INVESTIGATING THE SURFACE ELASTICITY AND TENSION EFFECTS ON CRITICAL BUCKLING BEHAVIOUR OF NANOTUBES BASED ON DIFFERENTIAL TRANSFORMATION METHOD

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ABSTRACT

By considering the coupled effects of surface and nonlocal elasticity theory, the critical buckling load response of silicon/aluminium nanotubes is investigated in this paper. The nonlocal Eringen theory takes into account the effect of small scale size while the Gurtin-Murdoch model is used to incorporate the surface effects. Governing equations are derived through Hamilton's principle. The differential transformation method (DTM) as an efficient and accurate numerical tool is employed to solve the governing equations of nanotubes subjected to different boundary conditions. The output results are compared favourably with available published works. The detailed mathematical derivations are presented and numerical investigations are performed while the emphasis is placed on investigating the effect of the nonlocal parameter, surface effect, aspect ratio, mode number and beam size on critical buckling loads of the nanotube in detail. The results show that increasing the nonlocal parameter increase the buckling ratio of the nanotubes.

***KEYWORDS**: Critical buckling behaviour, Nonlocal elasticity theory, Surface elasticity and tension effects, Differential transformation method*

1.0 INTRODUCTION

In order to study the mechanical behaviours of nanostructures, the surface effects and nonlocal elasticity theory are two important fields which are investigated by researchers separately, or simultaneously. The surface of a solid is a region with small thickness which has different properties from the bulk. If the surface energy-to-bulk energy ratio is large, for example in the case of nanostructures, the surface effects cannot be ignored (He et. al, 2004). On the other hand, the nonlocal elasticity theory which is initiated in the paper of Eringen

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(1983) expresses that the stress at a point is a function of strains at all points in the continuum. To account for the effect of surfaces/interfaces on mechanical deformation, the surface elasticity theory is presented by modelling the surface as a two dimensional membrane adhering to the underlying bulk material without slipping (Gurtin and Murdoch 1975),(Gurtin et al, 1998). There are many studies related to the wave propagation, static, buckling and free linear and nonlinear vibration analysis of nanobeams and carbon nanotubes based on different beam theories

Gurtin et al (1998) established the theory of surface elasticity to explain various size-dependent phenomena at the nanoscale, and the predictions fit well with atomistic simulations and experimental measurements. Wang and Feng (2007) analysed the surface effects on the axial buckling of nanowires. By using the surface Cauchy–Born model (Park, 2009) analysed the size-dependent effect of the residual surface stress on the resonant frequencies of silicon nanowires under finite deformation. Hosseini et al. (2013) studied the surface and nonlocal effects on free vibration of nanobeam based on both Timoshenko and Euler-Bernoulli beam theory (EBT) for different boundary conditions. In similar work, Malekzadeh et al. (2013) studied surface and nonlocal effect on free nonlinear vibration of non-uniform nanobeams based on EBT and Timoshenko beam theory. They expressed that the influence of surface and nonlocal effects depends on the boundary conditions of the nanobeam. Also, Eltaher et al. (2013) studied the coupling effects of nonlocal and surface energy on free vibration of nanobeam based on EBT for simply supported nanobeam. They showed that the surface effects depend on the size and the material of the nanobeam by calculating natural frequencies for two different materials. Recently, (Ansari and Sahmani, 2011) studied bending behaviour and buckling of nanobeams including surface stress corresponding to different beam theories without consideration of nonlocality effect.

Moreover, the governing motion equations are often solved by analytical method Hosseini et al. (2013) or finite element methods (Eltaher et al, 2013) or generalized differential quadrature (GDQ) method (Ansari and Sahmani, 2011) and other solutions which need high CPU time to solve. DTM is also used to find the exact solution of both linear and nonlinear equations and even partial differential equations with high precision and also is simpler in compare with other methods. Although this method comes from Taylor series expansion, but DTM is different from the traditional high order Taylor's series method. Because the traditional Taylor expansion requires symbolic competition of the necessary derivatives of the data functions and thus is computationally taken long time for large orders, while DTM takes less time to solve polynomial series. In other words, by applying DTM, governing equations for different boundary conditions reduces to algebraic equations, and finally all the calculations turn into simple iterative process. Also as seen in the literature, DTM has been used for solving a vast range of problems in different fields of engineering.

To the best knowledge of the authors, no research effort has been devoted so far to find the solution of critical buckling load of nanotubes considering both surface and small scale effects employing differential transformation method. Motivated by this fact, in this study, differential transformation method is applied in analysing the surface effects, including surface elasticity and stress, on critical buckling load of nanotubes, made of Aluminium and Silicon, using nonlocal elasticity theory. Hamilton's principle is employed to derive the governing equation and corresponding boundary conditions. DTM is then used to obtain the critical buckling load of nanotubes with various boundary conditions. To this end, the output results are compared favourably with those published works and influences of the surface effect, nonlocal parameter and size of nanotube on the critical buckling load are investigated.

2.0 THEORY AND FORMULATION

Nonlocal constitutive relation for Euler-Bernoulli beam is given as (Eringen, 1983):

$$\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E\,\varepsilon_{xx} \tag{1}$$

where σ_{xx} and ε_{xx} are the nonlocal stress and strain, respectively. The distributed transverse loading induced by the residual surface tension is (Farshi et al , 2010):

$$q = q_0 + H \frac{\partial^2 w}{\partial x^2} \tag{2}$$

and H is the surface effect constant is given by:

$$H = 2\tau^o D \tag{3}$$

Effective flexural rigidity, EI^* for nanotube is given by:

$$EI^* = \frac{1}{4}\pi E\,(R_o^4 - R_i^4) + \pi E^s\,(R_o^3 - R_i^3) \tag{4}$$

The general differential equation of Euler-Bernoulli beam based on nonlocal continuum model and surface effect are derived using the principle of Hamilton and expressed by (Reddy, (2007):

$$
\begin{aligned}
&\frac{\partial^2}{\partial x^2}\left(-EI^*\frac{\partial^2 w}{\partial x^2}\right)+\mu\frac{\partial^2}{\partial x^2}\left[\frac{\partial}{\partial x}\left(\bar{N}\frac{\partial w}{\partial x}\right)-q_0-H\frac{\partial^2 w}{\partial x^2}+\rho A\frac{\partial^2 w}{\partial t^2}-\rho I\frac{\partial^4 w}{\partial t^2\partial x^2}\right]+q_0\\
&+H\frac{\partial^2 w}{\partial x^2}-\frac{\partial}{\partial x}\left(\bar{N}\frac{\partial w}{\partial x}\right)=\rho A\frac{\partial^2 w}{\partial t^2}-\rho I\frac{\partial^4 w}{\partial t^2\partial x^2}
\end{aligned}
\tag{5}
$$

The coordinate system for nanotube is shown in Figure 1.

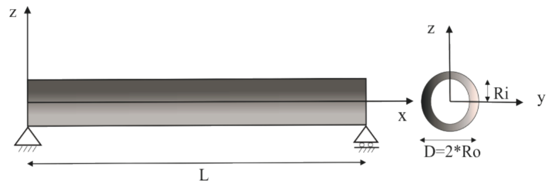

Figure 1. Geometry of nanotube with Length L , inner and outer radii Ri and Ro

3.0 DIFFERENTIAL TRANSFORMATION METHOD

Differential transformation method is one of the useful techniques to solve the differential equations with small calculation errors and ability to solve nonlinear equations with boundary conditions value problems. Abdel-Halim Hassan (2002) applied the DTM on eigenvalues and normalized eigenfunctions.

Table 1. Some of the transformation rules of the one-dimensional DTM

Original function	Transformed function
$f(x)=g(x)\pm h(x)$	$F(K)=G(K)\pm H(K)$
$f(x)=\lambda g(x)$	$F(K)=\lambda G(K)$
$f(x)=g(x)h(x)$	$F(K)=\sum_{l=0}^{K} G(K-l)H(l)$
$f(x)=\frac{d^n g(x)}{dx^n}$	$F(K)=\frac{(k+n)!}{k!}G(K+n)$
$f(x)=x^n$	$F(K)=\delta(K-n)=\begin{cases}1 & k=n\\ 0 & k\neq n\end{cases}$

Table 2. Transformed boundary conditions (B.C.) based on DTM

X=0 Original B.C.	Transformed B.C.	X=L Original B.C.	Transformed B.C.
$f(0)=0$	$F[0]=0$	$f(L)=0$	$\sum_{k=0}^{\infty} F[k]=0$
$\frac{df(0)}{dx}=0$	$\frac{dF[0]}{dx}=0$	$\frac{df(L)}{dx}=0$	$\sum_{k=0}^{\infty} k\,F[k]=0$
$\frac{d^2f(0)}{dx^2}=0$	$\frac{d^2F[0]}{dx^2}=0$	$\frac{d^2f(L)}{dx^2}=0$	$\sum_{k=0}^{\infty} k(k-1)F[k]=0$
$\frac{d^3f(0)}{dx^3}=0$	$\frac{d^3F[0]}{dx^3}=0$	$\frac{d^3f(L)}{dx^3}=0$	$\sum_{k=0}^{\infty} k(k-1)(k-2)F[k]=0$

In this method, certain transformation rules are applied to both the governing differential equations of motion and the boundary conditions of the system in order to transform them into a set of algebraic equations as presented in Table 1 and Table 2. The solution of these algebraic equations gives the desired results of the problem. The basic definitions and the application procedure of this method can be introduced as follows. The transformation equation of function can be defined as (Chen et al 2004),

$$F[k]=\frac{1}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0} \tag{6}$$

where $f(x)$ is the original function and $F[k]$ is the transformed function. The inverse transformation is defined as:

$$f(x)=\sum_{k=0}^{\infty}(x-x_0)^k F[k] \tag{7}$$

Combining equations (6) and (7) one obtains:

$$f(x)=\sum_{k=0}^{\infty}\frac{(x-x_0)^k}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0} \tag{8}$$

In actual application, the function f(x) is expressed by a finite series and Equation (8) can be written as follows:

$$f(x)=\sum_{k=0}^{N}\frac{(x-x_0)^k}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0} \tag{9}$$

which implies that the term in relation (9) is negligible:

$$f(x)=\sum_{k=N+1}^{\infty}\frac{(x-x_0)^k}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0} \tag{10}$$

3.1 Implementation of Differential Transform Method

While solving the Equation (5) authors preferred DTM approach which avoids solving complicated transcendental algebraic equations for general boundary conditions. In order to derive differential form of Equation (5) we refer Table 1 and the following expression is written as:

$$(EI^{*}+\mu H-\mu\bar{N})\frac{(k+4)!}{k!}\;W[k+4]+\left(\bar{N}-H\right)\frac{(k+2)!}{k!}\;W[k+2]=0 \tag{11}$$

And the various boundary condition for nanotube by using Table 2 can be expressed as:

- Simply supported–Simply supported:

$$W\,[0]=0\;,\;\;\mathrm{W}[2]=0$$
$$\sum_{k=0}^{\infty}W\,[k\,]=0\;,\;\;\sum_{k=0}^{\infty}k\,(k\,-1)\;W\,[k\,]=0 \tag{12}$$

- Clamped–Clamped:

$$\mathrm{W}[0]=0\;,\;\;\mathrm{W}[1]=0$$
$$\sum_{k=0}^{\infty}W\,[k\,]=0\;,\;\;\sum_{k=0}^{\infty}k\;\;W\,[k\,]=0 \tag{13}$$

- Clamped–Simply supported:

$$\mathrm{W}[0]=0\;,\;\;\mathrm{W}[1]=0$$
$$\sum_{k=0}^{\infty}W\,[k\,]=0\;,\;\;\sum_{k=0}^{\infty}k\,(k\,-1)\;W\,[k\,]=0 \tag{14}$$

By using Equation(11) and with the transformed boundary conditions one arrives at the following eigenvalue problem:

$$\begin{bmatrix} A_{11}(N\,) & A_{12}(N\,) \\ A_{21}(N\,) & A_{22}(N\,) \end{bmatrix}[C\,]=0 \tag{15}$$

Where correspond to the missing boundary conditions at x=0. For the non-trivial solutions of Equation(15), it is necessary that the determinant of the coefficient matrix is equal to zero:

$$\begin{vmatrix} A_{11}(N) & A_{12}(N) \\ A_{21}(N) & A_{22}(N) \end{vmatrix} = 0 \tag{16}$$

Solution of Equation (16) is simply a polynomial root finding problem. Many techniques such as Newton's method, Laguerre's method, etc. can be used to find the roots of this equation.

4.0 RESULTS AND DISCUSSIONS

A nanotube with circular cross section and two different materials, aluminium and silicon, are considered. The elastic bulk and surface properties of aluminium with crystallographic direction of [1 1 1] and silicon with crystallographic direction of [1 0 0] are tabulated in Table 3. The nondimensional buckling load is defined as:

$$\bar{N}_{cr} = \bar{N}\left(L^2 \big/ EI^* \right) \tag{17}$$

Table 4 presents the first critical buckling load where the length-to-thickness ratio is 10 while varying the nonlocal parameter. As can be noted, the obtained results are in good agreement with those of Reddy (2007) and even more conservative than those presented by Thai (2012). The critical buckling load decreases as the nonlocal parameter increases. This emphasizes the significance of the nonlocal effect on the buckling response of beams.

Table 3. Material properties of AL and Si

Material	E(Gpa)	ρ(kg/m^3)	υ	E_s(N/M)	τ_o(N/M)
AL	70	2700	0.3	5.1882	0.9108
Si	210	2370	0.24	-10.6543	0.6048

Table 4. The nondimensional buckling load for simply supported beam

L/h	μ	Thai[19]	Reddy[14]	Present
10	0	9.8696	9.8696	9.86960440
	1	8.9830	8.9830	8.98301623
	2	8.2426	8.2426	8.24258361
	3	7.6149	7.6149	7.61491765
	4	7.0761	7.0761	7.07607999

The effects of nonlocal effect (NE), nonlocal parameter and nonlocal surface effect (NSE) on the first three nondimensional buckling loads with different boundary conditions are presented in Table 5. It should be noted that $\mu = 0$ corresponds to local beam theory. From obtained results, it can be deduced that, when the nonlocal parameter increases, the buckling load decrease. Figure 2,3 and 4 depict the variation of the normalized nondimensional buckling load versus the nanotube length for three boundary conditions, i.e. Simply–Simply (S–S), Clamped–Simply (C–S) and Clamped–Clamped (C–C). It can be noted, buckling load ratio is decreased with increasing in the beam size. But, the increasing in nonlocality parameter leads to increase the buckling ratio for the same beam size.

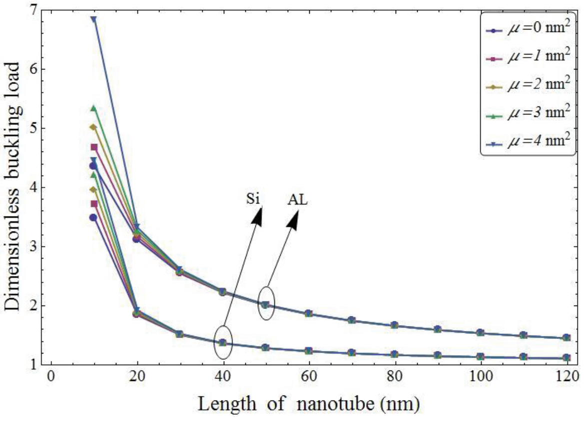

Figure 2. Buckling load ratio of the nanotube with various length and nonlocal parameters (S-S)

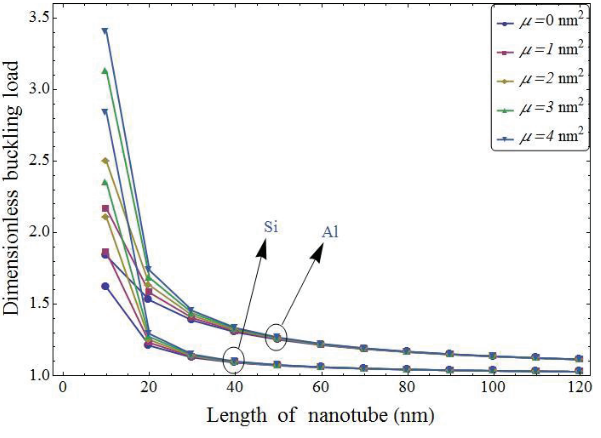

Figure 3. Buckling load ratio of the nanotube with various length and nonlocal parameters(C-C)

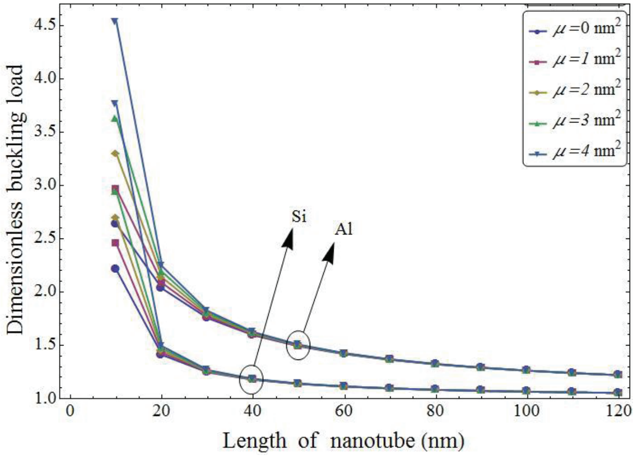

Figure 4. Buckling load ratio of the nanotube with various length and nonlocal parameters(C-S)

Table 5. The first three critical buckling loads versus nonlocal parameters for nanotubes with various boundary conditions

μ	$\bar{N}_i$	S-S		
		NE	NSE(AL)	NSE(Si)
0	i=1	9.8696	42.9089	34.2707
	i=2	39.4784	72.5177	63.7895
	i=3	88.8264	121.8660	113.2280
2	i=1	8.2425	41.2819	32.6437
	i=2	22.4976	55.0996	46.4614
	i=3	31.9919	65.0312	56.3930
4	i=1	7.0760	40.1154	31.4772
	i=2	15.3068	48.3461	39.7079
	i=3	19.5092	52.5485	43.9103
μ	$\bar{N}_i$	S-S		
		NE	NSE(AL)	NSE(Si)
0	i=1	20.1907	53.2300	44.5918
	i=2	59.6759	92.7188	84.0806
	i=3	118.9000	151.9390	143.3010
2	i=1	14.3828	47.3828	38.7839
	i=2	27.2063	60.2456	51.6074
	i=3	35.1838	68.2376	59.5994
4	i=1	11.1697	44.2090	35.5708
	i=2	17.6192	50.6585	42.0203

μ	$\bar{N}_i$	NE	NSE(AL)	NSE(Si)
	i=3	20.6567	53.6960	45.0578

μ	$\bar{N}_i$	C-S		
		NE	NSE(AL)	NSE(Si)
0	i=1	39.4784	72.5177	63.8795
	i=2	80.7629	113.8020	105.1640
	i=3	157.914	190.9530	182.3150
2	i=1	22.0603	55.0996	46.4614
	i=2	30.8814	63.9207	55.2825
	i=3	37.9758	71.0151	62.3769
4	i=1	15.3068	48.3461	39.7079
	i=2	19.0906	52.1298	43.4917
	i=3	21.5831	54.6224	45.9842

5.0 CONCLUSIONS

In the present study, the buckling behaviour of nanotubes including the effect of surface stress was predicted via linear partial differential equations of motion and related boundary conditions were derived. The nanotubes are considered to be made of Al with positive surface elasticity and Si with negative surface elasticity. Afterward, the differential transformation method as an efficient and accurate numerical tool was applied to solve the linear equations of nanotubes subjected to different boundary conditions. The good agreement between the results of this article and those available in literature validated the presented approach. Numerical results demonstrate that the small scale effects play an important role on the buckling behaviour of the nanotube. Also, it is observed that increasing the nonlocal parameter increased the buckling ratio of the nanotubes.

NOMENCLATURES

EI^*	Effective flexural rigidity	*Greek symbols*	
q	Distributed transverse loading	μ	Nonlocal parameter
$\bar{N}$	Critical buckling load	ρ	Mass density
A	Cross sectional area	τ^0	Surface tension
I	Mass moment of inertia	υ	Poisson's ratio
E^s	Surface elasticity modulus		
E	Elasticity Modulus		

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