7875}\left(\frac{3}{5}\right)^j - \frac{101}{15750}\left(\frac{4}{5}\right)^j + \frac{8}{7875}(1)^j \right\} \\ \sum_{j=0}^{\infty} \frac{\left(\frac{3}{5}h\right)^j}{j!} y_n^j - y_n - \frac{3}{5}hy_n' - \frac{123}{3500}h^2 y_n'' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \left\{ \frac{3501}{28000}\left(\frac{1}{5}\right)^j - \frac{9}{3500}\left(\frac{2}{5}\right)^j + \frac{87}{2800}\left(\frac{3}{5}\right)^j - \frac{9}{875}\left(\frac{4}{5}\right)^j + \frac{9}{5600}(1)^j \right\} \\ \sum_{j=0}^{\infty} \frac{\left(\frac{4}{5}h\right)^j}{j!} y_n^j - y_n - \frac{4}{5}hy_n' - \frac{376}{7875}h^2 y_n'' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \left\{ \frac{1424}{7875}\left(\frac{1}{5}\right)^j + \frac{176}{7875}\left(\frac{2}{5}\right)^j + \frac{608}{7875}\left(\frac{3}{5}\right)^j - \frac{16}{1575}\left(\frac{4}{5}\right)^j + \frac{16}{7875}(1)^j \right\} \\ \sum_{j=0}^{\infty} \frac{(h)^j}{j!} y_n^j - y_n - hy_n' - \frac{61}{1008}h^2 y_n'' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \left\{ \frac{475}{2016}\left(\frac{1}{5}\right)^j + \frac{25}{504}\left(\frac{2}{5}\right)^j + \frac{125}{1008}\left(\frac{3}{5}\right)^j + \frac{25}{1008}\left(\frac{4}{5}\right)^j + \frac{11}{2016}(1)^j \right\} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{20}$$

Table 2 displays the order and error constant of the proposed BHA.

3.2. Consistency of the BHA

Definition 3.3 [24]

"If a method has orderp ≥ 1*, then it is consistent".*

Thus, the proposed BHA is consistent since it is of order 6.

3.3. Zero-Stability of the BHA

Definition 3.4 [25]

"If no root of the characteristic polynomial has a modulus greater than one and every root with modulus one is simple, then such a method is called zero-stable".

In order words, the BHA is zero-stable if its first characteristic polynomial $\rho(z)$satisfies $|z_d| \leq 1$, $d = 1,2,3,...,n$. Therefore,

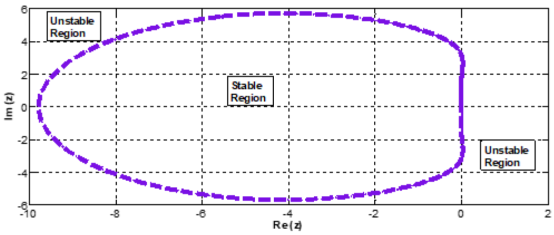

Figure 1. Region of absolute stability of the BHA

$$\rho(z) = \left\| \begin{bmatrix} z & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\| = z^4(z-1) = 0 \tag{21}$$

On the evaluation of equation (21), we obtain $z = 0, 0, 0, 0, 1$. Therefore, the proposed BHA is zero-stable.

3.4. Convergence of the BHA

Theorem 3.1 [25]

"For a method to be convergent, it is necessary and sufficient that it be consistent and zero-stable."

The BHA is convergent since it is consistent and zero-stable.

3.5. Region of Absolute Stability of the BHA

Definition 3.5 [26]

"The region of absolute stability of a method is the region in the complex z plane, where $z = \lambda h$."

In such region, the approximate solution $y'' = -\lambda^2 y$ satisfies $y_i \to 0$ as $i \to 0$ for any initial condition.

Definition 3.6 [26]

"If the region of absolute stability of method contains the whole of the left half-plane, that is $Re(h\lambda) < 0$, then the method is said to be A-stable."

Using the boundary locus method, the stability polynomial for the proposed BHA is

$$\begin{aligned} \overline{h}(w) = &-h^{10}\left(\left(8.13\times10^{-10}\right)w^5 + \left(1.00\times10^{-8}\right)w^4\right) \\ &- h^8\left(\left(5.02\times10^{-8}\right)w^5 + \left(5.04\times10^{-6}\right)w^4\right) \\ &- h^6\left(\left(1.32\times10^{-6}\right)w^5 + \left(7.18\times10^{-4}\right)w^4\right) \\ &- h^4\left(\left(3.71\times10^{-2}\right)w^4 - \left(2.00\times10^{-4}\right)w^5\right) \\ &- h^2\left(\left(2.00\times10^{-2}\right)w^5 + \left(6.27\times10^{-1}\right)w^4\right) + w^5 - 2w^4 \end{aligned} \tag{22}$$

For more details on boundary locus method, see [27-28]. The region of absolute stability of the BHA is shown in Figure 1.

Note that the absolute stability of the proposed BHA in Figure 1 is the region that lies inside the closed contour. From the figure, it is clear that the BHA is A-stable.

4. Numerical Examples

The newly derived BHA shall be applied in integrating some second-order physical systems with oscillatory solutions. The following abbreviations shall be used in the Tables 3-5 and Figures 3-10.

t: Point of evaluation
h: Step-size
NS: Number of steps taken
Err: Maximum global error
ErrH: Maximum global error for the Hamiltonian
ode 45: MATLAB inbuilt solver
VSSHM: Variable step-size hybrid method of order seven by [13]
BHM: Order eleven block hybrid method by [2]
IBM: Order eight implicit block method by [11]
BHA: Newly proposed sixth order block hybrid algorithm

Let $y(t_n)$ and y_n be the exact and approximate solutions respectively, then the maximum global error is computed as $Err = \max|y(t_n) - y_n|$ and the maximum global error for the Hamiltonian as $ErrH = \max|H_n - H_0|$.

Example 1. The system below describes the perturbed Kepler problem

$$\left.\begin{aligned} y_1''(t) &= -\frac{y_1(t)}{\left(y_1^2(t)+y_2^2(t)\right)^{3/2}} - \frac{\left(e^2+2e\right)y_1(t)}{\left(y_1^2(t)+y_2^2(t)\right)^{5/2}}, \ y_1(0) = 1,\ y_1'(0) = 0 \\ y_2''(t) &= -\frac{y_2(t)}{\left(y_1^2(t)+y_2^2(t)\right)^{3/2}} - \frac{\left(e^2+2e\right)y_2(t)}{\left(y_1^2(t)+y_2^2(t)\right)^{5/2}}, \ y_2(0) = 0,\ y_2'(0) = 1+e \end{aligned}\right\} \tag{23}$$

whose exact solution is given by

$$\left.\begin{aligned} y_1(t) &= \cos\left(e^t + t\right) \\ y_2(t) &= \sin\left(e^t + t\right) \end{aligned}\right\} \tag{24}$$

where e is the eccentricity taken as $e = 10^{-3}$. The problem (23) has an angular momentum

$$L = y_1(t)y_2'(t) - y_2(t)y_1'(t) \tag{25}$$

as a first integral and also has the Hamiltonian

$$H = \frac{1}{2}\left(y_1'^2(t) + y_2'^2(t)\right) - \frac{1}{\left(y_1^2(t)+y_2^2(t)\right)^{1/2}} - \frac{\left(e^2+2e\right)}{3\left(y_1^2(t)+y_2^2(t)\right)^{3/2}} \tag{26}$$

The author in [2] developed an eleventh order block hybrid method to compute the maximum global errors in the solution as well as the maximum global error in the Hamiltonian of Example 1 at different step-sizes. The newly derived BHA was also used to solve the same problem. The results obtained clearly showed that even though the BHA is of the sixth order, it performed better than the eleventh order block hybrid method proposed by [2], see Table 3. Figure 2 shows the plots for mean anomaly against eccentric anomaly at different eccentricities (e=0.01, 0.1, 0.2, 0.4, 0.6 and 0.9) using the newly derived BHA. On the other hand, Figure 3 shows the exact plots for mean anomaly against eccentric anomaly at the same eccentricities. The plots generated using the BHA (see Figure 2)

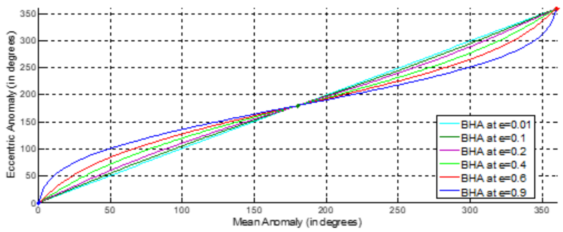

Figure 2. Mean anomaly against eccentric anomaly (for Example 1) at different values of eccentricity using the proposed BHA

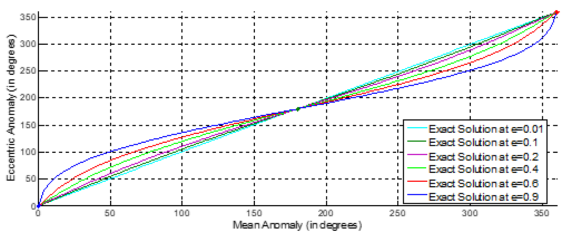

Figure 3. Exact plots for mean anomaly against eccentric anomaly (for Example 1) at different values of eccentricity

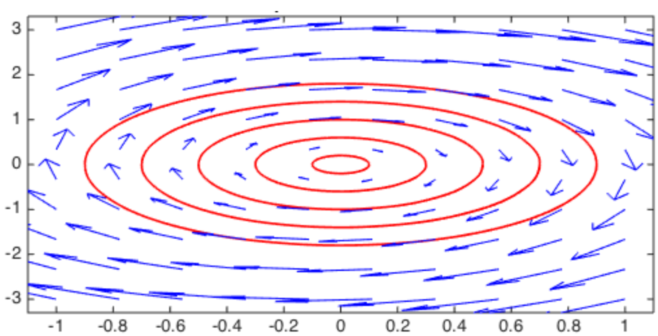

Figure 4. Showing the trajectories in the phase plane flow (for Example 1) using the proposed BHA at N=160

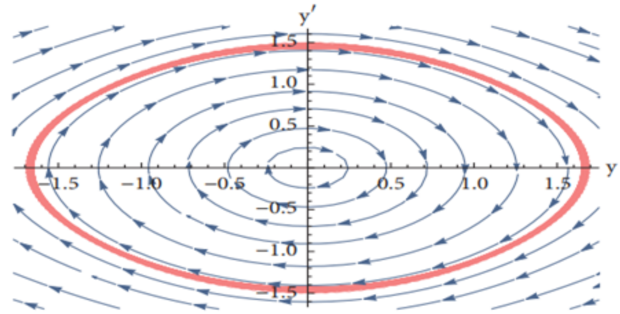

Figure 5. Showing the trajectories in the phase plane flow (for Example 1) using BHM at N=160

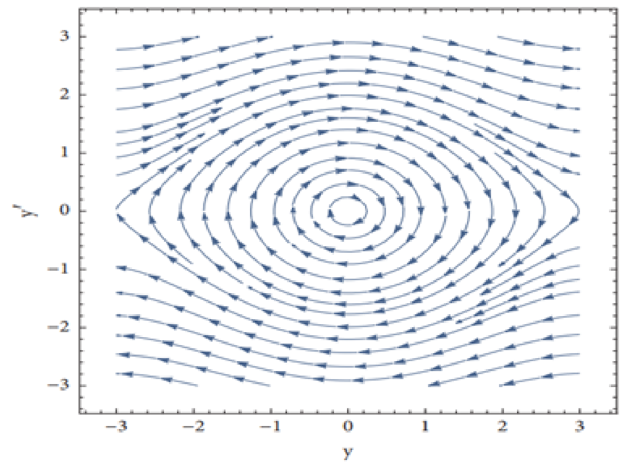

Figure 6. Showing the exact trajectories in the phase plane flow (for Example 1) at N=160

converges to those of the exact plots in Figure 3 at the respective eccentricities. It is important to state that the Figures 2 and 3 show the relationship between mean anomaly (that is, the angle between line drawn from the sun to perihelion and to a point moving in the orbit) and eccentric anomaly (that is, the angle that describes the position of a body that is moving along an elliptic orbit). Furthermore, Figures 4, 5 and 6 respectively depict the trajectories in the phase plane flow using proposed BHA, BHA developed by [2] and the exact flow $N = 160$. For more details on Kepler equations, see [29-35].

Example 2. The nonlinear system below describes the aerodynamic damping of a pendulum

$$\theta'' + \frac{g}{L}\sin\theta + \frac{c_a}{mL}(\theta')^2\, sgn\,(\theta') = 0 \tag{27}$$

and the linear system below describes the viscous damping of a pendulum

$$\theta'' + \frac{c_d}{m}(\theta')^2 + \frac{g}{L}\theta = 0 \tag{28}$$

To determine the quadratic damping in equations (27) and (28) respectively, the proposed BHA is employed in simulating the two equations and the simulation results are compared with that of author [1]. The author transformed (27) and (28) to their equivalent first order systems and then applied MATLAB solver (ode 45). Let $g = 981cms^{-2}$, $c_a = c_d = 14$, $L = 5cm$, $m = 10g$, $\theta(0) = 1.57$and $\theta'(0) = 0$. The initial value θ is in radians. Also, sgn in equation (27) represents signum function; c_a and c_d are the coefficients of aerodynamic damping and viscous damping respectively. Figure 7 shows a simple pendulum with two forces acting on the mass; that is, the weight mg and the tension T. The net force is found to be $F = mg\sin\theta$.

From simulation results presented in Figure 8, it is obvious that quadratic damping causes a rapid initial amplitude reduction but is less effective as time increases and amplitude decreases. On the other hand, linear damping is less effective initially but is more effective as time increases. The plots confirm that the proposed BHA is computationally reliable as they agree with those of ode 45.

Example 3. Consider the nonlinear oscillatory system

$$y''(t) + \begin{bmatrix} 13 & -12 \\ -12 & 13 \end{bmatrix} y(t) = \frac{\partial U}{\partial y},\ y(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix},\quad y'(0) = \begin{bmatrix} -5 \\ 5 \end{bmatrix} \tag{29}$$

where $U(y) = y_1(t)y_2(t)\left(y_1(t) + y_2(t)\right)^3$and the exact solution

Table 3. Juxtaposition of maximum global error for Example 1 on [0, 100]

h	BHA-Err	BHM-Err	BHA-ErrH	BHM-ErrH
1/2	4.39×10^{-15}	1.74×10^{-13}	2.62×10^{-17}	5.27×10^{-15}
1/4	3.46×10^{-14}	8.62×10^{-13}	5.89×10^{-17}	1.23×10^{-14}
1/8	5.09×10^{-14}	3.49×10^{-12}	9.22×10^{-17}	2.11×10^{-14}
1/16	8.13×10^{-14}	6.94×10^{-12}	2.16×10^{-16}	4.19×10^{-14}
1/32	1.67×10^{-13}	1.92×10^{-12}	6.65×10^{-16}	5.06×10^{-14}

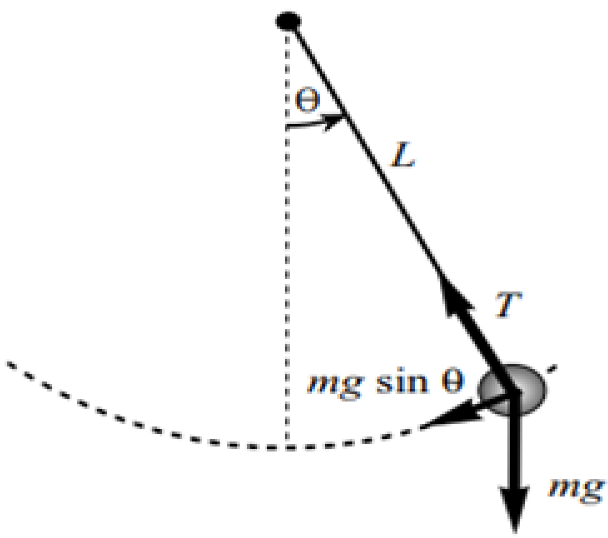

Figure 7. A simple pendulum

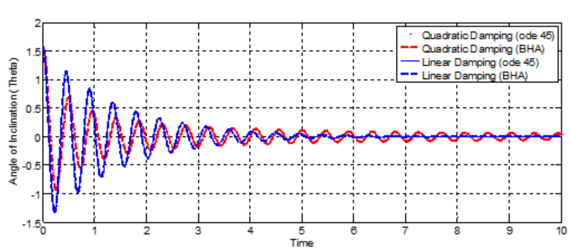

Figure 8. Solution curves for Example 2 depicting quadratic and linear damping

given by

$$y(t)=\begin{bmatrix}-\sin 5t-\cos 5t\\ \sin 5t+\cos 5t\end{bmatrix}\tag{30}$$

The system (29) has the Hamiltonian

$$\left.\begin{array}{l}H\left(y(t)+y'(t)\right)=\frac{1}{2}y'^{T}y'+\frac{1}{2}y^{T},\\ \begin{bmatrix}13 & -12\\ -12 & 13\end{bmatrix}y(t)+U(y)\end{array}\right\}\tag{31}$$

The eleventh order block hybrid method formulated by [2] was employed in computing the maximum global errors in the solution as well as the maximum global error in the Hamiltonian for Example 3 at different step sizes. The proposed BHA was also used to solve the same example. The results obtained clearly showed that even though the BHA is of the sixth order, it outperformed the block hybrid method of order eleven derived by [2], see Table 4.

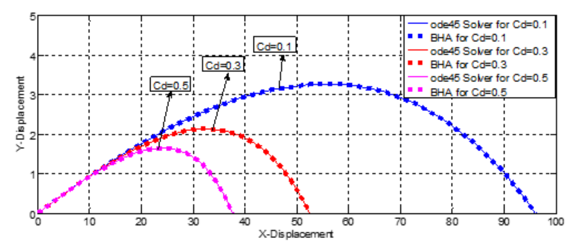

Figure 9. Solution curves for Example 4 showing the $X$and Y displacements

Example 4. The following nonlinear system describes the trajectory motion of an object in terms of X and Y displacements

$$mx''(t)+c_dx'(t)\left(\left(x'(t)\right)^2+\left(y'(t)\right)^2\right)^{(1/2)}=0\tag{32}$$

$$my''(t)+c_dy'(t)\left(\left(x'(t)\right)^2+\left(y'(t)\right)^2\right)^{(1/2)}=-w\tag{33}$$

The system was solved using the following parameters of aerodynamic drag $c_d=\{0.1,\,0.3,\,0.5\}\ N/m$. Let $x(0)=0$, $y(0)=0$,$x'(0)=100ms^{-1}$, $y'(0)=10ms^{-1}$,$m=10kg$,$w=mg$ and $g=9.81ms^{-2}$.

The proposed BHA was directly employed in simulating the system and the results compared with that of the author in [36] who applied ode 45 solver. It is obvious from the plots in Figure 9 that as c_d is increases, the maximum value of the displacement in the $y$direction decreases. The value of the displacement in the x direction, for which the displacement in the y direction is maximum, also decreases. The curves generated via the proposed BHA converge to those generated using the ode45 solver, implying that the BHA is computationally robust and accurate.

Example 5. Consider the Bessel's nonlinear oscillatory problem

$$t^2y''(t)+ty'(t)+\left(t^2-0.25\right)y(t)=0,\ y(1)=\sqrt{\frac{2}{\pi}}\sin(1),\quad y'(1)=\frac{2\cos(1)-\sin(1)}{\sqrt{2\pi}}\tag{34}$$

whose exact solution is given by

$$y(t)=\sqrt{\frac{2}{\pi t}}\sin(t)\tag{35}$$

The newly derived sixth order BHA was applied on Example 5 in the interval $[1,8]$. The results obtained were compared with

Table 4. Juxtaposition of maximum global error for Example 3 on [0, 100]

h	BHA-Err	BHM-Err	BHA-ErrH	BHM-ErrH
1/2	1.98×10^{0}	4.58×10^{0}	1.04×10^{3}	2.95×10^{3}
1/4	8.41×10^{-10}	7.54×10^{-8}	5.24×10^{-10}	3.13×10^{-8}
1/8	3.55×10^{-12}	2.20×10^{-11}	3.33×10^{-13}	5.61×10^{-12}
1/16	2.23×10^{-14}	4.95×10^{-14}	6.19×10^{-13}	1.62×10^{-12}
1/32	5.70×10^{-15}	9.15×10^{-14}	4.89×10^{-14}	3.78×10^{-12}

Table 5. Juxtaposition of maximum global error for Example 5 on [1, 8]

NS	BHA-Err	IBM-Err	VSSHM-Err
67	1.1984×10^{-17}	7.7716×10^{-16}	6.5286×10^{-11}
82	6.1189×10^{-18}	1.8874×10^{-15}	1.3679×10^{-11}
112	2.8346×10^{-18}	1.1601×10^{-15}	1.1897×10^{-12}

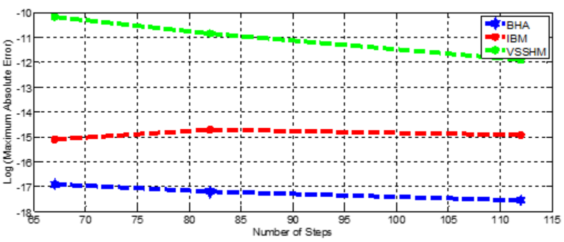

Figure 10. Accuracy curves for Example 5

those of order eight implicit block method (IBM) by [11] and order seven variable step-size hybrid method (VSSHM) developed by [13] using the same number of steps. The results of the new sixth order BHA performed better than the other two methods, see Table 5. Figure 10 also showed that the BHA is more accurate than the IBM and VSSHM.

For more details on oscillatory problems, refer to [37-45].

5. Conclusion

An attempt has been made in this research to derive an accuracy-preserving BHA for the solution of second-order physical systems with oscillatory solutions. Problems considered included Kepler, Bessel and damped systems. The results generated showed that the BHA is accuracy-preserving and computationally reliable. Simulation results obtained also showed that the new method was in agreement with the MATLAB inbuilt solver (ode 45). Properties such as order, error constant, zero-stability, stability, consistence and convergence of the new BHA were also validated. Finally, the proposed BHA is recommended for the integration of physical systems with oscillatory solutions. Further research will focus on the solution of physical models for higher order partial differential equations.

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