ALGEBRAIC OPTIMIZATION: MARGINAL ANALYSIS WITHOUT CALCULUS

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Abstract

Teaching students to conduct marginal analysis before they have studied calculus is a major challenge in introductory economics courses. This paper offers a simple algebraic approach to optimization that allows students to extract explicit marginal revenue and marginal cost functions from quadratic total revenue and total cost functions. For first- or second-degree polynomials, the algebraic results are identical to those derived from differential calculus. The technique offers students a deeper understanding of the profit maximization process than can be obtained from spreadsheets and other conventional teaching methods. The resulting functions can be used to develop related insights regarding issues such as deadweight loss and competitive market adjustments. Numerical examples of monopoly and perfect competition are used to illustrate the algebraic optimization technique.

Key Words: algebraic optimization; marginal analysis; profit maximization

JEL classifications: A22, C00, D40

Introduction

Marginal analysis is clearly among the most central notions in the entire canon of economics. Indeed, Saunders (1994) identified marginal analysis as one of the seven most important economic concepts for developing critical thinking and decision-making skills, and Karunaratne *et al*. (2016) list it as one of the ten threshold concepts in the discipline.[2] The topic receives a fairly thorough treatment in intermediate microeconomics courses, where students are assumed to be familiar with differential calculus (Carbaugh and Prante, 2011), but most college students do not major in economics, and thus never study this subject at the intermediate level.[3]

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[2] Threshold concepts are those that lead to a transformed way of thinking. The economic threshold concepts listed by Karunaratne, *et al*. (2016) include: economic models, opportunity cost, marginal analysis, equilibrium and disequilibrium, market structures and interactions, elasticity, efficiency, comparative advantage, real versus nominal values, and cumulative causation. Saunders' (1994) list of the most important concepts includes opportunity cost, marginal analysis, independence, exchange, productivity, money, and markets and prices.

[3] In reviewing the transcripts of more than 8,100 college students, Bosshardt and Watts (2008) found that nearly 60 percent had completed at least one economics course, but the average was only 1.5 courses, and only those majoring in economics or business averaged two or more courses. Mumford and Ohland (2011) and Bosshardt and Walstad (2017) obtained similar results. Thus, Perumal (2012, p. 3) notes, "The majority of such students are non-economics majors who often study no more than one or two compulsory economic principles courses."

Given the importance of marginal analysis and the fact that most individuals receive no more than a principles-level introduction to this concept, it is essential to present the topic as thoroughly, effectively, and efficiently as possible in basic courses. Moreover, a rewarding experience at the principles level can encourage students to pursue the economics major. Yet introductory textbooks are challenged to present optimization methods to students whose assumed level of mathematical competence is restricted to algebra and geometry. Most resort to rather unsatisfactory approaches that often lack realism, do not fully utilize the mathematical backgrounds they assume students have, and fail to engage students in actually conducting marginal analysis. Thus, while noting that "marginal analysis is at the heart of economics," Asano (2006, p. 46) observes, "the majority of first year students, however, seem to struggle in applying it to a firm's profit maximization problem."

The present note offers an alternative approach to the study of marginal analysis in the context of profit maximization. The method is quite simple—relying exclusively on algebra—and yet exceptionally rigorous. For first- or second-degree polynomial functions, it yields results identical to those obtained from the application of derivatives. The following section describes the problem to be addressed and the difficulties inherent in the customary approaches. The algebraic optimization method is then presented, extended, and illustrated with some brief numerical examples. The article ends with a short conclusion.

Existing Approaches to Profit Maximization

Consider the fundamental problem of maximizing a firm's profit, given a demand curve and a total cost function. To keep the analysis as general as possible, we will not assume that the firm is necessarily a competitive price-taker; rather, we can specify a linear demand curve as

$$P = a - bQ \tag{1}$$

where Q denotes the firm's output, P denotes price, and a and b are demand parameters such that $a > 0$ and $b \geq 0$. (If the firm is assumed to be a perfectly competitive price-taker, then $b = 0$ and the price is constant at $P = a$ regardless of the firm's output level). Total revenue (TR) is easily obtained as the multiplicative product of price and quantity:

$$TR(Q) = aQ - bQ^2 . \tag{2}$$

Although most principles texts present total cost as a schedule of numerical values, a number of textbooks and other pedagogical materials specify a total cost function. We follow examples in Cowen and Tabbarok (2013), Stengel (2011), Hirschey (2006), Cheung (2005) and others in assuming the firm has a quadratic total cost (TC) function.[4] Let

$$TC(Q) = f + vQ + wQ^2 \tag{3}$$

where f denotes fixed cost, and v and w are parameters of the variable cost portion of TC. To ensure that total cost and marginal cost are both increasing with output, we may for simplicity

[4] Davis (2014) finds that, even at the intermediate level, most cost functions are quadratic.

assume $f > 0$, $v \geq 0$ and $w > 0$. [5] Profit (π) is just the difference between total revenue and total cost: $\pi(Q) = TR(Q) - TC(Q)$.

Introductory textbooks (such as Krugman and Wells 2009, Mankiw 2012, or Brue and McConnell 2009) rightly explain that marginal cost (*MC*) is the change in *TC* from producing an extra unit of output, and marginal revenue (*MR*) is the change in *TR* from selling an extra unit; at the profit-maximizing output, *MC* equals *MR*, so that producing and selling one unit more or less would reduce profit. Yet the *MR* and *MC* functions are typically asserted to be unobtainable in the absence of differential calculus. Thus, second-best approaches are generally adopted.

A common practice asks students to construct tables, or spreadsheets, showing values of *P*, *TR*, *TC*, *MR*, *MC*, and π at various levels of *Q*. Such exercises demonstrate numerically that, at the maximum profit, *MR* = *MC*. This is a workable, but rather awkward and time-consuming trial-and-error approach. The values of *Q* are frequently limited to single or low-double digits, so that *MR* and *MC* can be calculated on the basis of single-unit changes; this restricts the analysis to extremely low output values, and implicitly assumes that output is a discrete, rather than continuous, variable. In some cases where discrete values are imposed, *MR* slightly exceeds *MC* at the optimum, without an explanation of how to handle continuous variables. Other exercises direct students to use software, requiring them to first acquire the necessary computer skills, to construct spreadsheets or graph solutions (Larson and Swofford, 2015).[6] Perhaps most worrisome, spreadsheets allow students to “cheat” by using *TR* and *TC* to find the highest profit first, and only afterwards verify that *MR* = *MC* at the maximum profit, rather than using *MR* and *MC* to find the optimum—essentially converting marginal analysis into marginal confirmation.[7]

Yet other approaches involve rather opaque methods. Some textbooks, such as those by Stengel (2011) and Hirschey (2006), simply give students the *MR* and *MC* functions directly, without showing the derivations, and ask them to equate these functions to solve for the optimal *Q*. Still others, including Cowen and Tabbarok (2013), employ software such as Excel macros or Solver, requiring even greater computer skills than those needed for spreadsheets. Such methods avoid the tedious computations of the spreadsheet, but by hiding the derivations of *MR* and *MC*, they represent “black boxes” (Larson and Swofford, 2015). In effect, these methods signal to principles students that marginal analysis is extremely important, but unless and until they study

[5] If $v = 0$, the total cost function in (3) can be obtained from the Cobb-Douglas production function $Q = \sqrt{LK}$ where *L* is labor and capital (*K*) is fixed at *K* = 1 in the short term, so that $L = Q^2$. Then with *w* as the wage rate and *f* as the price of capital, $TC = fK + wL = f + wQ^2$.

[6] Some research suggests that purely graphical presentations may be confusing and even counterproductive to learning for some students (Cohn, *et al*., 2001; Zetland, *et al*., 2010).

[7] Other issues may also arise with spreadsheets. Depending upon the structure of the exercise, the orders of magnitude that the price and quantity should be are not obvious to students. Consequently, obtaining appropriate values can involve substantial guesswork, unless hints are given about the range and intervals of *Q* to insert into the table. Hirschey’s (2006, p. 40) exercises, for example, provide hints such as, “Establish a range for *Q* from 0 to 10,000 in increments of 1,000.” Even then, extensive computation may be needed. Finding values for six variables at ten levels of output requires 60 calculations. In texts that dispense with *TR* and *TC* functions entirely, the exercises typically begin with some values already inserted in the table and direct students to deduce the remaining entries.

calculus and intermediate microeconomics, they will not fully comprehend how to actually conduct profit maximization.[8]

As an alternative to these pedagogies, the algebraic optimization shown below is an exceptionally simple technique that works with any first- or second-degree polynomials such as equations (2) and (3). It is neither a trial-and-error approach nor a black box. It utilizes the basic algebraic knowledge that principles students are assumed to have, giving students the satisfaction of solving such problems analytically.[9] We first present the analysis and extensions at a general level, then offer some numerical examples for classroom use.

Algebraic Optimization

Recall the total revenue function in (2) above. The marginal revenue of the last unit sold can be defined as the total revenue at Q units minus the total revenue at $Q - 1$ units. Likewise, starting from any level of Q, the marginal revenue from selling one extra unit is the total revenue at $Q + 1$ units minus the total revenue at Q units. Let us consider the additional revenue from those *two* units: one more and one less than Q, and denote that as $2MR$. Then

$$2MR = TR(Q+1) - TR(Q-1), \tag{4}$$

or, after substituting from equation (2),

$$2MR = [a(Q+1) - b(Q+1)^2] - [a(Q-1) - b(Q-1)^2]. \tag{5}$$

Many of the terms in (4) cancel, and elementary algebra reduces this expression to

$$2MR = 2a - 4bQ. \tag{6}$$

Because equation (6) represents the additional revenue from two units of output, we can divide it by 2 to obtain the marginal revenue function:

$$MR = a - 2bQ. \tag{7}$$

Notice, importantly, that this is exactly the same marginal revenue function that we would derive by applying differential calculus to the *TR* function in (2). One advantage of this approach is readily apparent: by explicitly obtaining the *MR* function in (7) and comparing it to the demand

8 For example, although Stengel (2011, p. 4) claims "an understanding of basic algebra will suffice," he later notes (Stengel, 2011, p. 20), "How to apply differential calculus is beyond the scope of this text; however, here are the functions that can be derived from the revenue, cost, and profit functions" and proceeds to write out the *MR* and *MC* functions needed to find the optimal output. A few authors, such as Heckner and Kretschmer (2008) and Doviak (2005), attempt to teach enough calculus in the principles course to have students differentiate the *TR* and *TC* functions.

9 Niven (1981) and Cadeddu and Lai (2015) have offered other techniques for finding maxima and minima without calculus, but their methods have not been applied to economic problems, and generally require mathematics that are well beyond basic algebra.

function in (1), students immediately see that the *MR* curve is exactly twice as steep as the linear demand curve—an insight that is not quite as obvious from spreadsheet calculations.[10]

Similarly, the additional cost of the two unit difference between $Q + 1$ and $Q - 1$ can be expressed as $2MC = TC(Q+1) - TC(Q-1)$; or, after substitution from equation (3),

$$2MC = [f + v(Q+1) + w(Q+1)^2] - [f + v(Q-1) + w(Q-1)^2]. \tag{8}$$

Again, algebraic rearrangement simplifies this expression to

$$2MC = 2v + 4wQ, \tag{9}$$

and dividing by 2 gives the marginal cost function as

$$MC = v + 2wQ, \tag{10}$$

exactly the same result that would be obtained from (3) by taking a first derivative.

Using equations (7) and (10), the optimal level of output, Q_0, can be found by setting *MR* = *MC*. This yields [11]

$$Q_0 = \frac{a - v}{2(b + w)}. \tag{11}$$

The optimal price to charge, P_0, is then found by substituting (11) into (1), while total revenue and total cost are obtained by substituting (11) into (2) and (3), respectively. Thus, this technique achieves a precise optimum from first principles without the use of differential calculus. The relationship between algebraic and differential optimization, including an algebraic determination of the second-order condition, is provided in the Appendix.

Extensions

The derivation of linear *MR* and *MC* curves also allows some rigor to be added to the customary lessons regarding market structure and costs that accompany profit maximization. These might be reserved as optional exercises, perhaps for an honors section of the course. For example, students are typically taught that a monopoly creates a deadweight loss, or inefficiency, by pricing above marginal cost. In the current framework, we can find the output level at which the *MC* curve intersects the demand curve by using (10) and (1) to set $v + 2wQ = a - bQ$; designate that level of output as

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[10] Alternatively, we could define $MR = [TR(Q + \Delta Q) - TR(Q - \Delta Q)]/2\Delta Q$ with any $\Delta Q \neq 0$ to obtain the same outcome. $\Delta Q = 1$ is used for simplicity; another convenient choice would be $\Delta Q = 1/2$.

[11] Equivalently, as shown in the Appendix, we could subtract (3) from (2) to establish the profit function, then apply algebraic optimization to obtain marginal profit, and set the latter expression equal to zero in order to obtain (11).

$$Q_1 = \frac{a - v}{b + 2w}. \tag{12}$$

Because the demand and marginal cost curves are both linear, students can use the basic geometry of Figure 1 (where *D* indicates the demand curve) to measure the deadweight loss (*DWL*) as the area of the triangle between the demand and marginal cost curves, from Q_0 to Q_1.

$$DWL = (1/2)[(P_0 - MC(Q_0)][Q_1 - Q_0], \tag{13}$$

as in the first example below. Consumer surplus can be measured in a similar manner.

Another familiar lesson is that average total cost (*ATC*) declines when *MC* < *ATC*, and rises when *MC* > *ATC*, so that *MC* crosses *ATC* at the minimum point of *ATC*. The level of output at which this occurs can now easily be quantified. Dividing (3) by *Q* gives *ATC* as

$$ATC = (f / Q) + v + wQ \tag{14}$$

and setting *MC* = *ATC* from (10) and (14), respectively, gives $f / Q = wQ$, or

$$Q_L = \sqrt{f / w} \tag{15}$$

where the subscript *L* denotes the lowest average total cost.[12] (If *P* = *MC* < *ATC*, the firm suffers a loss, but remains in operation if $a > v$. Note from (11) that the shutdown point occurs at $a \leq v$.

A third lesson enabled by having an explicit *MC* function is that the market supply curve is the horizontal summation of individual supply curves, and an individual firm's supply curve is the portion of the *MC* curve that lies above average variable cost. In the second example below, we show how this can be used to illustrate the effects of market entry or exit.

This method of conducting marginal analysis is fully transparent and can be taught easily and quickly, especially with numerical examples of the type shown below. It allows students the satisfaction of deriving a solution to the profit maximization problem analytically, rather than via trial-and-error or being given *MR* and *MC* functions that have been obliquely derived elsewhere. And importantly, it promotes realism by facilitating exercises in which the optimal output and price need not be small, round, or even whole, numbers. Indeed, by selecting appropriate values for *a*, *b*, *f*, *v*, and *w*, instructors can readily construct exercises with any desired outcomes.[13] We provide two examples below.

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[12] A limitation of using a quadratic total cost function is that average variable cost is not convex, as depicted in many texts; such a curve requires a cubic *TC* function. As Davis (2014, p. 184) notes however, "Many cubic functions are not plausible representations of a firm's costs."

[13] In particular, cases in which $\pi \geq (<)\ 0$ can be constructed by selecting parameter values such that $(a - v)^2 \geq (<)\ 4(b + w)f$. Classroom and homework exercises can be simplified by setting $v = 0$, and competitive firms can be assumed to face infinitely elastic demand curves such that $b = 0$.

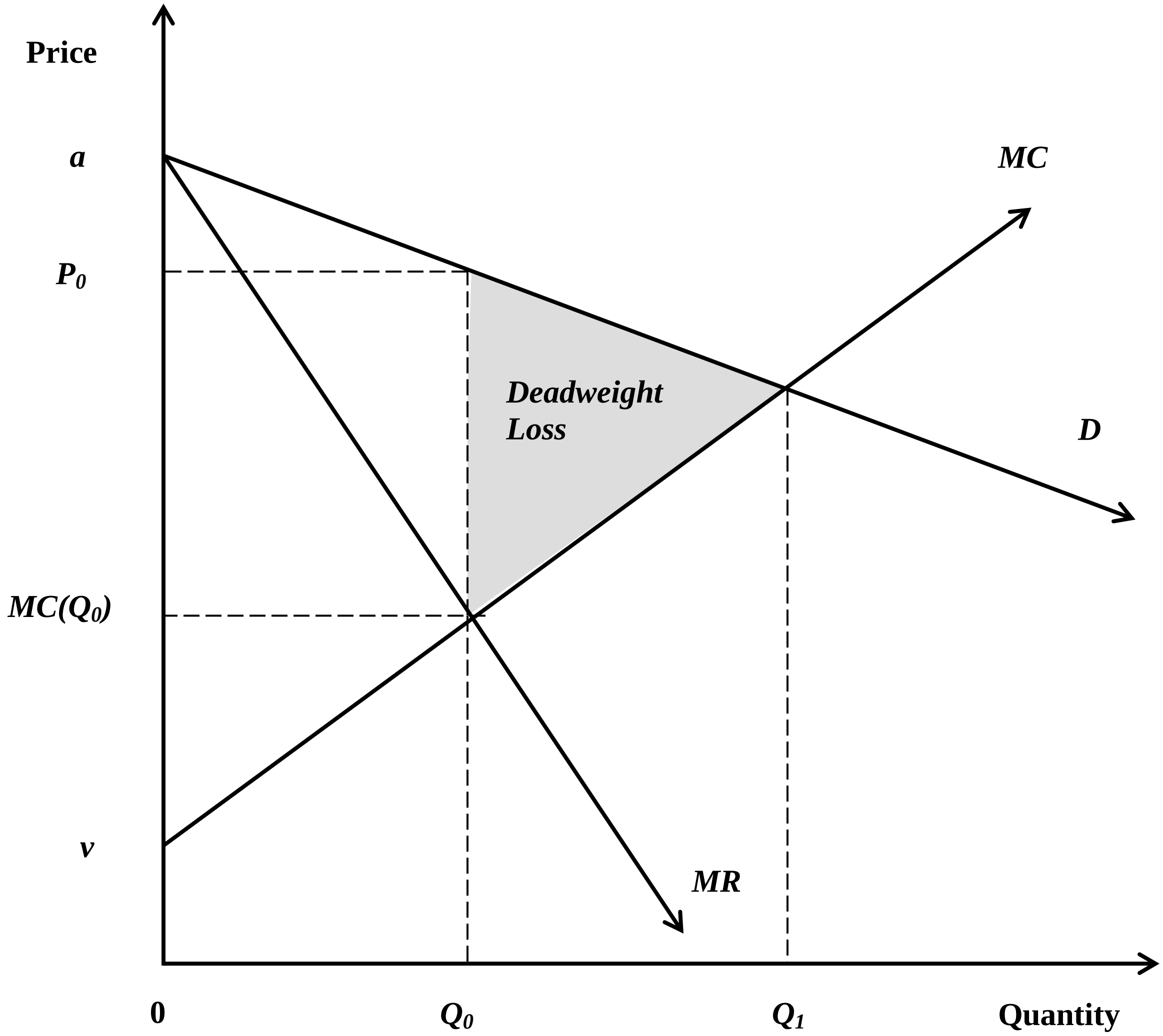

Figure 1. Monopoly Output, Price, and Deadweight Loss

Numerical Example: Monopoly

Consider the standard profit maximization problem assuming that a firm operating as a monopoly faces a demand function given by $P = 100 - 0.01Q$, so that total revenue is

$$TR = 100Q - 0.01Q^2 . \tag{16}$$

Let the total cost function be given by

$$TC = 2{,}800 + 4Q + 0.01Q^2 . \tag{17}$$

Suppose first that students attempt to find the optimum using a spreadsheet—that is, by calculating *TR* and *TC* at various levels of *Q*. Using an iterative, trial-and-error procedure, they eventually arrive at the output level that yields the maximum profit, but only after extensive calculations. Alternatively, they might resort to the black box of mathematical software. Neither practice ensures that they learn to obtain the solution by equating *MR* with *MC*.

A more efficient, transparent, and less frustrating, approach applies the analytic technique above. Using only algebra, students can readily calculate *MR* and *MC* as follows.

$$2MR = [100(Q+1) - 0.01(Q+1)^2] - [100(Q-1) - 0.01(Q-1)^2]$$
$$= 200 - 0.04Q. \tag{18}$$

$$MR = 100 - 0.02Q. \tag{19}$$

$$2MC = [2{,}800 + 4(Q+1) + 0.01(Q+1)^2] - [2{,}800 + 4(Q-1) + 0.01(Q-1)^2]$$
$$= 8 + 0.04Q. \tag{20}$$

$$MC = 4 + 0.02Q. \tag{21}$$

Finally, setting *MR* = *MC* yields the solution $Q_0 = 2{,}400$. Substituting this into the demand curve reveals that the optimal price to charge is $P_0 = 76$, and from *TR* and *TC* it is easy to calculate the maximum profit, $\pi_0 = 112{,}400$. If a student now chooses to use a spreadsheet to check the result, (s)he can immediately select values for *Q* near 2,400 and thus quickly confirm that profit is indeed maximized at that level of output, with *MR* = *MC* = 52. Note that in contrast to the more traditional approach, the solution is determined from the equivalence of *MR* and *MC*, and only confirmed through spreadsheet computations, rather than vice versa.

As suggested earlier, this example can be extended to calculate the deadweight loss from monopoly, by first finding the quantity of output at which the marginal cost curve intersects the demand curve. Setting $4 + 0.02Q = 100 - 0.01Q$ gives $Q_1 = 3{,}200$. Then the deadweight loss from monopoly is the triangular area $(1/2)(76 - 52)(3{,}200 - 2{,}400) = 9{,}600$.

Numerical Example: Perfect Competition

As a second example, consider a perfectly competitive firm confronting an initial market equilibrium price of *P* = 60, whose total revenue function is therefore *TR* = 60*Q*. Applying the technique above, students can calculate

$$MR = \frac{60(Q+1) - 60(Q-1)}{2} = 60 \tag{22}$$

which demonstrates that, for a perfectly competitive firm, the product price is the marginal revenue, so the optimality condition *MR* = *MC* becomes *P* = *MC*. Let the firm's total cost function be $TC = 100 + Q^2$ where for convenience *v* = 0 and *w* = 1. Using algebra, the marginal cost function is found as

$$MC = \frac{[100 + (Q+1)^2] - [100 + (Q-1)^2]}{2} = 2Q. \tag{23}$$

Setting *P* = *MC* yields an optimum at $Q_0 = 30$; then *TR* = 1,800 and *TC* = 1,000 so π = 800.

Such an example can also be used to illustrate the effects of market entry (or exit). Assuming that new (and identical) firms enter the market and erode profits, students can determine both the new market price and the output that each firm will produce, by using the following logic. Profit is eliminated when *TR* = *TC*, or equivalently, when *P* = *ATC*. Because the competitive firm optimizes where *P* = *MC*, the firm optimizes with zero profit when *P* = *ATC* = *MC*. In this example, average total cost is

$$ATC = (100/Q) + Q. \tag{24}$$

Setting (24) equal to (23) yields $Q_L = 10$. At 10 units of output, *ATC* = *MC* = *P* = 20. (Students can easily check that *ATC* is higher at any other level of output). Thus, in the absence of barriers to entry, new firms will enter this initially profitable market, increasing the market supply, until the equilibrium price falls to *P* = 20. Each firm will then experience *TR* = 200, *TC* = 200, and π = 0 in the long run.[14]

With additional details, this example can be extended even further. Assume that there are initially 40 firms in the market. Let the market demand be given by $P = 120 - 0.05q$, or $q = 2400 - 20P$, where market output *q* is the sum of output from *n* identical firms, so $q = nQ$. Then with each firm setting $P = 2Q$ or $Q = P/2$, the market supply from the 40 firms collectively was $q = 20P$. Setting market demand equal to market supply confirms that the initial equilibrium price is 60 and the equilibrium output from 40 firms is 1,200. When new firms enter the market, the price drops to *P* = 20 and each firm produces only *Q* = 10 units. The demand curve reveals that with *P* = 20, a total of *q* = 2,000 units are sold; thus, the final number of firms in the market can now be found as $n = q/Q = 200$. The market supply curve—the horizontal summation of individual firms' supply curves—has become $q = 100\,P$. Students can use the market supply and demand curves to verify that *P* = 20 is indeed the new equilibrium price. Of course, the example can easily be reconfigured (with $f > 900$) to illustrate market exit following initial losses.

As these examples suggest, enabling students to extract explicit *MC* and *MR* functions from *TC* and *TR* functions gives instructors the option to introduce as much or as little rigor into the course as they desire. Exercises may be as elaborate or elementary as needed. Indeed, output can be a continuous variable in fractional units at the optimum without increasing the complexity of student calculations.[15]

Conclusion

Principles-level economics students are expected to be familiar with algebra and geometry, yet in scrupulously avoiding calculus, introductory textbooks almost invariably resort to the simple arithmetic of spreadsheets to demonstrate profit maximization. That approach

[14] Textbooks often assert that all costs are variable in the long run, but as pointed out by Wang and Yang (2001), that is a misrepresentation caused by conflating fixed costs with sunk costs. Thus, our example follows Cheung (2005) in assuming that fixed cost persists as a component of total cost during the entry and exit process leading to the long run equilibrium.

[15] It should be emphasized that this technique is restricted to first- and second-degree polynomials, and should not be applied to other functional forms.

misses an opportunity to capitalize on the math skills that students already possess, and permits them to find the optimum through trial-and-error without actually using marginal analysis.

The main stumbling block is the derivation of marginal revenue and marginal cost functions, typically delayed until intermediate courses where optimization is taught with derivatives. Designed for measuring the effects of infinitesimal changes in variables, differential calculus is an ideal instrument for conducting marginal analysis, because it efficiently yields precise solutions to maximization and minimization problems. Certainly, algebra is not a perfect substitute for calculus, but applying algebra to quadratic revenue and cost functions can bring much of that efficiency, precision, and power to an introductory course. Working through profit maximization exercises like those above strengthens students' problem-solving ability and ensures that they perceive marginal analysis as the centerpiece of optimization, rather than as an afterthought. Thus, algebraic optimization can enrich students' understanding of one of the key concepts in economics. By facilitating examples in which output can be a continuous variable of any magnitude, this approach brings greater realism to the course. Greater realism, in turn, can both attract more students and lead to improved learning (Mearman, *et al*., 2014).

Naturally, because students differ in their responses to various teaching methods, no single technique is necessarily superior to all others. Tables or diagrams may be favored by some students, while algebraic optimization will be appreciated more by others, especially those with better algebraic skills. Likewise, some instructors may prefer to use conventional methods for teaching marginal analysis, but others will find it advantageous to add algebraic optimization to their pedagogical toolkits, especially because it need not replace traditional tools entirely. Algebraic optimization can easily be used in conjunction with spreadsheets and graphs.

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Appendix: Algebraic and Differential Optimization

This appendix elucidates the relationship between differential and algebraic optimization. Differential calculus measures the rate of change in a function for an infinitesimal increment to its argument. For any function $g(x)$, a right-hand side derivative can be expressed as

$$g'_R(x) = \lim_{\varepsilon \to 0} \frac{g(x+\varepsilon) - g(x)}{\varepsilon} \tag{A1}$$

and a left-hand side derivative is

$$g'_L(x) = \lim_{\varepsilon \to 0} \frac{g(x) - g(x-\varepsilon)}{\varepsilon}. \tag{A2}$$

The algebraic optimization technique is analogous to averaging the right-hand side and left-hand side derivatives, but with $\varepsilon = 1$ rather than vanishing. For any first-degree or second degree polynomial, this yields the same result as a derivative. Notice that for any $g(x) = \beta x^2$,

$$\frac{[g(x+1)-g(x)]+[g(x)-g(x-1)]}{2} = \frac{\beta(x^2+2x+1)-\beta(x^2-2x+1)}{2} = 2\beta x = g'(x). \quad \text{(A3)}$$

As an application, consider the total profit function obtained by subtracting (3) from (2):

$$\pi(Q) = [aQ - bQ^2] - [f + vQ + wQ^2]. \qquad \text{(A4)}$$

Applying algebraic optimization, we get the marginal profit function as follows:

$$M\pi(Q) = \frac{[\pi(Q+1)-\pi(Q)]+[\pi(Q)-\pi(Q-1)]}{2} = \frac{[\pi(Q+1)-\pi(Q-1)]}{2}$$
$$M\pi(Q) = a - v - 2(b+w)Q. \qquad \text{(A5)}$$

This is identical to the first derivative of (A4), and setting $M\pi(Q) = 0$ yields the optimum in (11).

Both Asano (2006) and Carbaugh and Prante (2011) have emphasized the need, at least in intermediate and advanced microeconomics courses, for checking the second-order condition to ensure that Q_0 represents a maximum rather than a minimum. Although it need not be taught at the principles level, it may be useful for instructors to recognize that algebraic optimization can also be used to check the second-order condition. By analogy to a second derivative, the "marginal marginal" profit function (for lack of a better phrase), can be written as

$$MM\pi(Q) = [M\pi(Q+1) - M\pi(Q-1)]/2. \qquad \text{(A6)}$$

After substitution from (A5), this becomes

$$MM\pi(Q) = \frac{[a-v-2(b+w)(Q+1)]-[a-v-2(b+w)(Q-1)]}{2}. \qquad \text{(A7)}$$

Rearranging gives the same second-order condition that would be obtained by using calculus:

$$MM\pi(Q) = -2(b+w) < 0. \qquad \text{(A8)}$$