Integration of Rational Functions

To cite this article before publication: L. Rathour, D. Obradovic, K. Khatri, S. K. Tiwari, L. N. Mishra, and V. N. Mishra. (2023). *J. Multidiscip. Appl. Nat. Sci.* in press. https://doi.org/10.47352/jmans.2774-3047.186.

Integration of Rational Functions

Laxmi Rathour[1,a)]; Dragan Obradovic[2,b)]; Kejal Khatri[3,c)]; Shiv Kant Tiwari[4,d*)]; Lakshmi Narayan Mishra[5,e)] ; Vishnu Narayan Mishra[6,f)]

[1]*Department of Mathematics, National Institute of Technology, Mizoram -796012 (India)*
[2]*Elementary School "Jovan Cvijic", Pozarevac-12208 (Serbia)*
[3]*Department of Mathematics, Government College, Simalwara, Dungarpur -314403 (India)*
[4]*Department of Mathematics, Lukhdhirji Engineering College, Morbi-363642 (India)*
[5]*Department of Mathematics, Vellore Institute of Technology, Vellore-632014 (India)*
[6]*Department of Mathematics, Indira Gandhi National Tribal University, Amarkantak-484887 (India)*
[a)] laxmirathour817@gmail.com
[b)]dragishaobradovic@yahoo.com
[c)]kejal0909@gmail.com
[d)] Correspondence: shivkant.math@gmail.com
[e)]lakshminarayanmishra04@gmail.com
[f)]vishnunarayanmishra@gmail.com

ORCIDs:

First Author	: https://orcid.org/0000-0002-2659-7568
Second Author	: https://orcid.org/0000-0001-5871-6958
Third Author	: https://orcid.org/0000-0002-3425-1727
Fourth Author	: https://orcid.org/0000-0003-0942-3467
Fifth Author	: https://orcid.org/0000-0001-7774-7290
Sixth Author	: https://orcid.org/0000-0002-2159-7710

AUTHOR CONTRIBUTIONS

All authors equally contributed to this paper.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest.

Integration of Rational Functions

Abstract. A rational function can always be integrated, that is, the integral of such a function is always an elementary function. The integration procedure is complex and consists of four steps: elimination of the common zero-points of the numerator and denominator, reduction to a true rational function, decomposition into partial fractions and integration of the obtained expressions using direct integration, substitution method or partial integration method. Integrating rational functions is important because integrals of rational functions of trigonometric functions as well as integrals of some irrational functions are reduced to integrals of rational functions by appropriate transformations.

Key words: mathematics; rational functions; polynomial functions; partial fractions.

1. INTRODUCTION

The tasks of teaching higher mathematics are to demonstrate to students the operation of the laws of materialistic dialectics, the essence of the scientific approach, the specifics of mathematics and its role in the realization of scientific and technological progress, using examples of mathematical concepts and methods. It is necessary to teach students the methods of research and solving mathematical formalized tasks, to develop in students the ability to analyze the obtained results, to inculcate in them the skills of independent study of literature on mathematics and its application.

Every rational fraction is integrated into elementary functions, but studying the general integration algorithm and alternative approaches is quite a difficult task. From lectures and seminars, it is impossible to sufficiently consider all the details, then the main part of the material is usually discussed in the classroom, and the rest is given to students for independent study. This paper was created within the educational and methodological complex of the discipline "Mathematical Analysis" and is dedicated to the topic "Integration of Rational Fractions". It considers those cases and methods that cannot be considered in their entirety in the classroom.

Integration of rational fractions. Expression of the form $\frac{Pm(x)}{Qn(x)}$, where *Pm(x)* and *Qn(x)* polynomials *m*-th and *n*-th degree, is called a rational fraction. A rational fraction is called proper if $m<n$ and improper if $m \geq n$. The problem of integrating a rational fraction can be reduced to the problem integration of a proper rational fraction, since any improper rational fraction can be represented by division by a column in as the sum of a polynomial and a proper rational fraction. We recite some research papers and books [1]-[7] for the right direction of integration of rational functions and its applications.

2. REPRESENTATION OF A RATIONAL FUNCTIONS AS A SUM OF POLYNOMIALS

Those who say that trigonometry is not needed in real life are not needed on the road. So, what are its common applied tasks? Measure the distance between inaccessible objects. The triangle technique is of great importance, which enables the measurement of distances to nearby stars in astronomy, between benchmarks in geography, control satellite navigation systems. The use of trigonometric techniques should also be noted, such as navigation techniques, music theory, acoustics, optics, financial market analysis, electronics, probability theory, statistics, biology, medicine (ultrasound and computed tomography), pharmacology, chemistry, number theory (and, as a result, cryptology, seismology, meteorology, oceanology, cartography, many parts of physics, topography and geodesy, architecture, phonetics, economics, electronic equipment, mechanical engineering, computer graphics, crystallography etc.

A rational function is a ratio of polynomials where the polynomial in the denominator should not be equal to zero. Doesn't that look like the definition of a rational number (which is of the form p/k, where k = 0)?

Polynomial functions play an important role in mathematics. They are generally simple to compute (requiring only calculations that can be performed manually) and can be used to model many real-world phenomena. In fact, scientists and mathematicians often simplify complex mathematical models by substituting a polynomial model that is "close enough" for their purposes.

Let it be $f(x) = \frac{P_n(x)}{Q_m(x)}$ rational function. If , it is $n \geq m$, i.e. not a real rational function, then by division $P_n(x)$ with $Q_m(x)$, f can be represented as a sum of polynomials and real rational functions in the form

$$f(x) = P(x) + \frac{R(x)}{Q_m(x)}$$

Let's say it is $f(x) = \frac{P_n(x)}{Q_m(x)}$ true rational function $(n < m)$. Then it can be decomposed into a sum of partial fractions of the form

$$\frac{A}{(x-a)^k} \text{ that is } \frac{Ax+B}{(x^2+bx+c)^k}$$

where the first type of fraction comes from real, and the second from the type of complex zero-points of polynomials in the denominator of the function f. Therefore, the problem is reduced to solving the integral of the form

$$\int \frac{A}{(x-a)^k}\,dx \text{ and } \int \frac{Ax+B}{(x^2+bx+c)^k}\,dx$$

where A and B are some real constants, is a natural number, and is a quadratic trinomial $x^2 + bx + c$ has complex-conjugate zero-points.

Integrals of the form $\int \frac{A}{(x-a)^k}\,dx$

1. If it is $k = 1$, he gives $\int \frac{A}{(x-a)}\,dx = A \ln|x-a| + C$.

2. If it is $k > 1$, then by changing the variable $x - a = t \Longrightarrow dx = dt$, we have it

$$\int \frac{A}{(x-a)^k}\,dx = \begin{Bmatrix} x - a = t \\ dx = dt \end{Bmatrix} = A\int t^{-k}dt = A\frac{t^{-k+1}}{-k+1} + C = \frac{A}{(1-k)(x-a)^{1-k}} + C.$$

3. REDUCTION TO TRUE RATIONAL FUNCTION

A rational function is a true rational function if the degree of the numerator is less than the degree of the denominator. If the degree of the numerator is greater than or equal to the degree of the denominator, we can divide the polynomials, so we have

$$f(x) = \frac{p(x)}{q(x)} = s(x) + \frac{r(x)}{q(x)}$$

where r and s are polynomials. The polynomial is the remainder when dividing, and the degree of r is $\int s\,(x)\,dx$ less than the power of q. Of course, it is solved by direct integration, so we can conclude the following:

Integrating rational functions is reduced to integrating real rational functions of the form

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q have no common zero-points and the degree of p is less than the degree of q.

Example: Dividing a polynomial give

$$\frac{x^3+x}{x^2-1} = x + \frac{2x}{x^2-1},$$

so, it is $\int \frac{x^3+x}{x^2-1}\, dx = \int \left(x + \frac{2x}{x^2-1}\right) dx$

$= \int x\, dx + \int \frac{2x}{x^2-1}\, dx = \frac{x^2}{2} + \int \frac{dt}{t}$, putting $\begin{Bmatrix} x^2 - 1 = t, \\ 2x\, dx = dt \end{Bmatrix}$

$= \frac{x^2}{2} + \ln|t| + C = \frac{x^2}{2} + \ln|x^2 - 1| + C.$

4. INTEGRATION BY PARTIAL FRACTIONS

Integration by partial fractions is a method used to decompose and then integrate a rational fraction that has complex terms in the denominator. By using a partial fraction, we calculate and decompose the expression into simpler terms so that we can easily calculate or integrate the resulting expression.

The basic idea in integration by partial fractions is to factor the denominator and then factor them into two different fractions where the denominators are the factors and the numerator is calculated accordingly. Let's learn more about the different forms used in integration by partial fractions, as well as the different methods.

4.1. We decompose the sub integral function into partial fractions

$$\frac{x-3}{x^3-x} = \frac{x-3}{x(x^2-1)} = \frac{x-3}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

Using the polynomial similarity method, we get that $A = 3, B = -1, C = -2$

$$\int \frac{x-3}{x^3-x}dx = \int \frac{x-3}{x(x-1)(x+1)}dx = \int \left[\frac{3}{x} - \frac{1}{x-1} - \frac{2}{x+1}\right]dx$$
$$= \int \frac{3}{x}dx - \int \frac{1}{x-1}dx - \int \frac{2}{x+1}dx = 3\ln|x| - \ln|x-1| - 2\ln|x+1| + C$$
$$= \ln\left|\frac{x^3}{(x-1)(x+1)^2}\right| + C$$

Integrals of the form $\int \frac{Ax+B}{(x^2+bx+c)}dx$. We have already solved integrals of this form using the shift method

$$\int \frac{Ax+B}{(x^2+bx+c)}dx = \int \frac{A(2x+b)+\left(B-\frac{Ab}{2}\right)}{ax^2+bx+c}dx$$
$$= \frac{A}{2}\int \frac{2x+b}{ax^2+bx+c}dx + \left(B - \frac{Ab}{2}\right).\int \frac{dx}{ax^2+bx+c}\,.$$

Where in $\int \frac{2ax+b}{x^2+bx+c}dx = \int \frac{(ax^2+bx+c)}{ax^2+bx+c}dx = \ln|ax^2+bx+c| + C$ and

$$\int \frac{dx}{x^2+bx+c} = \int \frac{dx}{(x+\frac{b}{2})^2+(c-\frac{b^2}{4^2})} \quad \text{putting} \begin{Bmatrix} x+\frac{b}{2}=t, c-\frac{b^2}{4^2}=k^2 \\ dx=dt \end{Bmatrix}$$
$$= \int \frac{dt}{t^2+k^2} = \frac{1}{k}\, arctgx\,\frac{t}{k} + C = \frac{1}{\sqrt{c-\frac{b^2}{4^2}}}\, arctgx\, \frac{x+\frac{b}{2}}{\sqrt{c-\frac{b^2}{4^2}}} + C.$$

Integrals of the form $\int \frac{Ax+B}{(x^2+bx+c)^k}dx$, wherein $k \geq 2, b^2-4c<0$. Let's perform the transformation of the sub integral function

$$\int \frac{Ax+B}{(x^2+bx+c)^k}dx = \int \frac{A(2x+b)+(B-\frac{Ab}{2})}{(x^2+bx+c)^k}dx$$
$$= \frac{A}{2}\int \frac{2x+b}{(x^2+bx+c)^k}dx + \left(B - \frac{Ab}{2}\right).\int \frac{dx}{(x^2+bx+c)^k}\,.$$

We can solve the first integral by shifting, where's he from $ax^2+bx+c=t$. So, we get $(2x+b)dx = dt$.

$$I_0 = \int \frac{Ax+B}{(x^2+bx+c)^k}dx = \int \frac{dt}{t^k} = \frac{1}{(1-k)t^{k-1}} + C = \frac{1}{(1-k)(x^2+bx+c)^{k-1}} + C.$$

Let's denote the second integral by $I_k = \int \frac{dx}{(ax^2+bx+c)^k}$ and perform the transformation

$$I_k = \int \frac{dx}{(ax^2+bx+c)^k} = \int \frac{dx}{\left[\left(x+\frac{b}{2}\right)^2+\left(c-\frac{b^2}{4^2}\right)\right]^k}$$

Introducing a shift $x+\frac{b}{2}=t, dx = dt$ and label $c-\frac{b^2}{4^2}=l^2$ and I_k we get

$$I_k = \int \frac{dx}{t^2+l^2}.$$

By combining the variable change method and partial integration, after a series of relations, a recursive formula for calculating the integral is obtained, which we will not derive here, but give it in its finished form

$$I_k = \frac{t}{2l(k-1)(t^2+l^2)^{k-1}} + \frac{2k-3}{2l^2(k-1)} \cdot \underbrace{\int \frac{dt}{(t^2+l^2)^{k-1}}}_{I_{k-1}},\ I_1 = \int \frac{dt}{t^2+l^2} = \frac{1}{l}\ arctgx\ \frac{t}{l} + C$$

Example: Let's calculate the integral $\int \frac{x-1}{(x^2+2x+3)^2}$

How it is $p=2$ and $q=3$, it is $p^2-4q=-8<0$, so the denominator has no real zero-points. Let's transform the integral

$$I = \int \frac{x-1}{(x^2+2x+3c)^2} dx = \int \frac{\frac{1}{2}(2x+2)+(-1-1)}{(x^2+2x+2)^2} dx = \frac{1}{2}\int \frac{2x+2}{(x^2+2x+3)^2} dx - 2 \cdot \int \frac{dx}{(x^2+2x+3c)^2}.$$

$$I_0 = \int \frac{2x+2}{(x^2+2x+3c)^2} dx = \left\{ \begin{matrix} x^2+2x+3c = t \\ (x^2+2x+3c)dx = (2x+2)dx = dt \end{matrix} \right\}$$

$$= \int \frac{dt}{t^2} = -\frac{1}{t} + C = \frac{1}{x^2+2x+3c} + C.$$

$$I_2 = \int \frac{dx}{(ax^2+bx+c)^2} = \int \frac{dx}{[(x+1)^2+(3-1)]^2} = \left\{ \begin{matrix} x+1 = t \\ dx = dt \end{matrix} \right\} = \frac{dt}{[t^2+2]^2}.$$

By substituting into the recurrent formula, we get

$$I_2 = \frac{t}{2\cdot 2(2-1)(t^2+2)^{2-1}} + \frac{2\cdot 2-3}{2\cdot 2(2-1)} \cdot \underbrace{\int \frac{dt}{(t^2+2)^{2-1}}}_{I_{2-1=1}}$$, where $I_1 = \int \frac{dt}{t^2+2} = \frac{1}{\sqrt{2}}\ arctg \frac{t}{2} + C,$

so, it is $I_2 = \frac{t}{4(t^2+2)} + \frac{1}{4} \cdot \frac{1}{\sqrt{2}}\ arctg\ \frac{t}{\sqrt{2}} + C = \{t = x+1\}$

$$= \frac{x+1}{4(x^2+2x+3)} + \frac{1}{4\sqrt{2}}\ arctg\ \frac{x+1}{\sqrt{2}} + C.$$

So, it is the final solution

$$I = \int \frac{x-1}{(x^2+2x+3c)^2} dx = \frac{1}{2} I_0 - 2I_2$$

$$= \frac{1}{x^2+2x+3c} + \frac{x+1}{4(x^2+2x+3)} + \frac{1}{4\sqrt{2}}\ arctg\ \frac{x+1}{\sqrt{2}} + C$$

$$= \frac{x+2}{4(x^2+2x+3)} + \frac{1}{4\sqrt{2}}\ arctg\ \frac{x+1}{\sqrt{2}} + C.$$

5. CONTRIBUTION TO THE INTEGRATION OF RATIONAL FUNCTIONS

Mathematical thinking and reasoning are the main goal of mathematics education. It is known from research that tasks involving problem solving, modelling and argumentation provide excellent opportunities to realize students' mathematical thinking [8]. Many topics in secondary education are already taught through tasks of this type (e.g. elementary algebra, exponential functions). However, there are still topics that are traditionally taught formally and mechanically, which makes it difficult for students to think and communicate in a conceptual sense. This is the case for the integral concept, which was introduced in most European countries in the last two years of secondary education and taught in a procedural way.

The integration of rational functions involves a number of cases that require different procedures. These procedures are based on operations such as calculating derivatives, dividing polynomials, factoring, and solving equations; everything is applied in order to decompose to the original integral in simpler cases that can be approached using already known techniques.

To find the rational part, we first need to know about square-free factorization. An important result in algebra is that every polynomial with rational coefficients can be uniquely factored into irreducible polynomials with rational coefficients, up to multiplication by a nonzero constant and rearrangement of factors, just as every integer can be uniquely factored into prime up to multiples of 1 and -1 and redistribution of factors (technically, it is with coefficients from the unique domain of factorization, for which rationales are a special case, up to the multiplication of unity, which for rationales is every constant except zero).

To integrate the corresponding rational function, we can apply the method of partial fractions. This method allows converting the integral of a complicated rational function into a sum of integrals of simpler functions. The denominators of partial fractions can contain unrepeated linear factors, repeated linear factors, unrepeated irreducible quadratic factors, and repeated irreducible quadratic factors.

6. CONCLUSION

The integration of rational functions includes a variety of cases requiring different procedures. These procedures are based on operations such as derivative calculation, division of polynomials, factorization and resolution of equations; all applied in order to decompose to the original integral in simpler cases that can be approached using the techniques already known.

REFERENCES

[1] C. Canuto and A. Tabacco. (2015). "Mathematical Analysis I". Springer Cham, Switzerland. 10.1007/978-3-319-12772-9.

[2] D. Lazard and R. Rioboo. (1990). "Integration of rational functions: Rational computation of the logarithmic part". *Journal of Symbolic Computation.* **9** (2): 113-115. 10.1016/s0747-7171(08)80026-0.

[3] R. H. C. Moir, R. M. Corless, M. M. Maza, and N. Xie. (2019). "Symbolic-numeric integration of rational functions". *Numerical Algorithms.* **83** (4): 1295-1320. 10.1007/s11075-019-00726-6.

[4] M.-T. Noda and E. Miyahiro. (1992). "A hybrid approach for the integration of a rational function". *Journal of Computational and Applied Mathematics.* **40** (3): 259-268. 10.1016/0377-0427(92)90182-w.

[5] M. T. Noda and E. Miyahiro. (1990). "On the symbolic/numeric hybrid integration". *Proceedings of the International Symposium on Symbolic and Algebraic Computation*. 10.1145/96877.96973.

[6] B. M. Trager. (1976). "Algebraic factoring and rational function integration". *Proceedings of the Third ACM Symposium on Symbolic and Algebraic Computation*. 10.1145/800205.806338.

[7] K. Gürlebeck, K. Habetha, and W. Sprößig. (2007). "Holomorphic functions in the plane and n-dimensional space". Springer Science & Business Media, Basel.

[8] J. Lithner. (2007). "A research framework for creative and imitative reasoning". *Educational Studies in Mathematics*. **67** (3): 255-276. 10.1007/s10649-007-9104-2.