Journal of Mountain Area Research
Vol. 8, 2023
https://doi.org/10.53874/jmar.v8i0.168

Full length article

A UNIQUE COMBINATION OF MASK IN BINARY FOUR-POINT SUBDIVISION SCHEME

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ABSTRACT

A unique binary four-point approximating subdivision scheme has been developed in which one part of binary formula have stationary mask and other part have the non-stationary mask. The resulting curves have the smoothness of C^3 continuous for the wider range of shape control parameter. The role of the parameter has been depicted using the square form of discrete control points.

Keywords: Binary, Stationary and Non-stationary, Approximating, Subdivision Scheme.

2010 AMS Subject Classifications: 41A05, 41A25, 41A30, 65D17, 65D10.

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1. INTRODUCTION

Geometric modelling is the heart of Computer Graphics and Computer Aided Geometric Design and covers a wide range of applications. Computer Aided Geometric Design is a branch of applied mathematics that designs smooth curves/ surfaces with algorithms. The Computer Aided Geometric Design field compiles the visual display. Computer Aided Geometric Design studies the design and handling of curves / surfaces provided by a collection of points. The development of new geometric objects and shapes is an important task in the field of Computer Aided Geometric Design, Computer Graphic, Computer Animation Industry, and Image Processing *etc*. Subdivision is an important tool to create different geometric objects and shapes in Computer Aided Geometric Design. Algorithms of subdivisions are suitable for computer applications. Subdivision develops different types of smooth curves and surfaces using refining rules by subdividing them from a collection of discrete data points.

For the development of subdivision field, G. de Rham firstly developed a subdivision scheme in 1947. Subdivision is a method that generates smooth curves and surfaces just as successive arrangement of refined control polygons. The current polygon is included to new points at each refining level and the basic points continue to remain or are thrown away across all resultant control polygons. The number of points placed from step k to level $k+1$ between two successive points is termed also the scheme's arity. If there are 2,3,4 ,…, n inserted then the subdivision schemes are known as binary, ternary,…, n-ary respectively.

Subdivision is also a technique for construction of smooth curves / surfaces, which first applied as an extension of splines to arbitrary topology control nets. Simplicity and flexibility of the subdivision algorithms make them suitable for many interactive computer graphics applications. Purity of the subdivision lies in the construction of smooth curves / surfaces. However, the uses such as special aspects of an objects, animations, and construction of composite geometric shapes which mostly like real world geometry.

An important characteristic of these schemes is that these schemes have local effect and not global effect on curve by changing the control points. Although the mathematical surface analysis resulting from subdivision algorithms is not always easy. Subdivision scheme takes the attention of scholars and researchers in this field because of its simplicity and understanding. In the last two decades, many papers have been published. There will be a lot of new Computer Aided Geometric Design applications in the future. In Computer Aided Geometric Design, the most frequently used and attractive schemes are subdivision schemes that draw different types of objects in the form of curves and surfaces. Iterative refinements use the subdivision schemes to build continuous curves/surfaces from the collection of specific control points. Subdivision schemes have been valued in various areas, just as Image Processing, Computer Graphic and Computer Animation, due to their clarity and easily handling.

Dyn *et al.* [1] conditions are algebraic and easy to check by considering the subdivision scheme symbol, but also relate to the scheme's parameterization. The convexity of four-point ternary interpolating subdivision scheme by Cai [2] is analysed with a tension parameter. It is shown that the resulting curve is C^2 for a certain range of the tension parameter.

Victoria *et al.* [3] shows that the subdivision curve converges and is continuous. In addition, starting with the initial polygon's chord-length parameterization. We get a subdivision curve parameterized by an arc-length multiple. The weights of the masks of the scheme are defined by Daniel and Shummugaraj [4] in terms of some values of trigonometric B-spline functions. Mustafa *et al.* [5] proposed and analysed the m-point approximating subdivision scheme with single parameter where $m > 1$. Compared to the existing binary and ternary subdivision scheme, smoothness of schemes is higher.

Zheng and Peng [6] presented an explicit formula that unifies the mask of ternary interpolation and approximation of subdivision scheme $(2n-1)$ points. Augsdorfer *et al.* [7] proposed a different form of binary four-point scheme that is helpful in computer designing. Deng and Guozhao, [8] developed an in-centre subdivision scheme for curve interpolation. Siddiqi and Rehan [9] improved the binary 4-point approximating subdivision scheme by introducing a global tension parameter. A new Subdivision Scheme for corner cutting was also proposed, which generates a limiting curve of C^1 continuity. Siddiqi and Rehan [10] presented ternary three-point non-stationary approximating subdivision scheme that generates a limiting curve C^2. Tan *et al.* [11] proposed a new four-point shape preserving C^3 subdivision technique. Tan *et al.* [12] developed a non-stationary three-point binary approximating subdivision scheme that generates C^3 smooth curves using the discrete control points. Kim *et al.* [13] proposed a binary subdivision scheme with four points that generates a smooth C^3-continuous limiting curve. With the help of local cubic polynomial fitting, Floater [14] derived an approximation property of four-point interpolatory curve subdivision.

2. STATIONARY BINARY FOUR-POINT SUBDIVISION TECHNIQUE

Kim *et al.* [13] introduced a binary subdivision technique with four points which creates a smooth C3-continuous limiting curve. Given the set of control points $q^0 = \{{q_i}^0\}_{i\epsilon Z}$ at step 0, the binary four-point subdivision scheme for the design of curves develops a new collection of control points $\{{q_i}^{k+1}\}_{i\epsilon Z}$ at step $k+1$ using the following subdivision rules

$$q_{2i}^{k+1} = -\beta q_{i-2}^k + 4\beta q_{i-1}^k + (1-6\beta)q_i^k + 4\beta q_{i+1}^k - \beta q_{i+2}^k$$

$$q_{2i+1}^{k+1} = -\frac{1}{16}q_{i-1}^k + \frac{9}{16}q_i^k + \frac{9}{16}q_{i+1}^k - \frac{1}{16}q_{i+2}^k \qquad (1)$$

Where $q^0 = \{q_i^0\}_{i\epsilon Z}$ is the collection of initial control points at step 0 and the sum of the mask of the scheme should be one as described in equation (1).

They found that the scheme is C1-continuous when $-0.12 < \beta < 0.21$ and the scheme is C2-continuous and C3-continuous when $-0.88 < \beta < 0.13$.

3. NON-STATIONARY BINARY FOUR-POINT SUBDIVISION TECHNIQUE

The refining rules of the binary non-stationary four points subdivision scheme is defined as

$$q_{2i}^{k+1} = -\beta_0^k q_{i-2}^k + \beta_1^k q_{i-1}^k + \beta_2^k q_i^k + \beta_3^k q_{i+1}^k - \beta_4^k q_{i+2}^k$$

$$q_{2i+1}^{k+1} = -\frac{1}{16}q_{i-1}^k + \frac{9}{16}q_i^k + \frac{9}{16}q_{i+1}^k - \frac{1}{16}q_{i+2}^k \qquad (2)$$

Where $q^k = \{q_i^k\}_{i\epsilon Z}$ is the collection of initial control point at any level 'k' and the mask of the scheme must satisfy the relationship $\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 = 1$.

The binary non-stationary four-point subdivision scheme (2) is counter part of stationary scheme [13]. The mask of the proposed scheme is given by

$$\beta_0^k = g(\beta^{k+1}) = \beta_4^k, \beta_1^k = 4g(\beta^{k+1}) = \beta_3^k,$$
$$\beta_2 = 1 - 6g(\beta^{k+1});$$

$$g(\beta^{k+1}) = \frac{1}{2}\left[\frac{(\beta^{k+1})^2-1}{(\beta^{k+1})^2+60}\right] \qquad (3)$$

With

$$\beta^{k+1} = \sqrt{\beta^k + 2},\ and\ \ \beta^0 \in [-2, +\infty).$$
(4)

In this way, the coefficients β_i^k at each different steps k can be calculated using the given formula and an initial parameter $\beta^0 \in [-2, +\infty)$. Starting with any $\beta^0 \geq -2$, we have $\beta^k + 2 \geq 0, for\ \ all\ \ k \in Z_{+,}$ so β^{k+1} is constantly characterized. A wide range of definitions enables to achieve significant modification in the form of the smooth curves.

Remark 1. As the initial values of β^0 extend in its definition range, the behaviour of the smooth curve develops significantly from the figure (1) and tends toward approximation the initial control polygon just as $\beta^0 \to +\infty$.

Remark 2. Equation (4), the sequence $\{\beta^k\}_{k\epsilon N}$ will be stationary, if we have $\beta^0 = 2$ then $\beta^k = 2$, $\forall\ k \in Z_+$. So that the non-stationary subdivision scheme is then retrograde to the stationary subdivision scheme.

If $\beta^0 > 2$, then $\beta^k > 2$, the sequence $\{\beta^k\}_{k\epsilon N}$ will be decreases strictly that β^k converges to 2 as the $k \to \infty$.

If $\beta^0 < 2$, then $\beta^k < 2$, the sequence $\{\beta^k\}_{k\epsilon N}$ will be increases strictly and β^k converges to 2 as the $k \to \infty$.

Therefore, in the definition domain of any β^0 we always have

$$\lim_{k\to+\infty} \beta^k = 2.$$

4. SMOOTHNESS

In this section, we will demonstrate that for the given initial discrete polygon, the subdivision scheme (3) gives a smooth C3 continuous curves for any selection of the parameter values β^0 in its definition range. Considering, Dyn and Levin's well-known results [14], smoothness about non-stationary scheme to its counterpart stationary scheme, asymptotically equivalent method has been used.

Theorem. A non-stationary subdivision scheme defined in equation (3), the masks are asymptotically equivalent to the stationary scheme (1). So the proposed scheme has C^3-continuous limit curves for the shape control parameter $\beta^0 \in [-2, +\infty)$.

Proof. To check the smoothness of the proposed scheme C^3-continuous, it is necessary to obtain the second divided difference mask. The mask of the scheme is

$$m^k = \left[-g(\beta^{k+1}), -\frac{1}{16}, 4g(\beta^{k+1}), \frac{9}{16}, 1 - 6g(\beta^{k+1}), \frac{9}{16}, 4g(\beta^{k+1}), -\frac{1}{16}, -g(\beta^{k+1})\right]$$

Then it turns out its first divided difference masks are

$$e_{(1)}^k = 2\left[-\beta, \left(\beta - \frac{1}{16}\right), \left(3\beta + \frac{1}{16}\right), \left(\frac{1}{2} - 3\beta\right), \left(\frac{1}{2} - 3\beta\right), \left(3\beta + \frac{1}{16}\right), \left(\beta - \frac{1}{16}\right), -\beta\right]$$

Then it turns out its 2nd divided difference masks are

$$e_{(2)}^k = 4\left[-\beta, \left(2\beta - \frac{1}{16}\right), \left(\beta + \frac{1}{8}\right), \left(\frac{3}{8} - 4\beta\right), \left(\beta + \frac{1}{8}\right), \left(2\beta - \frac{1}{16}\right), -\beta\right]$$

Then it turns out its 3rd divided difference masks are

$$e_{(3)}^k = 8\left[-\beta, \left(3\beta - \frac{1}{16}\right), \left(-2\beta + \frac{3}{16}\right), \left(-2\beta + \frac{3}{16}\right), \left(3\beta - \frac{1}{16}\right), -\beta\right]$$

The application of Remark 2 now provides

$$e_{(3)}^{\infty} = \lim_{k\to+\infty} e_3^k = 8\left[-\frac{3}{128}, \frac{1}{128}, \frac{18}{128}, \frac{18}{128}, \frac{1}{128}, -\frac{3}{128}\right]$$

This is only the coefficients of the stationary technique's third divided differences with equation coefficients in (1). In this case, the stationary technique is C^3-continuous, the technique associated with e_3^{∞} will be C^3 smooth. If it is as

$$\sum_{k=0}^{+\infty} \left\| e_{(3)}^k - e_3^{\infty} \right\|_{\infty} < +\infty. \quad (5)$$

The two techniques are then equivalent asymptotically. And one can conclude this that the e_3^{∞} of the technique is C^3, since then

$$e_{(3)}^k - e_3^{\infty} = 8\left[-2g(\beta^{k+1}) + \frac{6}{128}, -3\left(-2g(\beta^{k+1}) + \frac{6}{128}\right), 2\left(-2g(\beta^{k+1}) + \frac{6}{128}\right)\right]$$

$$\left\| e_{(3)}^k - e_3^{\infty} \right\|_{\infty} = 8max\left\{3\left|-2g(\beta^{k+1}) + \frac{6}{128}\right|, |-3|\left|-2g(\beta^{k+1}) + \frac{6}{128}\right|, 2\left|-2g(\beta^{k+1}) + \frac{6}{128}\right|\right\}$$

$$= 24\left|-2g(\beta^{k+1}) + \frac{6}{128}\right|$$

$$= 48\left|\frac{3}{128} - g(\beta^{k+1})\right|$$

Now we are proving the series smoothness

$$\sum_{k=0}^{+\infty} \left|\frac{3}{128} - g(\beta^{k+1})\right| \quad (6)$$

Which depends on the $g(\beta^{k+1})$ function. Now, since $g(\beta^{k+1})$ is expressed in terms of the β^{k+1}, the behaviour of β^{k+1} varies in the interval $[0, +\infty)$. From now on

$$\frac{3}{128} - g(\beta^{k+1}) = 0 \Leftrightarrow \beta^{k+1} = 2 \text{(i.e., } \beta^k = 2\text{)).}$$

$$\frac{3}{128} - g(\beta^{k+1}) > 0 \Leftrightarrow \beta^{k+1} \in (-1, 2)\left(i.e., \beta^0 \in [-2, 2)\right).$$

$$\frac{3}{128} - g(\beta^{k+1}) < 0 \Leftrightarrow \beta^{k+1} \in (2, +\infty).$$

Now, we are talking about smoothness of (6) in the three cases.

Case 1: $\boldsymbol{\beta^0 = 2 \ \ (i.e., \beta^{k+1} = 2)}$

Then $\left\|e_{(3)}^k - e_3^\infty\right\|_\infty = 0.$

Smoothness of (6) follows.

Case 2: $\boldsymbol{\beta^0 \in [-2,2)(i.e., \beta^{k+1} \in [0,2))}.$

In this case

$$\left\|e_{(3)}^k - e_3^\infty\right\|_\infty = 48\left[\frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^{k+1})^2-1}{(\beta^{k+1})^2+60}\right]\right]$$

Thus

$$\sum_{k=0}^{+\infty}\left(\frac{3}{128} - g(\beta^{k+1})\right) = \sum_{k=0}^{+\infty}\left(\frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^{k+1})^2-1}{(\beta^{k+1})^2+60}\right]\right) < +\infty.$$

We use the ratio test to this end. Since $\frac{3}{128} - g(\beta^{k+1}) > 0$ and the sequence $\{\beta^k\}_{k\in N}$ is increases surely in this case, we got it

$$\frac{\frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^{k+2})^2-1}{(\beta^{k+2})^2+60}\right]}{\frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^{k+1})^2-1}{(\beta^{k+1})^2+60}\right]} < 1.$$

The smoothness of (6) has therefore been proven.

Case 3: $\boldsymbol{\beta^0 \in (2,+\infty)(i.e., \beta^{k+1} \in (2,+\infty))}.$

In this case

$$\left\|e_{(3)}^k - e_3^\infty\right\|_\infty = 48\left[\frac{1}{2}\left\{\frac{(\beta^{k+1})^2-1}{(\beta^{k+1})^2+60}\right\} - \frac{3}{128}\right]$$

Thus

$$\sum_{k=0}^{+\infty}\left(g(\beta^{k+1}) - \frac{3}{128}\right)$$

$$= \sum_{k=0}^{+\infty}\left(\frac{1}{2}\left\{\frac{(\beta^{k+1})^2-1}{(\beta^{k+1})^2+60}\right\} - \frac{3}{128}\right) < +\infty.$$

Case 3.1

In this case, the sequence $\{\beta^k\}_{k\in N}$ is reducing strictly.

Use ratio test to this end. The sequence $\{\beta^k\}_{k\in N}$ reducing strictly as follows.

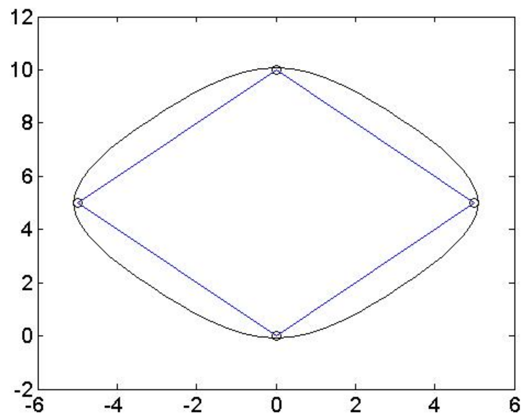
$$\frac{\frac{1}{2}\left[\frac{(\beta^{k+2})^2-1}{(\beta^{k+2})^2+60}\right] - \frac{3}{128}}{\frac{1}{2}\left[\frac{(\beta^{k+1})^2-1}{(\beta^{k+1})^2+60}\right] - \frac{3}{128}} < 1$$

The smoothness of (6) has therefore been proven.

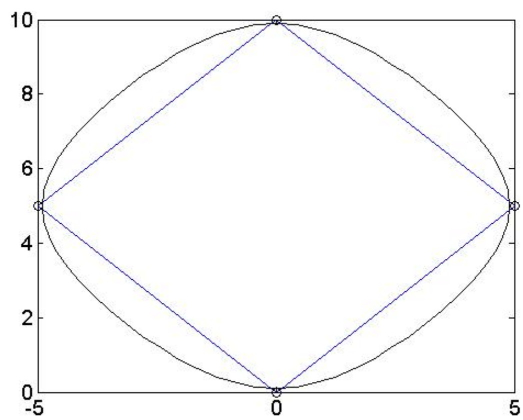
5. GRAPHICAL REPRESENTATION

We would like to give an example in this section to depict the benefit of the developed scheme (3). As discussed in section 2, generated curves appear to approximate the basic discrete control polygon while $\beta^0 \to +\infty$.

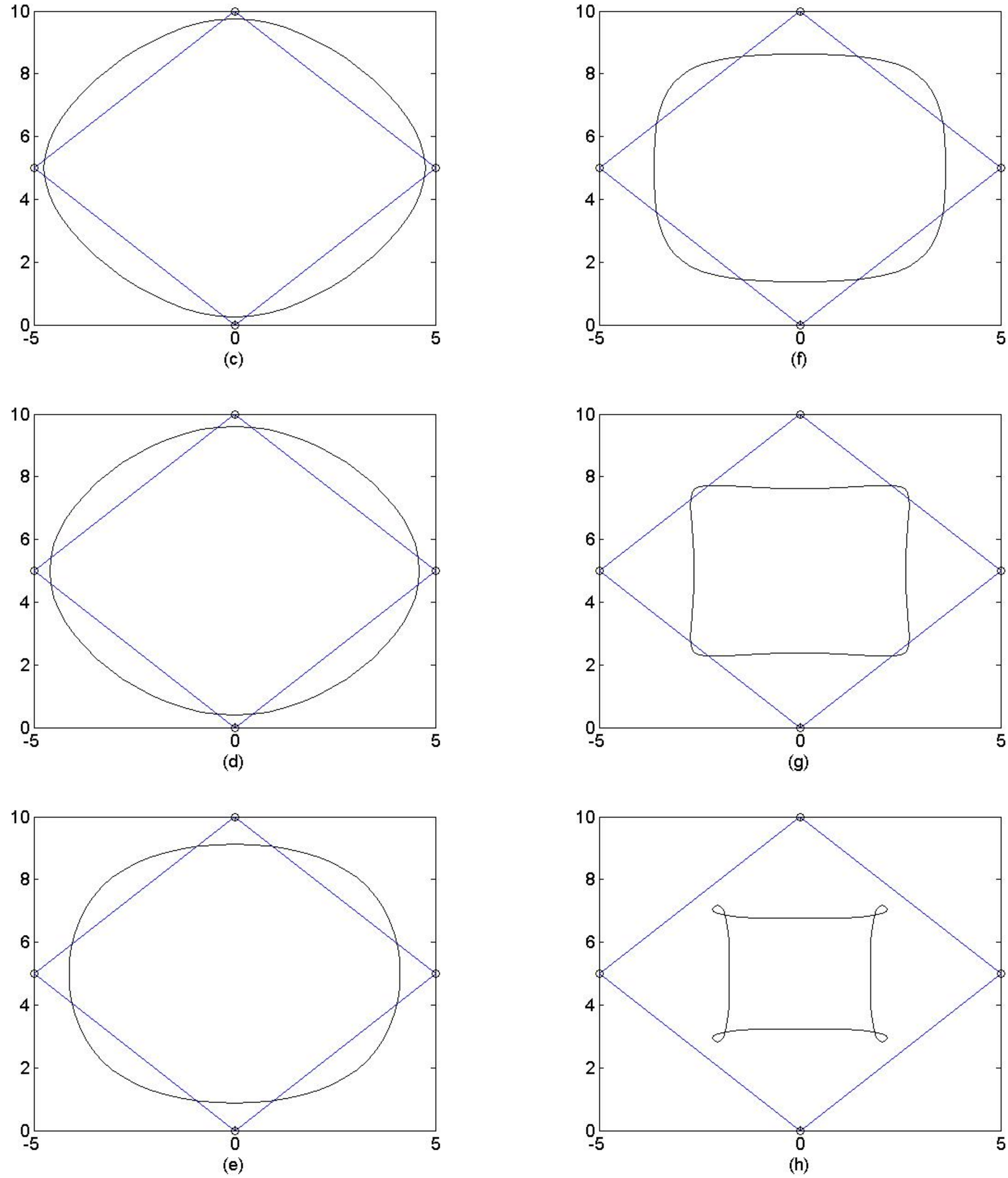
In Figure 3.1, Generating wide range of C[3]-continuous limiting curves for different values of parameters. (a) $\beta^0 = -2$ (b) $\beta^0 = -1$, (c) $\beta^0 = 0$, (d) $\beta^0 = 1$, (e) $\beta^0 = 5$, (f) $\beta^0 = 10$, (g) $\beta^0 = 25$, (h) $\beta^0 = 50$, (i) $\beta^0 = 100$.



(a)



(b)


(c)
(f)
(d)
(g)
(e)
(h)

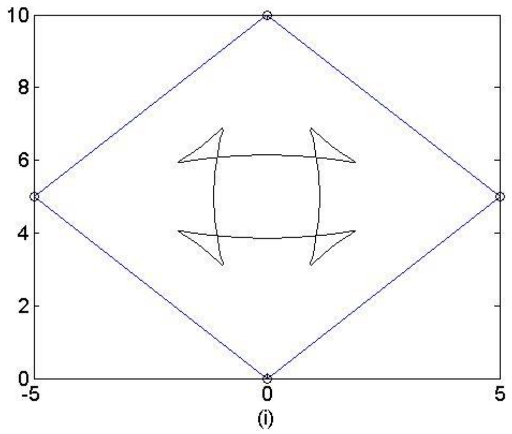


Figure 1. Generating wide range of C^3-continuous limiting curves using the proposed scheme (3) for different values of parameter. (a) $\beta^0 = -2$, (b) $\beta^0 = -1$, (c) $\beta^0 = 0$, (d) $\beta^0 = 1$, (e) $\beta^0 = 5$, (f) $\beta^0 = 10$, (g) $\beta^0 = 25$, (h) $\beta^0 = 50$, (i) $\beta^0 = 100$.

CONCLUSION

A nonstationary binary four-point approximating subdivision scheme with the shape control parameter has been developed that generates the smooth resulting curve of C^3 continuous for the wider range of shape control parameter $\beta^0 \in [-2, +\infty)$.

DECLARATIONS

Funding: No funding was received to conduct this study.

Conflicts of interest/Competing interests: The authors declare no conflict of interest or competing interest.

Authors' contributions:

The authors Uzma Mukhtar and Kashif Rehan have equally contributed to this study in all stages from conceptualization to the write up of the final draft.

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Received: 04 October 2022. Revised/Accepted: 20 February 2023.